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A coupled Ritz-finite element method for free vibration of rectangular thin and thick plates with general boundary conditions

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Abstract. A coupled method, that combines the Ritz method and the finite element (FE) method, is proposed to solve the vibration problem of rectangular thin and thick plates with general boundary conditions. The eigenvalue partial differential equation(s) of the plate is (are) first reduced to a set of eigenvalue ordinary differential equations by the application of the Ritz method. The resulting eigenvalue differential equations are then reduced to an eigenvalue algebraic equation system using the finite element method. The natural boundary conditions of the plate problem including the free edge and free corner boundary conditions are also implemented in a simple and accurate manner. Various boundary conditions including simply supported, clamped and free boundary conditions are considered. Comparisons with existing numerical and analytical solutions show that the proposed mixed method can produce highly accurate results for the problems considered using a small number of Ritz terms and finite elements. The proposed mixed Ritz-FE formulation is also compared with the mixed FE-Ritz formulation which has been recently proposed by the present author and his co-author. It is found that the proposed mixed Ritz-FE formulation is more efficient than the mixed FE-Ritz formulation for free vibration analysis of rectangular plates with Levy-type boundary conditions.

Keywords: Ritz method; higher order FEM; rectangular thin plates; rectangular thick plates; free edges; free corners; Levy-type boundary conditions

1. Introduction

Rectangular thin and thick plates are important structural elements that are widely used in various fields of engineering including mechanical, civil, aerospace, marine and structural engineering. Thus, a good understanding of the vibration characteristics of such structural components is crucial to the structural designers.

There are generally two kinds of methods which can be employed to determine the natural frequencies and mode shapes of rectangular plates, namely analytical and numerical methods. Analytical methods are very desirable since they provide valuable information about solution behavior over problem domain. But, their applications to plate problems are limited to rectangular plates with Levy-type boundary conditions (Leissa 1973, Xiang *et al*. 2002, Hashemi *et al*. 2012). This is due to the complexities introduced by the imposition of free edges and free corner boundary conditions. So, various approximate or numerical methods such as the Ritz method (Leissa 1973, Bassily and Dickinson 1975, Dawe and Roufaeil 1980, Bhat 1985, Felix *et al*. 2011, Eftekhari and Jafari 2012a, Chakraverty and Pradhan 2014), the method of superposition (Gorman 1978, Gorman and Ding 1996), the finite strip method (Dawe 1987, Ashour 2003, Akhras and Li 2007, Ovesy and Ghannadpour 2009, Azhari and Heidarpour 2011), the finite integral transform method (Zhong and Yin 2008), the extended Kantorovich approach (Jones and Milne 1976, Fallah *et al*. 2013), the Fourier and power series expansion methods (Bhaskar and Dhaoya 2009, Zhang and Li 2009), the finite difference method (Rajasekaran and Wilson 2013), the finite element method (Valizadeh *et al*. 2013, Pachenari and Attarnejad 2014), the differential quadrature method (DQM) (Bert *et al*. 1988, Darvizeh *et al*. 2002, Karami and Malekzadeh 2003, Civalek 2004, Malekzadeh *et al*. 2004, Lal and Saini 2013), the meshless method (Tsiatas and Yiotis 2013, Ragb *et al*. 2014), the spectral element method (Wu *et al*. 2014), and the discrete singular convolution (DSC) method (Ng *et al*. 2004, Wang and Xu 2010) have been developed to study the dynamic and stability behaviors of rectangular plates with general boundary conditions.

Among the approximate methods used for addressing the present problem, the finite element method (FEM) is one of the most popular and versatile techniques that has been widely used by many researchers to obtain approximate solutions for the natural frequencies of rectangular thin and thick plates (Reddy 1993, Zienkiewicz and Taylor 2000). It is powerful computationally due to its flexibility in handling complex geometries and boundary conditions. However, the number of unknowns and the amount of input data are very large in the FEM. A way for overcoming this limitation is to combine the FEM with other known analytical or approximate methods. In this regard, some researchers have combined the FEM with

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higher order methods such as the Ritz technique (Cheung 1968, 1976, Cheung *et al*. 1996), and the differential quadrature method (DQM) (Eftekhari and Jafari 2012b, 2014a). Although these mixed methods were shown to work well for some plate problems, their accuracy and convergence are not guaranteed for handling the plate problem with general boundary conditions. For example, in conventional finite strip method (FSM) (say mixed Ritz-FEM), only the geometric boundary conditions can be incorporated into the solution process. Hence, the conventional FSM may encounter some difficulties when dealing with rectangular plates involving free edges and free corners (Jafari and Eftekhari 2011). The mixed FE-DQM can be used to overcome some of the drawbacks of the FSM (Eftekhari and Jafari 2014a). For instance, the free edge boundary conditions can be easily implemented in this mixed method. However, similar to the mixed Ritz-DQM (Eftekhari and Jafari 2012c), the implementation of free corner boundary conditions is still an important issue in the mixed FE-DQ formulation.

To overcome the above-mentioned limitations, the present author and his co-author proposed recently a simple mixed method in which all the natural boundary conditions, including the free edge and free corner boundary conditions, are exactly implemented (Eftekhari and Jafari 2012d). In this method, the original plate problem is first reduced to two simple beam problems. One beam problem is discretized by the FEM while the other by the Ritz method. An analogue procedure was also proposed to implement the natural boundary conditions of the plate. The mixed FE-Ritz method has been successfully applied to study the free vibration of various benchmark thin plate problems. It has been found that the mixed FE-Ritz method can produce highly accurate solutions for vibration problem of plates involving free edges, free corners and irregular boundaries (Eftekhari and Jafari 2012d). However, as we will show in this paper, the mixed FE-Ritz method is not very efficient for solving plate problems with Levy-type boundary conditions. Besides, the application of the mixed FE-Ritz method was limited to *thin plate problems*. To improve its efficiency and applicability, this paper presents a mixed Ritz-FE formulation which is more suitable for analysis of such type of plates. This formulation is similar to the FE-Ritz approach, but the order of combined methods (FEM and Ritz method) is only reversed. Its stability, rate of convergence, and accuracy are challenged through the solution of some benchmark vibration problems. It is shown that the mixed Ritz-FEM can be used as an efficient tool for vibration analysis of rectangular thin and thick plates with general boundary conditions. Furthermore, it requires less computational effort compared with the FE-Ritz approach for vibration analysis of Levy-type rectangular plates.

2. Formulation for free vibration of thin rectangular plates

2.1 Governing equation and Ritz formulation

For an elastic isotropic thin rectangular plate with length a, width b, thickness h, mass density ρ, Young's modulus E, Poisson's ratio μ, bending stiffness $D = Eh^3 / [12(1-\mu^2)]$, the governing non-dimensional differential equation for free vibration is

$$w_{,XXXX} + 2\lambda^2 w_{,XXYY} + \lambda^4 w_{,YYYY} = \Omega^2 w \tag{1}$$

where a subscript comma denotes differentiation; w is the dimensionless mode function of the lateral deflection; $X = x / a$ and $Y = y / b$ are dimensionless coordinates; $\lambda = a / b$ is the aspect ratio; and $\Omega = \omega a^2 \sqrt{\rho h / D}$ is the dimensionless frequency, wherein ω is the circular frequency.

The boundary conditions of the rectangular thin plate are:

(I) Simply-supported edge (S)

$$w = w_{,XX} = 0 \quad \text{at} \quad X = 0 \quad \text{and/or} \quad X = 1 \tag{2}$$

$$w = w_{,YY} = 0 \quad \text{at} \quad Y = 0 \quad \text{and/or} \quad Y = 1 \tag{3}$$

(II) Clamped edge (C)

$$w = w_{,X} = 0 \quad \text{at} \quad X = 0 \quad \text{and/or} \quad X = 1 \tag{4}$$

$$w = w_{,Y} = 0 \quad \text{at} \quad Y = 0 \quad \text{and/or} \quad Y = 1 \tag{5}$$

(III) Free edge (F)

$$\begin{gathered} w_{,XXX} + (2-\mu)\lambda^2 w_{,XYY} = w_{,XX} + \mu\lambda^2 w_{,YY} = 0 \\ \text{at} \quad X = 0 \quad \text{and/or} \quad X = 1 \end{gathered} \tag{6}$$

$$\begin{gathered} w_{,YYY} + \frac{2-\mu}{\lambda^2} w_{,YXX} = w_{,YY} + \frac{\mu}{\lambda^2} w_{,XX} = 0 \\ \text{at} \quad Y = 0 \quad \text{and/or} \quad Y = 1 \end{gathered} \tag{7}$$

For a free corner formed by the intersection of two free edges, the additional condition

$$w_{,XY} = 0 \tag{8}$$

must also be satisfied at the corner (Leissa 1973).

Using the separation of variable technique, the transverse deflection of the plate is approximated along the X-axis by using the following series

$$w(X,Y) = \sum_{j=1}^{n} \Psi_j(Y)\Phi_j(X) \tag{9}$$

where $\Psi_j(Y)$ are undetermined parameters, $\Phi_i(X)$ are orthonormal approximation functions that satisfy the geometric boundary conditions of the plate in the X-direction (see Bhat (1985) for details), and n is the number of solution terms. Substituting Eq. (9) into Eq. (1), multiplying both sides of the resulting equation by $\Phi_i(X)$, and performing the integration over the length of the plate ($0 \le X \le 1$), we obtain

$$\begin{gathered} [A]\{\Psi\} + 2\lambda^2[C]\{\Psi_{,YY}\} + \lambda^4[B]\{\Psi_{,YYYY}\} = \\ \Omega^2[B]\{\Psi\} \end{gathered} \tag{10}$$

where (i, j = 1, 2,..., n)

$$A_{ij} = \int_0^1 \Phi_i \Phi_{j,XXXX} \mathrm{d}X = [\Phi_i \Phi_{j,XXX}]_0^1 - [\Phi_{i,X} \Phi_{j,XX}]_0^1 + \int_0^1 \Phi_{i,XX} \Phi_{j,XX} \mathrm{d}X = A_{ij}^{*} + A_{ij}^{**} + A_{ij}^{***} \tag{11}$$

$$B_{ij} = \int_0^1 \Phi_i \Phi_j \mathrm{d}X \tag{12}$$

$$C_{ij} = \int_0^1 \Phi_i \Phi_{j,XX} \mathrm{d}X = [\Phi_i \Phi_{j,X}]_0^1 - \int_0^1 \Phi_{i,X} \Phi_{j,X} \mathrm{d}X = C_{ij}^{*} + C_{ij}^{**} \tag{13}$$

$$\{\Psi\} = [\Psi_1 \;\; \Psi_2 \;\; . \; . \; . \;\; \Psi_n]^T \tag{14}$$

$$\{\Psi_{,YY}\} = [\Psi_{1,YY} \;\; \Psi_{2,YY} \;\; . \; . \; . \;\; \Psi_{n,YY}]^T \tag{15}$$

$$\{\Psi_{,YYYY}\} = [\Psi_{1,YYYY} \;\; \Psi_{2,YYYY} \;\; . \; . \; . \;\; \Psi_{n,YYYY}]^T \tag{16}$$

It can be seen that the application of separation of variable technique with Ritz procedure results in a system of eigenvalue ordinary differential equations of the fourth-order. This system will be further discretized by the application of the FEM. The details will be given in the next sub-section. It is interesting to note that since we used orthonormal functions as the Ritz trial functions, the matrix [B] in Eq. (10) is an identity matrix. This will improve the efficiency of the proposed formulation significantly. However, the formulation will be presented for the general case where this matrix is not an identity matrix.

2.2 FEM analogues of resulting system of eigenvalue ordinary differential equations

At this stage, we apply the FEM to discretize the resulting system of eigenvalue ordinary differential equations given in Eq. (10). The domain of the problem along the Y-axis is first divided into N number of equal length finite elements. The polynomial approximation of the solution within a typical finite element is then assumed as ($Y_e \le Y \le Y_{e+1}$)

$$\Psi_i^e(Y) = \sum_{j=1}^{p+1} \alpha_j^{i,e} \psi_j^e(Y), \qquad i = 1,2,\ldots,n \tag{17}$$

where $\alpha_j^{i,e}$ (f, i = 1, 2,..., n, e = 1, 2,..., N) are nodal values of the eth finite element (displacements and slopes) and $\psi_j^e(Y)$ are the interpolation functions of degree p. Now, consider the kth equation of the system (10)

$$\begin{aligned} &(A_{k1}\Psi_1 + A_{k2}\Psi_2 + \ldots + A_{kn}\Psi_n) + \\ &2\lambda^2(C_{k1}\Psi_{1,YY} + C_{k2}\Psi_{2,YY} + \ldots + C_{kn}\Psi_{n,YY}) + \\ &\lambda^4(B_{k1}\Psi_{1,YYYY} + B_{k2}\Psi_{2,YYYY} + \ldots + B_{kn}\Psi_{n,YYYY}) = \\ &\Omega^2(B_{k1}\Psi_1 + B_{k2}\Psi_2 + \ldots + B_{kn}\Psi_n) \end{aligned} \tag{18}$$

Substituting Eq. (17) into Eq. (18), multiplying both sides of resulting equation by $\psi_j^e(Y)$ and performing the integration over the length of the eth finite element ($Y_e \le Y \le Y_{e+1}$), we obtain

$$\begin{aligned} &[\overline{B}^e]\left(A_{k1}\{\alpha^{1,e}\} + A_{k2}\{\alpha^{2,e}\} + \ldots + A_{kn}\{\alpha^{n,e}\}\right) + \\ &2\lambda^2[\overline{C}^e]\left(C_{k1}\{\alpha^{1,e}\} + C_{k2}\{\alpha^{2,e}\} + \ldots + C_{kn}\{\alpha^{n,e}\}\right) + \\ &\lambda^4[\overline{A}^e]\left(B_{k1}\{\alpha^{1,e}\} + B_{k2}\{\alpha^{2,e}\} + \ldots + B_{kn}\{\alpha^{n,e}\}\right) = \\ &\Omega^2[\overline{B}^e]\left(B_{k1}\{\alpha^{1,e}\} + B_{k2}\{\alpha^{2,e}\} + \ldots + B_{kn}\{\alpha^{n,e}\}\right) \end{aligned} \tag{19}$$

where (i, j = 1, 2,..., p+1, k = 1, 2,..., n)

$$\overline{A}_{ij}^e = \int_{Y_e}^{Y_{e+1}} \psi_i^e \psi_{j,YYYY}^e \mathrm{d}Y = [\psi_i^e \psi_{j,YYY}^e]_{Y_e}^{Y_{e+1}} - [\psi_{i,Y}^e \psi_{j,YY}^e]_{Y_e}^{Y_{e+1}} + \int_{Y_e}^{Y_{e+1}} \psi_{i,YY}^e \psi_{j,YY}^e \mathrm{d}Y = \hat{A}_{ij}^e + \hat{\hat{A}}_{ij}^e + \hat{\hat{\hat{A}}}_{ij}^e \tag{20}$$

$$\overline{B}_{ij}^e = \int_{Y_e}^{Y_{e+1}} \psi_i^e \psi_j^e \mathrm{d}Y \tag{21}$$

$$\overline{C}_{ij}^e = \int_{Y_e}^{Y_{e+1}} \psi_i^e \psi_{j,YY}^e \mathrm{d}Y = [\psi_i^e \psi_{j,Y}^e]_{Y_e}^{Y_{e+1}} - \int_{Y_e}^{Y_{e+1}} \psi_{i,Y}^e \psi_{j,Y}^e \mathrm{d}Y = \hat{C}_{ij}^e + \hat{\hat{C}}_{ij}^e \tag{22}$$

$$\{\alpha^{k,e}\} = [\alpha_1^{k,e} \;\; \alpha_2^{k,e} \;\; . \; . \; . \;\; \alpha_{p+1}^{k,e}]^T \tag{23}$$

The assembly of finite element equations (19) and imposition of geometric boundary conditions are similar to those of one-dimensional beam element equations. By doing so, we obtain the following assembled equations

$$\begin{aligned} &[\overline{B}]\left(A_{k1}\{\alpha^1\} + A_{k2}\{\alpha^2\} + \ldots + A_{kn}\{\alpha^n\}\right) + \\ &2\lambda^2[\overline{C}]\left(C_{k1}\{\alpha^1\} + C_{k2}\{\alpha^2\} + \ldots + C_{kn}\{\alpha^n\}\right) + \\ &\lambda^4[\overline{A}]\left(B_{k1}\{\alpha^1\} + B_{k2}\{\alpha^2\} + \ldots + B_{kn}\{\alpha^n\}\right) = \\ &\Omega^2[\overline{B}]\left(B_{k1}\{\alpha^1\} + B_{k2}\{\alpha^2\} + \ldots + B_{kn}\{\alpha^n\}\right) \end{aligned} \tag{24}$$

Eq. (24) can be expressed for all the k-values in the compact form as

$$[\widetilde{K}]\{\widetilde{\alpha}\} = \Omega^2[\widetilde{M}]\{\widetilde{\alpha}\} \tag{25}$$

where the sub-matrices $[\widetilde{K}_{ij}]$ and $[\widetilde{M}_{ij}]$ are given by (i, j = 1, 2,..., n)

$$[\widetilde{K}_{ij}] = A_{ij}[\overline{B}] + 2\lambda^2 C_{ij}[\overline{C}] + \lambda^4 B_{ij}[\overline{A}] \tag{26}$$

$$[\widetilde{M}_{ij}] = B_{ij}[\overline{B}] \tag{27}$$

and

$$\{\tilde{\alpha}\}=[\{\alpha^1\}^T\ \{\alpha^2\}^T\ \ldots\ \{\alpha^n\}^T]^T \tag{28}$$

The eigenvalue problem (25) can be solved for the eigenvalues Ω, if the natural boundary conditions of the plate problem are also applied. The procedure is detailed in the following section.

2.3 Implementation of natural boundary conditions of the rectangular thin plate

According to the new variational formulation recently proposed by the present author and his co-author (Eftekhari and Jafari 2012a, 2014b, c), the natural boundary conditions of the problem can be integrated along its boundaries or analogized by the approximate method. The resulting analog equations can then be used to modify the stiffness matrix of the problem. It is noted that for the case of rectangular plates with combination of simply supported and clamped edges, there is no need to any modification of the stiffness matrix. Therefore, in this section, the required boundary analog equations are derived only for the case of rectangular plates involving free edges.

2.3.1 Implementation of free edge boundary conditions in the X-direction

Consider a rectangular thin plate having free edges at $X = 0$ and $X = 1$. The boundary conditions of the plate for this case are given in Eq. (6). The main idea of the proposed variational method (Eftekhari and Jafari 2012a) is that these boundary conditions can be related to the boundary terms A^*_{ij} and A^{**}_{ij} defined in Eq. (11). Substituting the Ritz approximation (given in Eq. (9)) into Eq. (6) and performing some mathematical manipulations, we obtain the following boundary analog equations

$$[A^*]\{\Psi\}=-(2-\mu)\lambda^2[C^*]\{\Psi_{,YY}\} \tag{29}$$

$$[A^{**}]\{\Psi\}=\mu\lambda^2[C^*]^T\{\Psi_{,YY}\} \tag{30}$$

where the matrix $[C^*]$ is defined in Eq. (13). Eqs. (29) and (30) can be used to modify the stiffness matrix of the plate.

2.3.2 Implementation of free edge boundary conditions in the Y-direction

Consider a rectangular thin plate having free edges at $Y = 0$ and $Y = 1$. The boundary conditions of the plate for this case are given in Eq. (7). To implement the free edge boundary conditions at $Y = 0$ and $Y = 1$, these boundary conditions should be expressed in terms of the boundary terms $\hat{A}^e_{ij}$ and $\hat{\hat{A}}^e_{ij}$ defined in Eq. (20). Substituting the Ritz approximation (given in Eq. (9)) and the finite element approximation (given in Eq. (17)) into Eq. (7) and performing some mathematical manipulations, we obtain the following boundary analog equations

$$\begin{aligned}&B_{kl}\left([\tilde{A}^3]-[\tilde{A}^4]\right)\{\alpha^l\}=\\&-\frac{\mu}{\lambda^2}C_{kl}\left([\tilde{D}^1]^T-[\tilde{D}^2]^T\right)\{\alpha^l\},\end{aligned}\qquad k,l=1,2,\ldots,n \tag{31}$$

$$\begin{aligned}&B_{kl}\left([\tilde{A}^1]-[\tilde{A}^2]\right)\{\alpha^l\}=\\&-\frac{(2-\mu)}{\lambda^2}C_{kl}\left([\tilde{D}^1]-[\tilde{D}^2]\right)\{\alpha^l\}\end{aligned} \tag{32}$$

wherein B_{kl} and C_{kl} are defined in Eqs. (12) and (13), respectively, and

$$\begin{aligned}&[\tilde{A}^1]=[\bar{A}^1]_{\exp},\quad [\tilde{A}^2]=[\bar{A}^2]_{\exp},\quad [\tilde{A}^3]=[\bar{A}^3]_{\exp},\\&[\tilde{A}^4]=[\bar{A}^4]_{\exp},\quad [\tilde{D}^1]=[\bar{D}^1]_{\exp},\quad [\tilde{D}^2]=[\bar{D}^2]_{\exp}\end{aligned} \tag{33}$$

$$\bar{A}^1_{ij}=[\psi^e_i\psi^e_{j,YYY}]^{e=N}_{Y=Y_{e+1}},\qquad \bar{A}^2_{ij}=[\psi^e_i\psi^e_{j,YYY}]^{e=1}_{Y=Y_e} \tag{34}$$

$$\bar{A}^3_{ij}=[\psi^e_{i,Y}\psi^e_{j,YY}]^{e=N}_{Y=Y_{e+1}},\qquad \bar{A}^4_{ij}=[\psi^e_{i,Y}\psi^e_{j,YY}]^{e=1}_{Y=Y_e} \tag{35}$$

$$\bar{D}^1_{ij}=[\psi^e_i\psi^e_{j,Y}]^{e=N}_{Y=Y_{e+1}},\qquad \bar{D}^2_{ij}=[\psi^e_i\psi^e_{j,Y}]^{e=1}_{Y=Y_e} \tag{36}$$

where the subscript "exp" means that the associated matrix is written in the *global coordinate system*. Eqs. (31) and (32) can be used to modify the stiffness matrix of the plate.

2.3.3 Implementation of free corner boundary condition

As pointed out earlier, for the case of corners formed by the intersection of two free edges, the additional condition $w_{,XY} = 0$ must also be satisfied at the corners. Consider a rectangular thin plate having such a corner at $(X, Y) = (X^*, Y^*)$ (where (X^*, Y^*) = (0, 0), (0, 1), (1, 0) or (1, 1)). Substituting the Ritz approximation (given in Eq. (9)) and the finite element approximation (given in Eq. (17)) into this equation and performing some mathematical manipulations, we obtain the following boundary analog equations

$$\begin{aligned}C^1_{k1}[\tilde{C}^1]\{\alpha^1\}+C^1_{k2}[\tilde{C}^1]\{\alpha^2\}+\ldots+C^1_{kn}[\tilde{C}^1]\{\alpha^n\}=\{0\},\\ k=1,2,\ldots,n\end{aligned} \tag{37}$$

where

$$C^1_{ij}=[\Phi_i\Phi_{j,X}]_{X=X^*},\quad X^*=0\ \text{ or }\ 1 \tag{38}$$

$$[A^{**}]\{\Psi\}=\mu\lambda^2[C^*]^T\{\Psi_{,YY}\} \tag{39}$$

$$[\tilde{C}^1]=[\bar{C}^1]_{\exp} \tag{40}$$

Eq. (37) can be used to modify the stiffness matrix of the plate.

2.4 Numerical results

To demonstrate the stability, rate of convergence and accuracy of the proposed coupled approach, natural frequencies of square thin plates with different boundary conditions are evaluated and the results are shown in Tables 1-4. To simplify the notation, the boundary conditions of the plate are denoted by letters S (simply supported), C

Table 1 Convergence and comparison of natural frequencies of a clamped square thin plate (N = 1)

n	p	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
3	7	35.9900	73.4201	74.1843	108.4270	137.2938
	9	35.9881	73.4037	74.1797	108.4111	132.0998
	13	35.9881	73.4037	74.1797	108.4111	131.9019
	15	35.9881	73.4037	74.1797	108.4111	131.9019
5	7	35.9881	73.4121	73.4201	108.2574	132.0998
	9	35.9855	73.3947	73.4121	108.2387	131.7789
	13	35.9854	73.3946	73.4118	108.2386	131.6646
	15	35.9854	73.3946	73.4118	108.2386	131.6646
7	7	35.9881	73.4037	73.4118	108.2387	131.9035
	9	35.9854	73.3941	73.3947	108.2179	131.6656
	13	35.9852	73.3939	73.3941	108.2174	131.5816
	15	35.9852	73.3939	73.3941	108.2174	131.5816
9	7	35.9881	73.4037	73.4118	108.2386	131.9019
	9	35.9854	73.3941	73.3946	108.2175	131.6646
	13	35.9852	73.3939	73.3939	108.2166	131.5808
	15	35.9852	73.3939	73.3939	108.2166	131.5808
Leissa (1973)		35.992	73.413	73.413	108.27	131.64

Table 2 Convergence and comparison of natural frequencies of a clamped square thin plate (n = 11)

p	n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
3	2	36.4831	74.1918	92.3989	124.8356	132.9960
	6	35.9935	73.4170	73.5077	108.3535	131.8358
	10	35.9864	73.3976	73.4090	108.2365	131.6376
	20	35.9853	73.3941	73.3948	108.2179	131.5847
	30	35.9852	73.3939	73.3940	108.2168	131.5816
	40	35.9852	73.3939	73.3939	108.2166	131.5810
	45	35.9852	73.3939	73.3939	108.2166	131.5809
	50	35.9852	73.3939	73.3939	108.2166	131.5809
5	1	35.9995	73.5345	74.1797	108.4111	132.2604
	2	35.9870	73.4019	73.4286	108.2275	131.8262
	4	35.9853	73.3942	73.3942	108.2177	131.5816
	6	35.9852	73.3939	73.3939	108.2167	131.5809
	8	35.9852	73.3939	73.3939	108.2166	131.5808
7	1	35.9881	73.4037	73.4118	108.2386	131.9019
	2	35.9852	73.3941	73.3941	108.2175	131.5852
	3	35.9852	73.3939	73.3939	108.2166	131.5808
	4	35.9852	73.3939	73.3939	108.2165	131.5808
Leissa (1973)		35.992	73.413	73.413	108.27	131.64

(clamped), and F (free). For instance, the symbol SFCF denotes that the plate has a simply supported edge at $X = 0$, a free edge at $Y = 0$, a clamped edge at $X = 1$, and a free edge at $Y = 1$. In all computation, the Poisson's ratio $\mu = 0.3$ is taken.

In applying the proposed method to vibration problem of thin plates, the domain of the problem along the Y axis is divided into N equal length finite elements with pth order interpolation functions, and n number of Ritz terms is considered for approximation of the solution in the X-direction.

Table 1 illustrates the convergence study for the first

Table 3 Convergence and comparison of natural frequencies of Levy-type square thin plates (N = 1)

Plate	$p = n$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
SSSS	7	19.7392	49.3481	49.3486	78.9573	98.7013
	9	19.7392	49.3480	49.3480	78.9568	98.6961
	11	19.7392	49.3480	49.3480	78.9568	98.6960
	Leissa (1973)	19.7392	49.3480	49.3480	78.9568	98.6960
SCSS	7	23.6464	51.6752	58.6582	86.1493	100.2807
	9	23.6463	51.6743	58.6464	86.1345	100.2699
	11	23.6463	51.6743	58.6464	86.1345	100.2698
	Leissa (1973)	23.6463	51.6743	58.6464	86.1345	100.2698
SCSC	7	28.9514	54.7478	69.3439	94.6021	102.2458
	9	28.9509	54.7431	69.3270	94.5853	102.2168
	11	28.9509	54.7431	69.3270	94.5853	102.2162
	Leissa (1973)	28.9509	54.7431	69.3270	94.5853	102.2162
SSSF	7	11.6845	27.7563	41.1969	59.0662	61.8711
	9	11.6845	27.7563	41.1967	59.0655	61.8607
	11	11.6845	27.7563	41.1967	59.0655	61.8606
	Leissa (1973)	11.6845	27.7563	41.1967	59.0655	61.8606
SCSF	7	12.6874	33.0654	41.7027	63.0159	72.4153
	9	12.6874	33.0651	41.7019	63.0148	72.3977
	11	12.6874	33.0651	41.7019	63.0148	72.3976
	Leissa (1973)	12.6874	33.0651	41.7019	63.0148	72.3976
SFSF	7	9.6314	16.1348	36.7257	38.9453	46.7383
	9	9.6314	16.1348	36.7256	38.9450	46.7381
	11	9.6314	16.1348	36.7256	38.9450	46.7381
	Leissa (1973)	9.6314	16.1348	36.7256	38.9450	46.7381

five natural frequencies, $\Omega = \omega a^2\sqrt{\rho h/D}$, of clamped square thin plates with respect to p (order of FEM interpolation functions) and n (number of Ritz terms). Only *one finite element* is considered in the Y-direction (i.e., N = 1). The results are also compared with the results obtained by the conventional Ritz method (Leissa 1973). It can be seen from Table 1 that the present results converge very quickly and agree well with those of Ritz approach.

Table 2 shows the convergence of solutions for natural frequencies of clamped square thin plates with respect to p and N (number of finite elements). These results are obtained using n = 11. It can be seen from Table 2 that the results of proposed method converge uniformly with increasing number of finite elements to their final values. It can also be seen that when lower order algorithms are used, a larger number of finite elements are required to achieve accurate solutions. For example, when FEM with Hermit interpolation functions is employed (i.e., when p = 3), the convergence is achieved with 45 finite elements. By increasing the order of interpolation functions, the convergence rate is improved significantly and accurate solutions are achieved by using a smaller number of finite elements.

Table 3 shows the convergence study of the first five dimensionless natural frequencies of Levy-type square plates (i.e., plates with two opposite sides simply supported). The number of Ritz terms (i.e., n) and the order of interpolation functions (i.e., p) are taken to be the same (i.e., we assumed that n = p). These results are obtained using only *one finite element*. The analytical solutions of Leissa (1973) are also shown in this Table for comparison purposes. It can be clearly seen from table 3 that the present results converge very quickly and agree very closely with the exact solution values of Leissa (1973) even to all available significant digits.

The first five non-dimensional frequency parameters for square plates involving free corners are tabulated in Table 4. These results are obtained using *two eleventh-order finite elements* and different values of n. The results are also compared with the results obtained by the conventional Ritz method (Leissa 1973). It can be seen from Table 4 that the results of proposed method have a close agreement with the results of conventional Ritz method (Leissa 1973). However, the results of proposed approach are slightly smaller than the Ritz solution values of Leissa (1973). The reason for this is the lack of satisfaction of free edge and free corner boundary conditions in the Leissa's Ritz formulation. These results show the effectiveness of the proposed mixed Ritz-FE approach for vibration analysis of thin square plates with general boundary conditions.

Table 4 Convergence and comparison of natural frequencies of square thin plates involving free corners (N = 2, p = 11)

Plate	n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
SSSS	7	3.3671	17.3164	19.2930	38.2113	51.0392
	9	3.3670	17.3164	19.2929	38.2112	51.0354
	11	3.3670	17.3164	19.2929	38.2112	51.0354
	Leissa (1973)	3.3687	17.407	19.367	38.291	51.324
SCSS	7	5.3525	19.0775	24.6757	43.0936	52.7079
	11	5.3513	19.0757	24.6714	43.0889	52.7075
	13	5.3511	19.0753	24.6707	43.0879	52.7075
	15	5.3511	19.0751	24.6704	43.0873	52.7075
SCSC	17	5.3510	19.0750	24.6701	43.0869	52.7075
	Leissa (1973)	5.364	19.171	24.768	43.191	53.000
	7	6.9217	23.9127	26.5871	47.6628	62.7224
	11	6.9198	23.9054	26.5853	47.6537	62.7076
SSSF	13	6.9196	23.9044	26.5851	47.6524	62.7066
	15	6.9195	23.9038	26.5850	47.6517	62.7062
	17	6.9194	23.9035	26.5850	47.6512	62.7059
	Leissa (1973)	6.942	24.034	26.681	47.785	63.039
SCSF	7	3.4722	8.5098	21.2921	27.1995	30.9675
	11	3.4712	8.5073	21.2863	27.1990	30.9585
	13	3.4711	8.5069	21.2853	27.1989	30.9568
	15	3.4710	8.5066	21.2847	27.1988	30.9558
SFSF	17	3.4710	8.5065	21.2844	27.1988	30.9553
	Leissa (1973)	3.9417	8.5246	21.429	27.331	31.111
	7	6.6437	14.9015	25.3758	26.0006	48.4538
	9	6.6437	14.9015	25.3757	26.0005	48.4495

3. Formulation for free vibration of thick rectangular plates

3.1 Governing equations and Ritz formulation

Consider the free vibration of an elastic isotropic thick rectangular plate with length a, width b, thickness h, mass density ρ, Young's modulus E, Poisson's ratio μ, shear correction factor k^2, bending stiffness $D = Eh^3 / [12(1-\mu^2)]$, and shear modulus $G = E / [2(1+\mu)]$, governed by the following non-dimensional differential equations (Mindlin 1945, Rao 2007)

$$\varphi_{X,XX} + \mu_1\lambda^2\varphi_{X,YY} + \mu_2\lambda\varphi_{Y,XY} - \tau_1(\varphi_X + w_{,X}) + \tau_2\Omega^2\varphi_X = 0 \tag{41}$$

$$\mu_1\varphi_{Y,XX} + \lambda^2\varphi_{Y,YY} + \mu_2\lambda\varphi_{X,XY} - \tau_1(\varphi_Y + \lambda w_{,Y}) + \tau_2\Omega^2\varphi_Y = 0 \tag{42}$$

$$w_{,XX} + \lambda^2 w_{,YY} + \varphi_{X,X} + \lambda\varphi_{Y,Y} + \tau_3\Omega^2 w = 0 \tag{43}$$

where φ_X and φ_Y are dimensionless mode functions of rotations due to plate bending, W is the dimensionless mode function of the lateral deflection; $X = x / a$ and $Y = y / b$ are dimensionless coordinates; $\lambda = a / b$ is the aspect ratio; and $\Omega = \omega a^2\sqrt{\rho h/D}$ is the dimensionless frequency parameter wherein ω is the circular frequency of the plate. Furthermore

$$\mu_1 = (1-\mu)/2, \quad \mu_2 = (1+\mu)/2, \\ \tau_1 = 12k^2\mu_1/\delta^2, \quad \tau_2 = \delta^2/12, \quad \tau_3 = 1/\tau_1, \\ \delta = h/a \tag{44}$$

The boundary conditions of the rectangular plate are:

(I) Simply-supported edge (S)

$$w = \varphi_Y = \varphi_{X,X} = 0 \quad \text{at} \quad X = 0 \quad \text{and/or} \quad X = 1 \tag{45}$$

$$\begin{cases} \varphi_{X,X} + \mu\lambda\varphi_{Y,Y} = 0 \\ \lambda\varphi_{X,Y} + \varphi_{Y,X} = 0 \quad \text{at} \quad Y = 0 \quad \text{and/or} \quad Y = 1 \\ \varphi_X + w_{,X} = 0 \end{cases} \tag{46}$$

(II) Clamped edge (C)

$$w=\varphi_X=\varphi_Y=0 \quad \text{at} \quad X=0 \quad \text{and/or} \quad X=1 \quad (\text{also at} \quad Y=0 \quad \text{and/or} \quad Y=1) \tag{47}$$

(III) Free edge (F)

$$\begin{cases} \varphi_{X,X}+\mu\lambda\varphi_{Y,Y}=0 \\ \lambda\varphi_{X,Y}+\varphi_{Y,X}=0 \quad \text{at} \quad X=0 \quad \text{and/or} \quad X=1 \\ \varphi_X+w_{,X}=0 \end{cases} \tag{48}$$

$$\begin{cases} \lambda\varphi_{Y,Y}+\mu\varphi_{X,X}=0 \\ \lambda\varphi_{X,Y}+\varphi_{Y,X}=0 \quad \text{at} \quad Y=0 \quad \text{and/or} \quad Y=1 \\ \varphi_Y+\lambda w_{,Y}=0 \end{cases} \tag{49}$$

Using the separation of variable technique, the solutions to Eqs. (41)-(43) are assumed to be in the following forms

$$\varphi_X(X,Y)=\sum_{j=1}^{n}\Lambda_j(Y)\Theta_j(X) \tag{50}$$

$$\varphi_Y(X,Y)=\sum_{j=1}^{n}\Gamma_j(Y)\Phi_j(X) \tag{51}$$

$$w(X,Y)=\sum_{j=1}^{n}\Psi_j(Y)\Phi_j(X) \tag{52}$$

where $\Lambda_j(Y)$, $\Gamma_j(Y)$ and $\Psi_j(Y)$ (j = 1, 2,..., n) are undetermined parameters; $\Theta_j(X)$ and $\Phi_j(X)$ are orthonormal approximation functions that satisfy the geometric boundary conditions of the plate in the X-direction; and n is the number of solution terms. Substituting Eqs. (50)-(52) into Eqs. (41)-(43), multiplying both sides of resulting equations, respectively, by $\Theta_i(X)$, $\Phi_i(X)$ and $\Phi_i(X)$ and performing the integration over the length of the plate ($0 \le X \le 1$), we obtain

$$\begin{aligned}&[A]\{\Lambda\}+\mu_1\lambda^2[B]\{\Lambda_{,YY}\}+\mu_2\lambda[C]\{\Gamma_{,Y}\}- \\ &\tau_1[B]\{\Lambda\}-\tau_1[D]\{\Psi\}+\tau_2\Omega^2[B]\{\Lambda\}=\{0\}\end{aligned} \tag{53}$$

$$\begin{aligned}&\mu_1[\tilde{A}]\{\Gamma\}+\lambda^2[\tilde{B}]\{\Gamma_{,YY}\}+\mu_2\lambda[\tilde{C}]\{\Lambda_{,y}\}- \\ &\tau_1[\tilde{B}]\{\Gamma\}-\tau_1\lambda[\tilde{D}]\{\Psi_{,Y}\}+\tau_2\Omega^2[\tilde{B}]\{\Gamma\}=\{0\}\end{aligned} \tag{54}$$

$$\begin{aligned}&[\tilde{\tilde{A}}]\{\Psi\}+\lambda^2[\tilde{\tilde{B}}]\{\Psi_{,YY}\}+[\tilde{\tilde{C}}]\{\Lambda\}+ \\ &\lambda[\tilde{\tilde{D}}]\{\Gamma_{,Y}\}+\tau_3\Omega^2[\tilde{\tilde{B}}]\{\Psi\}=\{0\}\end{aligned} \tag{55}$$

where (i, j = 1, 2,..., n)

$$\begin{aligned}&A_{ij}=\int_0^1\Theta_i\Theta_{j,XX}\,\mathrm{d}X=[\Theta_i\Theta_{j,X}]_0^1-\int_0^1\Theta_{i,X}\Theta_{j,X}\,\mathrm{d}X= \\ &A_{ij}^*+A_{ij}^{**}\end{aligned} \tag{56}$$

$$B_{ij}=\int_0^1\Theta_i\Theta_j\,\mathrm{d}X \tag{57}$$

$$B_{ij}=\int_0^1\Theta_i\Theta_j\,\mathrm{d}X \tag{58}$$

$$\begin{aligned}&\tilde{A}_{ij}=\int_0^1\Phi_i\Phi_{j,XX}\,\mathrm{d}X=[\Phi_i\Phi_{j,X}]_0^1-\int_0^1\Phi_{i,X}\Phi_{j,X}\,\mathrm{d}X= \\ &\tilde{A}_{ij}^*+\tilde{A}_{ij}^{**}\end{aligned} \tag{59}$$

$$\tilde{B}_{ij}=\tilde{D}_{ij}=\int_0^1\Phi_i\Phi_j\,\mathrm{d}X \tag{60}$$

$$\tilde{C}_{ij}=\int_0^1\Phi_i\Theta_{j,X}\,\mathrm{d}X \tag{61}$$

$$\begin{aligned}&\tilde{\tilde{A}}_{ij}=\tilde{A}_{ij}=\int_0^1\Phi_i\Phi_{j,XX}\,\mathrm{d}X=[\Phi_i\Phi_{j,X}]_0^1-\int_0^1\Phi_{i,X}\Phi_{j,X}\,\mathrm{d}X \\ &=\tilde{\tilde{A}}_{ij}^*+\tilde{\tilde{A}}_{ij}^{**}\end{aligned} \tag{62}$$

$$\tilde{\tilde{B}}_{ij}=\tilde{\tilde{D}}_{ij}=\tilde{B}_{ij}=\tilde{D}_{ij}=\int_0^1\Phi_i\Phi_j\,\mathrm{d}X \tag{63}$$

$$\tilde{\tilde{C}}_{ij}=\tilde{C}_{ij}=\int_0^1\Phi_i\Theta_{j,X}\,\mathrm{d}X \tag{64}$$

$$\{\Lambda\}=[\Lambda_1 \;\; \Lambda_2 \;\; \cdots \;\; \Lambda_n]^T \tag{65}$$

$$\{\Lambda_{,Y}\}=[\Lambda_{1,Y} \;\; \Lambda_{2,Y} \;\; \cdots \;\; \Lambda_{n,Y}]^T \tag{66}$$

$$\{\Lambda_{,YY}\}=[\Lambda_{1,YY} \;\; \Lambda_{2,YY} \;\; \cdots \;\; \Lambda_{n,YY}]^T \tag{67}$$

$$\{\Gamma\}=[\Gamma_1 \;\; \Gamma_2 \;\; \cdots \;\; \Gamma_n]^T \tag{68}$$

$$\{\Gamma_{,Y}\}=[\Gamma_{1,Y} \;\; \Gamma_{2,Y} \;\; \cdots \;\; \Gamma_{n,Y}]^T \tag{69}$$

$$\{\Gamma_{,YY}\}=[\Gamma_{1,YY} \;\; \Gamma_{2,YY} \;\; \cdots \;\; \Lambda_{n,YY}]^T \tag{70}$$

$$\{\Psi\}=[\Psi_1 \;\; \Psi_2 \;\; . \; . \; . \;\; \Psi_n]^T \tag{71}$$

$$\{\Psi_{,Y}\}=[\Psi_{1,Y} \;\; \Psi_{2,Y} \;\; \cdots \;\; \Psi_{n,Y}]^T \tag{72}$$

$$\{\Psi_{,YY}\}=[\Psi_{1,YY} \;\; \Psi_{2,YY} \;\; . \; . \; . \;\; \Psi_{n,YY}]^T \tag{73}$$

It can be seen that the application of separation of variable technique with Ritz procedure results in a system of eigenvalue ordinary differential equations of second-order. This system will be further discretized by the application of the FEM. The details will be given in the next sub-section. It should be pointed out that due to orthonormality property of the Ritz trial functions, the matrices $[B]$, $[\tilde{B}]$, $[\tilde{D}]$, $[\tilde{\tilde{B}}]$ and $[\tilde{\tilde{D}}]$ are identity matrices of order $n \times n$. Therefore the proposed approach, at this stage, requires the evaluation of the matrices $[A]$, $[\tilde{A}]$, $[C]$, and $[\tilde{C}]$ only.

3.2 FEM analogues of resulting system of eigenvalue ordinary differential equations

To discretize Eqs. (53)-(55) using the FEM, the domain of the plate problem along the Y-axis is first discretized into N number of equal length finite elements. The polynomial approximation of the solutions within a typical finite element is then assumed as (i = 1, 2,..., n, $Y_e \le Y \le Y_{e+1}$)

$$\Lambda_i^e(Y) = \sum_{j=1}^{q+1} \alpha_j^{i,e} \psi_j^e(Y) \tag{74}$$

$$\Gamma_i^e(Y) = \sum_{j=1}^{q+1} \beta_j^{i,e} \psi_j^e(Y) \tag{75}$$

$$\Psi_i^e(Y) = \sum_{j=1}^{q+1} \gamma_j^{i,e} \psi_j^e(Y) \tag{76}$$

where $\alpha_j^{i,e}$, $\beta_j^{i,e}$ and $\gamma_j^{i,e}$ (j = 1, 2,..., q+1; i = 1, 2,…, n; e = 1,2,…, N) are nodal values of the eth finite element (rotations and displacements) and $\psi_j^e(Y)$ are the interpolation functions of degree q. Now, consider the kth equation of the system of Eqs. (53)-(55)

$$\begin{aligned}
&(A_{k1}\Lambda_1 + A_{k2}\Lambda_2 + \ldots + A_{kn}\Lambda_n) + \\
&\mu_1\lambda^2(B_{k1}\Lambda_{1,YY} + B_{k2}\Lambda_{2,YY} + \ldots + B_{kn}\Lambda_{n,YY}) + \\
&\mu_2\lambda(C_{k1}\Gamma_{1,Y} + C_{k2}\Gamma_{2,Y} + \ldots + C_{kn}\Gamma_{n,Y}) - \\
&\tau_1(B_{k1}\Lambda_1 + B_{k2}\Lambda_2 + \ldots + B_{kn}\Lambda_n) - \\
&\tau_1(D_{k1}\Psi_1 + D_{k2}\Psi_2 + \ldots + D_{kn}\Psi_n) + \\
&\tau_2\Omega^2(B_{k1}\Lambda_1 + B_{k2}\Lambda_2 + \ldots + B_{kn}\Lambda_n) = 0
\end{aligned} \tag{77}$$

$$\begin{aligned}
&\mu_1(\tilde{A}_{k1}\Gamma_1 + \tilde{A}_{k2}\Gamma_2 + \ldots + \tilde{A}_{kn}\Gamma_n) + \\
&\lambda^2(\tilde{B}_{k1}\Gamma_{1,YY} + \tilde{B}_{k2}\Gamma_{2,YY} + \ldots + \tilde{B}_{kn}\Gamma_{n,YY}) + \\
&\mu_2\lambda(\tilde{C}_{k1}\Lambda_{1,Y} + \tilde{C}_{k2}\Lambda_{2,Y} + \ldots + \tilde{C}_{kn}\Lambda_{n,Y}) - \\
&\tau_1(\tilde{B}_{k1}\Gamma_1 + \tilde{B}_{k2}\Gamma_2 + \ldots + \tilde{B}_{kn}\Gamma_n) - \\
&\tau_1\lambda(\tilde{D}_{k1}\Psi_{1,Y} + \tilde{D}_{k2}\Psi_{2,Y} + \ldots + \tilde{D}_{kn}\Psi_{n,Y}) + \\
&\tau_2\Omega^2(\tilde{B}_{k1}\Gamma_1 + \tilde{B}_{k2}\Gamma_2 + \ldots + \tilde{B}_{kn}\Gamma_n) = 0
\end{aligned} \tag{78}$$

$$\begin{aligned}
&(\tilde{\tilde{A}}_{k1}\Psi_1 + \tilde{\tilde{A}}_{k2}\Psi_2 + \ldots + \tilde{\tilde{A}}_{kn}\Psi_n) + \\
&\lambda^2(\tilde{\tilde{B}}_{k1}\Psi_{1,YY} + \tilde{\tilde{B}}_{k2}\Psi_{2,YY} + \ldots + \tilde{\tilde{B}}_{kn}\Psi_{n,YY}) + \\
&(\tilde{\tilde{C}}_{k1}\Lambda_1 + \tilde{\tilde{C}}_{k2}\Lambda_2 + \ldots + \tilde{\tilde{C}}_{kn}\Lambda_n) + \\
&\lambda(\tilde{\tilde{D}}_{k1}\Gamma_{1,Y} + \tilde{\tilde{D}}_{k2}\Gamma_{2,Y} + \ldots + \tilde{\tilde{D}}_{kn}\Gamma_{n,Y}) + \\
&\tau_3\Omega^2(\tilde{\tilde{B}}_{k1}\Psi_1 + \tilde{\tilde{B}}_{k2}\Psi_2 + \ldots + \tilde{\tilde{B}}_{kn}\Psi_n) = 0
\end{aligned} \tag{79}$$

Substituting the finite element approximations, given in Eqs. (74)-(76), into Eqs. (77)-(79), multiplying both sides of resulting equation by $\psi_j^e(Y)$ and performing the integration over the length of the eth finite element ($Y_e \le Y \le Y_{e+1}$), we obtain

$$\begin{aligned}
&[E^e]\left(A_{k1}\{\alpha^{1,e}\} + A_{k2}\{\alpha^{2,e}\} + \ldots + A_{kn}\{\alpha^{n,e}\}\right) + \\
&\mu_1\lambda^2[F^e]\left(B_{k1}\{\alpha^{1,e}\} + B_{k2}\{\alpha^{2,e}\} + \ldots + B_{kn}\{\alpha^{n,e}\}\right) + \\
&\mu_2\lambda[G^e]\left(C_{k1}\{\beta^{1,e}\} + C_{k2}\{\beta^{2,e}\} + \ldots + C_{kn}\{\beta^{n,e}\}\right) - \\
&\tau_1[E^e]\left(B_{k1}\{\alpha^{1,e}\} + B_{k2}\{\alpha^{2,e}\} + \ldots + B_{kn}\{\alpha^{n,e}\}\right) - \\
&\tau_1[E^e]\left(D_{k1}\{\gamma^{1,e}\} + D_{k2}\{\gamma^{2,e}\} + \ldots + D_{kn}\{\gamma^{n,e}\}\right) + \\
&\tau_2\Omega^2[E^e]\left(B_{k1}\{\alpha^{1,e}\} + B_{k2}\{\alpha^{2,e}\} + \ldots + B_{kn}\{\alpha^{n,e}\}\right) = \\
&\{0\}
\end{aligned} \tag{80}$$

$$\begin{aligned}
&\mu_1[E^e]\left(\tilde{A}_{k1}\{\beta^{1,e}\} + \tilde{A}_{k2}\{\beta^{2,e}\} + \ldots + \tilde{A}_{kn}\{\beta^{n,e}\}\right) + \\
&\lambda^2[F^e]\left(\tilde{B}_{k1}\{\beta^{1,e}\} + \tilde{B}_{k2}\{\beta^{2,e}\} + \ldots + \tilde{B}_{kn}\{\beta^{n,e}\}\right) + \\
&\mu_2\lambda[G^e]\left(\tilde{C}_{k1}\{\alpha^{1,e}\} + \tilde{C}_{k2}\{\alpha^{2,e}\} + \ldots + \tilde{C}_{kn}\{\alpha^{n,e}\}\right) - \\
&\tau_1[E^e]\left(\tilde{B}_{k1}\{\beta^{1,e}\} + \tilde{B}_{k2}\{\beta^{2,e}\} + \ldots + \tilde{B}_{kn}\{\beta^{n,e}\}\right) - \\
&\tau_1\lambda[G^e]\left(\tilde{D}_{k1}\{\gamma^{1,e}\} + \tilde{D}_{k2}\{\gamma^{2,e}\} + \ldots + \tilde{D}_{kn}\{\gamma^{n,e}\}\right) + \\
&\tau_2\Omega^2[E^e]\left(\tilde{B}_{k1}\{\beta^{1,e}\} + \tilde{B}_{k2}\{\beta^{2,e}\} + \ldots + \tilde{B}_{kn}\{\beta^{n,e}\}\right) = \\
&\{0\}
\end{aligned} \tag{81}$$

$$\begin{aligned}
&[E^e]\left(\tilde{\tilde{A}}_{k1}\{\gamma^{1,e}\} + \tilde{\tilde{A}}_{k2}\{\gamma^{2,e}\} + \ldots + \tilde{\tilde{A}}_{kn}\{\gamma^{n,e}\}\right) + \\
&\lambda^2[F^e]\left(\tilde{\tilde{B}}_{k1}\{\gamma^{1,e}\} + \tilde{\tilde{B}}_{k2}\{\gamma^{2,e}\} + \ldots + \tilde{\tilde{B}}_{kn}\{\gamma^{n,e}\}\right) + \\
&[E^e]\left(\tilde{\tilde{C}}_{k1}\{\alpha^{1,e}\} + \tilde{\tilde{C}}_{k2}\{\alpha^{2,e}\} + \ldots + \tilde{\tilde{C}}_{kn}\{\alpha^{n,e}\}\right) + \\
&\lambda[G^e]\left(\tilde{\tilde{D}}_{k1}\{\beta^{1,e}\} + \tilde{\tilde{D}}_{k2}\{\beta^{2,e}\} + \ldots + \tilde{\tilde{D}}_{kn}\{\beta^{n,e}\}\right) + \\
&\tau_3\Omega^2[E^e]\left(\tilde{\tilde{B}}_{k1}\{\gamma^{1,e}\} + \tilde{\tilde{B}}_{k2}\{\gamma^{2,e}\} + \ldots + \tilde{\tilde{B}}_{kn}\{\gamma^{n,e}\}\right) = \\
&\{0\}
\end{aligned} \tag{82}$$

where (i, j = 1, 2,..., q+1, k = 1, 2,..., n)

$$E_{ij}^e = \int_{Y_e}^{Y_{e+1}} \psi_i^e \psi_j^e \mathrm{d}Y \tag{83}$$

$$F_{ij}^e = \int_{Y_e}^{Y_{e+1}} \psi_i^e \psi_{j,YY}^e \mathrm{d}Y = [\psi_i^e \psi_{j,Y}^e]_{Y_e}^{Y_{e+1}} - \int_{Y_e}^{Y_{e+1}} \psi_{i,Y}^e \psi_{j,Y}^e \mathrm{d}Y = \hat{F}_{ij}^e + \hat{\hat{F}}_{ij}^e \tag{84}$$

$$G_{ij}^e = \int_{Y_e}^{Y_{e+1}} \psi_i^e \psi_{j,Y}^e \mathrm{d}Y \tag{85}$$

$$\{\alpha^{k,e}\} = [\alpha_1^{k,e} \;\; \alpha_2^{k,e} \;\; . \; . \; . \;\; \alpha_{q+1}^{k,e}]^T \tag{86}$$

$$\{\beta^{k,e}\}=[\beta_1^{k,e}\ \ \beta_2^{k,e}\ \ ...\ \ \beta_{q+1}^{k,e}]^T \tag{87}$$

$$\{\gamma^{k,e}\}=[\gamma_1^{k,e}\ \ \gamma_2^{k,e}\ \ ...\ \ \gamma_{q+1}^{k,e}]^T \tag{88}$$

The assembly of finite element equations (80)-(82) and imposition of geometric boundary conditions are similar to those of one-dimensional bar element equations (see Reddy (1993) for details). By doing so, we obtain the following assembled equations

$$\begin{aligned}&[E]\left(A_{k1}\{\alpha^1\}+A_{k2}\{\alpha^2\}+...+A_{kn}\{\alpha^n\}\right)+\\&\mu_1\lambda^2[F]\left(B_{k1}\{\alpha^1\}+B_{k2}\{\alpha^2\}+...+B_{kn}\{\alpha^n\}\right)+\\&\mu_2\lambda[G]\left(C_{k1}\{\beta^1\}+C_{k2}\{\beta^2\}+...+C_{kn}\{\beta^n\}\right)-\\&\tau_1[E]\left(B_{k1}\{\alpha^1\}+B_{k2}\{\alpha^2\}+...+B_{kn}\{\alpha^n\}\right)-\\&\tau_1[E]\left(D_{k1}\{\gamma^1\}+D_{k2}\{\gamma^2\}+...+D_{kn}\{\gamma^n\}\right)+\\&\tau_2\Omega^2[E]\left(B_{k1}\{\alpha^1\}+B_{k2}\{\alpha^2\}+...+B_{kn}\{\alpha^n\}\right)\\&=\{0\}\end{aligned} \tag{89}$$

$$\begin{aligned}&\mu_1[E]\left(\tilde{A}_{k1}\{\beta^1\}+\tilde{A}_{k2}\{\beta^2\}+...+\tilde{A}_{kn}\{\beta^n\}\right)+\\&\lambda^2[F]\left(\tilde{B}_{k1}\{\beta^1\}+\tilde{B}_{k2}\{\beta^2\}+...+\tilde{B}_{kn}\{\beta^n\}\right)+\\&\mu_2\lambda[G]\left(\tilde{C}_{k1}\{\alpha^1\}+\tilde{C}_{k2}\{\alpha^2\}+...+\tilde{C}_{kn}\{\alpha^n\}\right)-\\&\tau_1[E]\left(\tilde{B}_{k1}\{\beta^1\}+\tilde{B}_{k2}\{\beta^2\}+...+\tilde{B}_{kn}\{\beta^n\}\right)-\\&\tau_1\lambda[G]\left(\tilde{D}_{k1}\{\gamma^1\}+\tilde{D}_{k2}\{\gamma^2\}+...+\tilde{D}_{kn}\{\gamma^n\}\right)+\\&\tau_2\Omega^2[E]\left(\tilde{B}_{k1}\{\beta^1\}+\tilde{B}_{k2}\{\beta^2\}+...+\tilde{B}_{kn}\{\beta^n\}\right)=\{0\}\end{aligned} \tag{90}$$

$$\begin{aligned}&[E]\left(\tilde{\tilde{A}}_{k1}\{\gamma^1\}+\tilde{\tilde{A}}_{k2}\{\gamma^2\}+...+\tilde{\tilde{A}}_{kn}\{\gamma^n\}\right)+\\&\lambda^2[F]\left(\tilde{\tilde{B}}_{k1}\{\gamma^1\}+\tilde{\tilde{B}}_{k2}\{\gamma^2\}+...+\tilde{\tilde{B}}_{kn}\{\gamma^n\}\right)+\\&[E]\left(\tilde{\tilde{C}}_{k1}\{\alpha^1\}+\tilde{\tilde{C}}_{k2}\{\alpha^2\}+...+\tilde{\tilde{C}}_{kn}\{\alpha^n\}\right)+\\&\lambda[G]\left(\tilde{\tilde{D}}_{k1}\{\beta^1\}+\tilde{\tilde{D}}_{k2}\{\beta^2\}+...+\tilde{\tilde{D}}_{kn}\{\beta^n\}\right)+\\&\tau_3\Omega^2[E]\left(\tilde{\tilde{B}}_{k1}\{\gamma^1\}+\tilde{\tilde{B}}_{k2}\{\gamma^2\}+...+\tilde{\tilde{B}}_{kn}\{\gamma^n\}\right)=\{0\}\end{aligned} \tag{91}$$

Eqs. (89)-(91) can be expressed for all the k-values in the compact form

$$\begin{bmatrix}[\tilde{K}^{11}] & [\tilde{K}^{12}] & [\tilde{K}^{13}]\\ [\tilde{K}^{21}] & [\tilde{K}^{22}] & [\tilde{K}^{23}]\\ [\tilde{K}^{31}] & [\tilde{K}^{32}] & [\tilde{K}^{33}]\end{bmatrix}\begin{Bmatrix}\{\tilde{\alpha}\}\\ \{\tilde{\beta}\}\\ \{\tilde{\gamma}\}\end{Bmatrix}=-\Omega^2\begin{bmatrix}[\tilde{M}^{11}] & [0] & [0]\\ [0] & [\tilde{M}^{22}] & [0]\\ [0] & [0] & [\tilde{M}^{33}]\end{bmatrix}\begin{Bmatrix}\{\tilde{\alpha}\}\\ \{\tilde{\beta}\}\\ \{\tilde{\gamma}\}\end{Bmatrix} \tag{92}$$

where (i, j = 1, 2,..., n)

$$\begin{aligned}&[\tilde{K}_{ij}^{11}]=A_{ij}[E]+\mu_1\lambda^2B_{ij}[F]-\tau_1B_{ij}[E],\\&[\tilde{K}_{ij}^{12}]=\mu_2\lambda C_{ij}[G],\qquad [\tilde{K}_{ij}^{13}]=-\tau_1D_{ij}[E]\end{aligned} \tag{93}$$

$$\begin{aligned}&[\tilde{K}_{ij}^{21}]=\mu_2\lambda\tilde{C}_{ij}[G],\\&[\tilde{K}_{ij}^{22}]=\mu_1\tilde{A}_{ij}[E]+\lambda^2\tilde{B}_{ij}[F]-\tau_1\tilde{B}_{ij}[E],\\&[\tilde{K}_{ij}^{23}]=-\tau_1\lambda\tilde{D}_{ij}[G]\end{aligned} \tag{94}$$

$$\begin{aligned}&[\tilde{K}_{ij}^{31}]=\tilde{\tilde{C}}_{ij}[E],\qquad [\tilde{K}_{ij}^{32}]=\lambda\tilde{\tilde{D}}_{ij}[G],\\&[\tilde{K}_{ij}^{33}]=\tilde{\tilde{A}}_{ij}[E]+\lambda^2\tilde{\tilde{B}}_{ij}[F]\end{aligned} \tag{95}$$

$$\begin{aligned}&[\tilde{M}_{ij}^{11}]=\tau_2B_{ij}[E],\qquad [\tilde{M}_{ij}^{22}]=\tau_2\tilde{B}_{ij}[E],\\&[\tilde{M}_{ij}^{33}]=\tau_3\tilde{\tilde{B}}_{ij}[E]\end{aligned} \tag{96}$$

and

$$\{\tilde{\alpha}\}=[\{\alpha^1\}^T\ \ \{\alpha^2\}^T\ \ ...\ \ \{\alpha^n\}^T]^T \tag{97}$$

$$\{\tilde{\beta}\}=[\{\beta^1\}^T\ \ \{\beta^2\}^T\ \ \cdots\ \ \{\beta^n\}^T]^T \tag{98}$$

$$\{\tilde{\gamma}\}=[\{\gamma^1\}^T\ \ \{\gamma^2\}^T\ \ \cdots\ \ \{\gamma^n\}^T]^T \tag{99}$$

After applying the natural boundary conditions of the plate problem, the eigenvalue problem (92) can be solved for the eigenvalues Ω. The procedure is detailed in the following sub-section.

3.3 Implementation of natural boundary conditions of the rectangular thick plate

The procedure for implementation of natural boundary conditions of rectangular thick plates is similar to that described in Section 2.3 for rectangular thin plates. Furthermore, similar to rectangular thin plates, the case of rectangular thick plates with combination of simply supported and clamped edges does not require any modification of the stiffness matrix.

3.3.1 Implementation of free edge boundary conditions in the X-direction

Consider a rectangular thick plate having free edges at X = 0 and X = 1. The boundary conditions of the plate for this case are given in Eq. (48). To apply these boundary conditions, we substitute the Ritz approximations (given in Eqs. (50)-(52)) into Eq. (48) and perform some mathematical manipulation to obtain

$$[A^*]\{\Lambda\}=-\mu\lambda[A^1]\{\Gamma_{,Y}\} \tag{100}$$

$$[\tilde{A}^*]\{\Gamma\}=-\lambda[A^1]^T\{\Lambda_{,Y}\} \tag{101}$$

$$[\tilde{\tilde{A}}^*]\{\Psi\}=-[A^1]^T\{\Lambda\} \tag{102}$$

where $[A^*]$, $[\tilde{A}^*]$, and $[\tilde{\tilde{A}}^*]$ are defined in Eqs. (56), (59) and (62), respectively. Furthermore

$$A_{ij}^1=[\Theta_i\Phi_j]_0^1 \tag{103}$$

Substituting Eqs. (100)-(102) into Eqs. (53)-(55) and the results into Eq. (92) gives the modified stiffness matrices of the plate.

Table 5 Convergence and comparison of natural frequencies of a clamped square thick plate (N = 1, h/b = 0.1, k^2 = 5/6, μ = 0.3)

n	q	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
7	9	3.2955	6.2860	6.2864	8.8108	10.3836
	10	3.2955	6.2860	6.2863	8.8108	10.3822
	11	3.2955	6.2859	6.2863	8.8107	10.3821
8	9	3.2955	6.2860	6.2860	8.8100	10.3816
	10	3.2954	6.2859	6.2860	8.8100	10.3802
	11	3.2954	6.2859	6.2859	8.8099	10.3802
9	9	3.2954	6.2859	6.2860	8.8100	10.3802
	10	3.2954	6.2859	6.2859	8.8100	10.3789
	11	3.2954	6.2858	6.2859	8.8099	10.3789
Liew *et al*. (1993)		3.2954	6.2858	6.2858	8.8098	10.3788

Table 6 Convergence and comparison of natural frequencies of a clamped square thick plate (n = 10, h/b = 0.1, k^2 = 5/6, μ = 0.3)

q	N	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
2	3	3.4343	6.3662	7.5192	9.6435	10.4761
	7	3.3019	6.2902	6.3314	8.8428	10.4174
	10	3.2971	6.2871	6.2975	8.8185	10.3963
	20	3.2955	6.2859	6.2866	8.8104	10.3802
	30	3.2954	6.2858	6.2860	8.8099	10.3791
	40	3.2954	6.2858	6.2859	8.8098	10.3789
	45	3.2954	6.2858	6.2859	8.8098	10.3789
3	3	3.2974	6.2886	6.3120	8.8334	10.4246
	7	3.2955	6.2859	6.2862	8.8103	10.3800
	9	3.2954	6.2859	6.2859	8.8099	10.3791
	11	3.2954	6.2858	6.2859	8.8098	10.3789
	13	3.2954	6.2858	6.2858	8.8098	10.3788
4	3	3.2956	6.2862	6.2871	8.8112	10.3816
	4	3.2955	6.2859	6.2860	8.8102	10.3799
	5	3.2954	6.2859	6.2859	8.8099	10.3790
	6	3.2954	6.2858	6.2859	8.8098	10.3789
Liew *et al*. (1993)		3.2954	6.2858	6.2858	8.8098	10.3788

3.3.2 Implementation of free edge boundary conditions in the Y-direction

Consider a rectangular thick plate having free edges at Y = 0 and Y = 1. The boundary conditions of the plate for this case are given in Eq. (49). Substituting the Ritz approximations (given in Eqs. (50)-(52)) and the finite element approximations (given in Eqs. (74)-(76)) into Eq. (49) and performing some mathematical manipulations, we obtain (k, l = 1, 2,..., n)

$$B_{kl}\left([\tilde{A}^1]-[\tilde{A}^2]\right)\{\alpha^l\}=-\frac{1}{\lambda}C_{kl}\left([\tilde{H}^1]-[\tilde{H}^2]\right)\{\beta^l\} \tag{104}$$

$$\tilde{B}_{kl}\left([\tilde{A}^1]-[\tilde{A}^2]\right)\{\beta^l\}=-\frac{\mu}{\lambda}\tilde{C}_{kl}\left([\tilde{H}^1]-[\tilde{H}^2]\right)\{\alpha^l\} \tag{105}$$

$$\tilde{\tilde{B}}_{kl}\left([\tilde{A}^1]-[\tilde{A}^2]\right)\{\gamma^l\}=-\frac{1}{\lambda}\tilde{\tilde{B}}_{kl}\left([\tilde{H}^1]-[\tilde{H}^2]\right)\{\beta^l\} \tag{106}$$

wherein B_{kl}, C_{kl}, $\tilde{B}_{kl}$, $\tilde{C}_{kl}$, $\tilde{\tilde{B}}_{kl}$ and $\tilde{\tilde{C}}_{kl}$ are defined in Eqs. (57), (58), (60), (61), (63) and (64), respectively, and

$$\begin{aligned}&[\tilde{A}^1]=[\bar{A}^1]_{\exp}, \qquad [\tilde{A}^2]=[\bar{A}^2]_{\exp},\\ &[\tilde{H}^1]=[\bar{H}^1]_{\exp}, \qquad [\tilde{H}^2]=[\bar{H}^2]_{\exp}\end{aligned} \tag{107}$$

$$\bar{A}^1_{ij}=[\psi^e_i\psi^e_{j,Y}]^{e=N}_{Y=Y_{e+1}}, \qquad \bar{A}^2_{ij}=[\psi^e_i\psi^e_{j,Y}]^{e=1}_{Y=Y_e} \tag{108}$$

$$\overline{H}_{ij}^{1} = [\psi_i^e \psi_j^e]_{Y=Y_{e+1}}^{e=N}, \qquad \overline{H}_{ij}^{2} = [\psi_i^e \psi_j^e]_{Y=Y_e}^{e=1} \qquad (109)$$

Table 7 Convergence and comparison of natural frequencies of Levy-type square thick plates (N = 1, h/a = 0.2, k^2 = 0.86667, μ = 0.3)

Plate	n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
SSSS	8	17.5055	38.3848	38.3851	55.5862	65.7270
	9	17.5055	38.3847	38.3847	55.5860	65.7211
	10	17.5055	38.3847	38.3847	55.5860	65.7193
	Hashemi and Arsanjani (2005)	17.5055	38.3847	38.3847	55.5860	65.7193
SCSS	8	19.7988	39.2036	41.7815	57.3384	66.0401
	9	19.7988	39.2032	41.7813	57.3381	66.0395
	10	19.7988	39.2032	41.7813	57.3380	66.0323
	Hashemi and Arsanjani (2005)	19.7988	39.2032	41.7813	57.3380	66.0322
SCSC	8	22.5099	40.1388	45.0574	59.1234	66.3785
	9	22.5099	40.1384	45.0569	59.1227	66.3780
	10	22.5099	40.1384	45.0569	59.1227	66.3706
	Hashemi and Arsanjani (2005)	22.5099	40.1384	45.0569	59.1227	66.3706
SSSF	8	10.7218	23.2432	32.8930	43.8590	45.5859
	9	10.7218	23.2430	32.8923	43.8580	45.5853
	10	10.7218	23.2429	32.8922	43.8579	45.5852
	Hashemi and Arsanjani (2005)	10.7218	23.2429	32.8922	43.8579	45.5852
SCSF	8	11.3932	25.8980	33.0755	45.0461	48.8922
	9	11.3932	25.8975	33.0748	45.0446	48.8912
	10	11.3931	25.8975	33.0747	45.0445	48.8911
	Hashemi and Arsanjani (2005)	11.3931	25.8975	33.0747	45.0445	48.8911
SFSF	8	8.9997	14.1345	29.2560	31.4344	36.1664
	9	8.9997	14.1341	29.2560	31.4338	36.1647
	10	8.9997	14.1341	29.2558	31.4338	36.1647
	Hashemi and Arsanjani (2005)	8.9997	14.1341	29.2558	31.4338	36.1646

where the subscript "exp" means that the associated matrix is written in the *global coordinate system*. Eqs. (104)-(106) can be used to modify the stiffness matrix of the plate.

3.4 Numerical results

To validate the proposed mixed methodology and its effectiveness, natural frequencies of square thick plates with different boundary conditions are evaluated and the results are tabulated in Tables 5-8. In all computation, the Poisson's ratio μ = 0.3 is taken.

Tables 5 and 6 shows the convergence and comparison studies for the first five frequency parameters, $\Omega = (\omega b^2/\pi^2)\sqrt{\rho h/D}$, for clamped thick square plates. The results are compared with those obtained by the Pb2-Ritz method (Liew *et al*. 1993).

An excellent agreement is observed between the present results and those of the Pb2-Ritz approach. It can also be observed that a better convergence rate can be achieved by increasing the order of interpolation functions.

Table 7 shows the convergence and comparison studies of the first five dimensionless frequency parameters, $\Omega = \omega a^2\sqrt{\rho h/D}$, for thick square plates with Levy-type boundary conditions. These results are obtained using only *one finite element* (i.e., N = 1). Furthermore, the number of Ritz terms (i.e., n) and the order of interpolation functions (i.e., q) are assumed to be the same (i.e., q = n). The analytical solutions of Hashemi and Arsanjani (2005) are also shown in this Table for comparison. It can be seen from Table 7 that the present results converge very quickly and agree well with analytical solutions.

The numerical results for different boundary conditions of the thick plate are presented in Table 8. An excellent agreement is observed between the results of present study and those presented in the literature. These results show the effectiveness of the proposed mixed Ritz-FE approach for vibration analysis of thick square plates with general boundary conditions.

4. Comparison with mixed FE-Ritz approach

The present author and his co-author proposed recently a mixed FE-Ritz approach for free vibration analysis of thin plates with general boundary conditions (Eftekhari and

Jafari 2012d). This mixed approach first uses the FEM to discretize the spatial partial derivatives with respect to a co-ordinate direction of the plate. It then employs the Ritz method to solve the resulting system of ordinary differential equations. The mixed FE-Ritz method has been claimed to overcome some of limitations of both the FEM and the Ritz method. However, as we will show in this section, it is not very efficient for solving plate problems with Levy-type boundary conditions. To demonstrate this, we consider the free vibration of a Levy-type thin rectangular plate. The governing equation for this problem is given in Eq. (1). For this case, it is most appropriate to choose the Ritz trial functions as the beam eigenfunctions, i.e.

$$\Phi_j(\xi) = \sin j\pi\zeta, \quad j = 1,2,\ldots,n, \quad \xi = X \text{ or } Y \tag{110}$$

Note that the Levy-type plate is assumed to have simply supported edge in the ξ direction, and the Ritz method with above eigenfunctions will be employed in that direction.

To better understand the differences between the mixed Ritz-FE and mixed FE-Ritz approaches, the applications of these approaches to free vibration analysis of Levy-type thin plates are described in the following sections. We assume that the size of Ritz matrices [A], [B] and [C] is $n \times n$ while the FEM matrices $[\bar{A}]$, $[\bar{B}]$ and $[\bar{C}]$ are of order $n_f \times n_f$.

Table 8 Convergence and comparison of natural frequencies of square thick plates with different boundary conditions (N = 1, h/b = 0.1, k^2 = 5/6, μ = 0.3)

Plate	$q = n$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
CSSF	9	1.6195	2.9167	4.6614	5.7679	5.9717
	10	1.6194	2.9165	4.6612	5.7676	5.9712
	11	1.6194	2.9165	4.6611	5.7675	5.9711
	Liew *et al.* (1993)	1.6195	2.9165	4.6612	5.7675	5.9711
CFSF	9	1.4735	1.9491	3.6461	4.5019	5.0397
	10	1.4734	1.9491	3.6452	4.5016	5.0395
	11	1.4734	1.9489	3.6452	4.5016	5.0392
	Liew *et al.* (1993)	1.4735	1.9491	3.6452	4.5017	5.0395
CFFF	9	0.3476	0.8168	2.0357	2.5840	2.8623
	10	0.3476	0.8168	2.0355	2.5837	2.8621
	11	0.3476	0.8167	2.0355	2.5836	2.8619
	Liew *et al.* (1993)	0.3476	0.8168	2.0356	2.5836	2.8620
FFFF	9	1.2895	1.9194	2.3633	3.2353	3.2362
	10	1.2889	1.9194	2.3633	3.2346	3.2353
	11	1.2888	1.9194	2.3633	3.2344	3.2345
	Liew *et al.* (1993)	1.2887	1.9194	2.3633	3.2344	3.2344
CSSF	9	1.6195	2.9167	4.6614	5.7679	5.9717
	10	1.6194	2.9165	4.6612	5.7676	5.9712
	11	1.6194	2.9165	4.6611	5.7675	5.9711
	Liew *et al.* (1993)	1.6195	2.9165	4.6612	5.7675	5.9711
CFSF	9	1.4735	1.9491	3.6461	4.5019	5.0397
	10	1.4734	1.9491	3.6452	4.5016	5.0395
	11	1.4734	1.9489	3.6452	4.5016	5.0392
	Liew *et al.* (1993)	1.4735	1.9491	3.6452	4.5017	5.0395

4.1 Ritz-FE formulation for Levy-type thin rectangular plates

When beam eigenfunctions (110) are used as the Ritz trial functions, the matrices [A], [B] and [C] in Eq. (10) become diagonal matrices. Their diagonal elements are

$$\begin{aligned} A_{ii} &= i^4\pi^4/2, \qquad B_{ii} = 1/2, \\ C_{ii} &= -i^2\pi^2/2, \qquad i = 1,2,\ldots,n \end{aligned} \tag{111}$$

Using Eq. (111), the eigenvalue Eq. (25) is simplified to

$$[K]_i\{\alpha\}_i = \Omega_i^2[M]_i\{\alpha\}_i, \qquad i = 1,2,\ldots,n \tag{112}$$

where

$$[K]_i = A_{ii}[\bar{B}] + 2\lambda^2 C_{ii}[\bar{C}] + \lambda^4 B_{ii}[\bar{A}] \tag{113}$$

$$[M]_i = B_{ii}[\bar{B}] \tag{114}$$

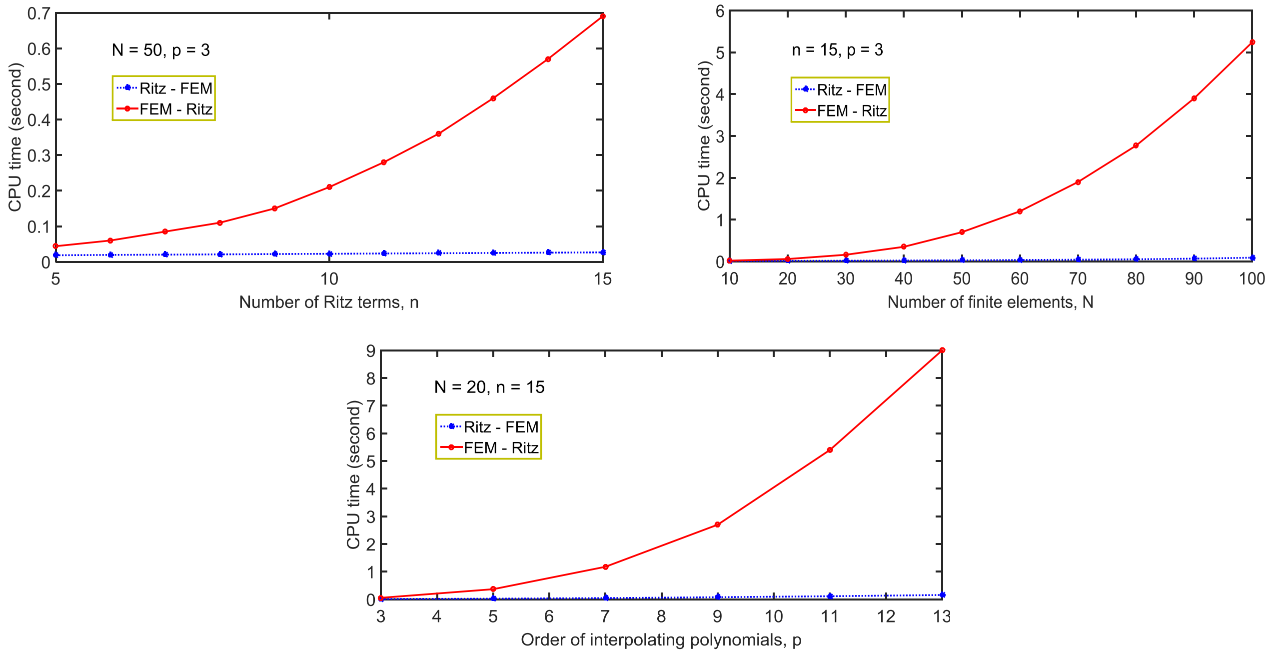

Fig. 1 Comparison of the CPU time of the Ritz-FEM with that of the FEM-Ritz for free vibration analysis of a simply supported square thin plate

$$\{\alpha\}_i = [\alpha_1^i \;\; \alpha_2^i \;\; . \; . \; . \;\; \alpha_{n_f}^i]^T \tag{115}$$

It can be seen from Eq. (112) that the ith natural frequency of Levy-type rectangular plates can be obtained by the present mixed Ritz-FE approach by solving an $n_f \times n_f$ eigenvalue equation. Noting that the size of resulting eigenvalue problem for plates with general boundary conditions is $nn_f \times nn_f$ (see Eq. (25)), one may conclude that the computational time of the mixed Ritz-FE approach for Levy-type rectangular thin plates is much less than that for plates with general boundary conditions.

4.2 FE-Ritz formulation for Levy-type thin rectangular plates

In the FE-Ritz approach, the FEM is first employed in on co-ordinate direction of the plate and the Ritz method is then employed in another one. Application of the FE-Ritz approach to the present problem gives (Eftekhari and Jafari 2012d)

$$[\hat{K}]\{\hat{\alpha}\} = \Omega^2[\hat{M}]\{\hat{\alpha}\} \tag{116}$$

where the sub-matrices $[\hat{K}_{ij}]$ and $[\hat{M}_{ij}]$ are given by

$$[\hat{K}_{ij}] = \overline{A}_{ij}[B] + 2\lambda^2\overline{C}_{ij}[C] + \lambda^4\overline{B}_{ij}[A] \tag{117}$$

$$[\hat{M}_{ij}] = \overline{B}_{ij}[B], \qquad i, j = 1,2,\ldots,n_f \tag{118}$$

and

$$\{\hat{\alpha}\} = [\{\alpha^1\}^T \;\; \{\alpha^2\}^T \;\; . \; . \; . \; \{\alpha^{n_f}\}^T]^T \tag{119}$$

$$\{\alpha^k\} = [\alpha_1^k \;\; \alpha_2^k \;\; . \; . \; . \;\; \alpha_n^k]^T, \qquad k = 1,2,\ldots,n_f \tag{120}$$

It can be seen from Eq. (116) that the natural frequencies of general rectangular plates can be obtained by the mixed FE-Ritz approach by solving an $n_f \times n_f$ eigenvalue equation. It can also be seen that even if the Ritz matrices $[A]$, $[B]$ and $[C]$ are diagonal matrices, the resulting mass and stiffness matrices are not diagonal matrices. Therefore, the computational cost of the FE-Ritz approach for calculation of natural frequencies of Levy-type rectangular plates is approximately equal to that of non-Levy-type rectangular plates.

4.3 Comparison of computing costs for vibration analysis of Levy-type thin plates

Consider the free vibration problem of a Levy-type square thin plate. When FE-Ritz approach is applied to the present problem, one should solve finally an eigenvalue equation of size $nn_f \times nn_f$ where $n_f = N(p-1) + 2$ (see Eq. (116)). Note that, N and p are, respectively, the number of finite elements and the order of interpolating functions. The computing cost for solving such an eigenvalue equation is approximately $(nn_f)^3$ or $n^3\, n_f^3$.

Such a cost can be significantly reduced if the problem is solved using the Ritz-FE approach. When Ritz-FE approach is applied, one should solve an eigenvalue equation of size $n_f \times n_f$, n times (see Eq. (112)). This requires nn_f^3 scalar multiplications. Therefore, we can conclude

$$\frac{\text{Cost}_{FEM-Ritz}}{\text{Cost}_{Ritz-FEM}} \approx \frac{n^3 n_f^3}{nn_f^3} = n^2 \tag{121}$$

Fig. 1 illustrates the results for CPU times of the two approaches (Ritz-FEM and FEM-Ritz) for free vibration analysis of a simply supported square plate. Note that the results are obtained using different values of *n*, *N*, and *p*. Needless to say the Ritz-FEM is much more efficient than the FEM-Ritz approach.

5. Conclusions

A simple and accurate mixed Ritz-FE formulation is introduced and developed to study the free vibration of the thin and thick rectangular plates with general boundary conditions. The proposed formulation reduces the original 2-D problem to two simple 1-D problems whose formulation is much easier than the case where the Ritz method or the FEM is fully applied to the plate problem. Its accuracy, convergence and stability are challenged through the solution of some benchmark vibration problems of thin and thick rectangular plates under different boundary conditions. It is shown that the mixed Ritz-FEM with only one higher order finite element can obtain the lower order natural frequencies of thin and thick square plates accurately.

The proposed mixed Ritz-FE formulation is also compared with the mixed FE-Ritz formulation which has been recently proposed by the present author and his co-author. It is found that the proposed mixed Ritz-FE formulation is more efficient than the mixed FE-Ritz formulation for free vibration analysis of rectangular plates with Levy-type boundary conditions.

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