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Buckling analysis of new quasi-3D FG nanobeams based on nonlocal strain gradient elasticity theory and variable length scale parameter

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Abstract. A size-dependent novel hyperbolic shear deformation theory of simply supported functionally graded beams is presented in the frame work of the non-local strain gradient theory, in which the stress accounts for only the nonlocal strain gradients stress field. The thickness stretching effect ($\varepsilon_z \neq 0$) is also considered here. Elastic coefficients and length scale parameter are assumed to vary in the thickness direction of functionally graded beams according to power-law form. The governing equations are derived using the Hamilton principle. The closed-form solutions for exact critical buckling loads of nonlocal strain gradient functionally graded beams are obtained using Navier's method. The derived results are compared with those of strain gradient theory.

Keywords: new hyperbolic shear theory; nonlocal strain gradient elasticity; stretching effect; buckling analysis

1. Introduction

Functionally graded materials (FGMs) are microscopically inhomogeneous composite materials that possess continuous variation of material properties from one surface to another with a spatial gradient of macroscopic material properties of mechanical strength and thermal conductivity. FGMs have various advantages, for instance, minimization or elimination of stress concentration developed in classical laminated composites, and increased bonding strength along the interface of two different materials. Subsequently, a number of studies have been carry out to analyze the static, vibration, and buckling of advanced composite structures due to the increased relevance of the FGMs structural components in the design of engineering structures (Ait Amar Meziane *et al*. 2014, Mahi *et al*. 2015, Ait Yahia *et al*. 2015, Bourada *et al*. 2015, Kar and Panda 2016, Kar *et al*. 2016, 2017, Mahapatra *et al*. 2017, Boukhari *et al*. 2016, Madani *et al*. 2016, Bennoun *et al*. 2016, Bouderba *et al*. 2016, Bellifa *et al*. 2016, Kolahchi *et al*. 2016a, 2017a, Tounsi *et al*. 2016, Hajmohammad *et al*. 2017, Shokravi 2017a, b). Recently, the high-order polynomial shear deformation theory (HSDT) has been utilized in a different environment to achieve the structural responses of the functionally graded carbon nanotube reinforced composite plate and shell structures, see for example (Mehar and Panda 2016a, b, 2017a, b, Mehar *et al*. 2016, 2017a, b). Today, engineering nanostructures such as nanorods, nanobeams and nanoplates have large kinds of applications due to their high mechanical, thermal, chemical, and electronic characteristics (Ekinci and Roukes 2005, Rahmani *et al*. 2017). Amid the application of nanostructures can be referred to micro/nano electro-mechanical systems (MEMS/NEMS) and nano actuators. Led to an abrupt momentum in modeling of micro and nano scale structures. However, size effects on the mechanical and physical properties of nanostructures have been observed at small sizes both in experimental investigation, (Lam *et al*. 2003, McFarlan *et al*. 2005, Babaei *et al*. 2009) and in numerical simulations (Duan and Wang 2007, Agrawal *et al*. 2008). In these applications, size effects become very prominent. It should be noted that classical continuum mechanics theory does not suitable for nanostructures due to neglecting size influence in nanosize structures. To overcome this problem, various non-classical continuum theories imply additional material length scale parameters were developed, such as nonlocal elasticity theory (Eringen 1972, 1983), strain gradient theory (Mindlin 1964, 1965, Papargyri-Beskou *et al*. 2003) and nonlocal strain gradient

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theory (Askes and Aifantis 2009). However, the studies of size-dependence effects on the buckling behaviour of FG elastic materials with micro/nano-structure should be related to both internal lengths and external dimensions and are always of fundamental significance. In contrast to classical elasticity theory, the stress of non local elasticity theory at a reference point accounts for not only the strain at the reference point, but also the strains at all points in the whole body (Eringen 1983). In this context, a large number of studies have been performed to study the mechanical responses of nanostructures. Peddieson *et al*. (2003) first used nonlocal Eringen elasticity theory (Eringen 1983) to analyze Euler-Bernoulli nanobeams. Wang and Liew (2007) investigate static response of size-dependent structures based on nonlocal elasticity using two different beam theories. Various shear deformation beam theories were also reformulated by (Reddy 2007) using nonlocal differential constitutive relations. A generalized nonlocal beam theory, to study mechanical analysis of nanobeams, was presented by Aydogdu (2009). Pradhan and Murmu (2010) have used the differential quadrature method (DQM) to obtain the numerical solutions of nondimensional frequencies of rotating nanocantilever beam. Civalek *et al*. (2010) studies a size-dependent Euler-Bernoulli beam for mechanical analysis of cantilever microtubules using differential quadrature method (DQM). The effect of the size-dependent, and shear deformation of the forced vibration analysis of single-walled carbon nanotubes (SWCNT) under moving nanoparticles have been proposed by (Simsek 2011). Based on nonlocal Euler-Bernoulli model, bending analysis of nanobeam has been studied by Nguyen *et al*. (2014). Benguediab *et al*. (2014) proposed a comprehensive nonlocal shear deformation beam theory for bending, buckling and vibration analysis of homogeneous nanobeams founded on Eringen’s nonlocal elasticity theory. Jamali *et al*. (2016) carried out a buckling study of nanocomposite plate with square cut out reinforced by carbon nanotubes (CNTs) resting on an elastic medium. Tounsi and his colleagues (Zemri *et al*. 2015, Larbi Chaht *et al*. 2015, Ahouel *et al*. 2016, Bounouara *et al*. 2016, Bellifa *et al*. 2017, Besseghier *et al*. 2017, Khetir *et al*. 2017, Yazid *et al*. 2018) have presented the nonlocal model based on Eringen’s theory for the bending, buckling and free vibration of FG nanobeams and nanoplates. Also, Kolahchi and his co-workers (Kolahchi *et al*. 2016b, 2017b, c, Kolahchi 2017, Kolahchi and Cheraghbak 2017) studied the bending, vibration, buckling and dynamic stability of composite nanoplates using higher-order plates theories. Based on nonlocal sinusoidal shear deformation plate model, dynamic pull-in and pull-out analysis of viscoelastic nanoplates has been studied by Shokravi (2017c). Eltaher *et al*. (2016) studied the static stability of nonlocal nanobeams using higher-order beam theories. Recently, Bouafia *et al*. (2017) investigated nonlocal bending and free vibration behaviors of FG nanobeams using quasi-3D theory in which both shear deformation and thickness stretching effects are introduced. These studies pointed that the nonlocal size-dependent effect plays an important role in studying the static and dynamic behavior of isotropic and FG beams at small-scale. The strain gradient theory (Mindlin 1964, 1965, Aifantis 1992) is a microstructure-dependent continuum theory which can capture the stiffness enhancement effect. It states that the total stress field must account for additional strain gradient terms to consider microstructural deformation mechanism. By using the Mindlin’s strain gradient theory. Yang *et al*. (2002) proposed the modified couple stress theory including only one additional parameter and considering a symmetric couple stress tensor. Reddy (2011) studied the bending, vibration and buckling problems of functionally graded beams by using Euler-Bernoulli and Timoshenko beams models as well as modified couple stress theory. Akgöz and Civalek (2014) studied the thermomechanical buckling behavior of an embedded FG simply supported micro-scaled beam based on sinusoidal shear deformation beam theory. Ebrahimi and Beni (2016) has presented analytical solutions of free vibrations of a short cylindrical nanotube made of piezoelectric material based on the consistent couple stress theory and using the shear deformable cylindrical theory. However, the modified couple stress theory can be considered as the special case of the modified strain gradient theory. Recently, considerable interests have been devoted to theoretical works of the mechanical and dynamic behaviors of small-scaled structures within the framework of strain gradient theory and modified couple stress theory (Ma *et al*. 2008, Kong *et al*. 2009, Ansari *et al*. 2011, Nateghi *et al*. 2012, Ansari *et al*. 2013, Akgöz and Civalek 2013, Sahmani and Ansari 2013 Şimşek and Reddy 2013, Kolahchi and Bidgoli 2016, Li *et al*. 2016, 2017). A stiffness enhancement effect has been observed for these gradient elasticity models. From the discussions above, it is clear that the nonlocal elasticity model and the strain gradient models describe two entirely different size-dependent mechanical and physical characteristics of small-scaled materials and structures. To assess the true effects of the two size-dependent problems on the structural responses, some nonlocal strain gradient models have been elaborated to evaluate the two length scales effects on mechanical behaviors. Papargyri-Beskou *et al*. (2003) studied the bending and buckling problems of Euler–Bernoulli beam based on the strain gradient elasticity with surface energy. More recently, Lim *et al*. (2015) proposed a model combining both strain gradient and nonlocal elasticity models named nonlocal strain gradient theory and tried to show the impacts of nonlocal and length scale parameters on the mechanical and physical responses of size-dependent structures. This model attracted a many researchers since to model small scale structures. Li and Hu (2015) showed the influences of using nonlocal strain gradient theory on buckling behaviors of nanobeams and showed that the stiffness softening effects or the stiffness enhancement effects are shown to depend on the values of the two length scales parameter. Li *et al*. (2015) presented an analytical model for the flexural wave propagation analysis of small-scaled functionally graded beams based on the nonlocal strain gradient theory. Akgöz and Civalek (2012) carried out static analysis of microbeams of Bernoulli–Euler model based on the modified strain gradient theory. It should be noted that the thickness stretching effect is ignored in these previous works and the transverse displacement is considered constant in the thickness direction, as in different beam theories including those of Euler–Bernoulli, Timoshenko, and various higher-order shear deformation beam theories without stretching effect. This appears quite inadequate since functionally graded nanobeams are characterized by a strong variation of material properties in the thickness direction (Carrera *et al*. 2017, Abualnour *et al*. 2018). This paper aims to improve the beam theory developed by Tounsi and his coworkers (Larbi Chaht *et al*. 2015, Bouafia *et al*. 2017) by including the so-called stretching effect. By searching the literature, it is found that in all works on functionally graded nanobeams mentioned above, the length scale parameters employed in the formulation are considered as constants. The only work in the open literature that takes into consideration the variations in the length scale parameters seems to be that by Al-Basyouni *et al*. (2015). In the present work, the buckling analysis of simply supported functionally graded nanobeams is presented using a nonlocal strain gradient elasticity theory based on a novel hyperbolic quasi-3D theory in which both shear deformation and thickness stretching effects are introduced. Elastic coefficients and length scale parameter of functionally graded nanobeams are assumed to change continuously along the thickness according to the power-law form. The most interesting feature of this theory is that it accounts for a hyperbolic variation of the transverse shear strains across the thickness and satisfies the zero traction boundary conditions on the top and bottom surfaces of the beam without using shear correction factors. By using the Hamilton's principle the governing equations of motion are derived. The closed-form solutions for exact critical buckling loads of nonlocal strain gradient (FG) beams are obtained using Navier's method. The derived results are compared with those of strain gradient theory. Selected numerical results are presented to indicate the effects of the power-law index, nonlocal parameter, slenderness ratio on the buckling of FG nanobeam with graded nonlocality.

2. Model and theoretical formulations

A quasi-3D hyperbolic shear deformation beam theory considering shear and normal deformations is adopted in this study. The displacement field of the proposed theory is chosen based on the following assumptions (Houari *et al*. 2013, Bessaim *et al*. 2013, Belabed *et al*. 2014, Hamidi *et al*. 2015): (1) The transverse displacement is partitioned into bending, shear and stretching components; (2) the axial displacement consists of extension, bending and shear components; (3) the bending component of axial displacement is similar to that given by the Euler–Bernoulli beam theory; and (4) the shear component of axial displacement gives rise to the hyperbolic variation of shear strain and hence to shear stress through the thickness of the beam in such a way that shear stress vanishes on the top and bottom surfaces. The material properties of the FG nanobeam are assumed to vary in the thickness direction. Based on the of nonlocal strain gradient theory (Askes and Aifantis 2009, Ansari *et al*. 2012) to consider the small scale effects, the governing equations are derived using the principle of minimum total potential energy. The length scale parameter is assumed to vary in the thickness direction. To illustrate the accuracy of the present theory, the obtained results are compared with those predicted by the Euler–Bernoulli beam theory (EBT) and Timoshenko beam theory (TBT). Finally, the influences of nonlocal parameter, power law index, and aspect ratio on the buckling of FGM nanobeam are discussed.

2.1 Material properties

Consider a uniform FG nanobeam of thickness h, length L, and width b made by mixing two distinct materials (metal and ceramic). The coordinate x is along the longitudinal direction and z is along the thickness direction. It is assumed that material properties of the FGM nanobeam, such as Young's modulus (E), Poisson's ratio (v), and length scale parameter vary continuously through the nanobeam thickness according to a power-law form (Kolahchi *et al*. 2015, Kar and Panda 2015, Ghorbanpour *et al*. 2016, Bousahla *et al*. 2016, Houari *et al*. 2016, Beldjelili *et al*. 2016, Ait Atmane *et al*. 2017, Hachemi *et al*. 2017), which can be described by

$$P(z) = (P_c - P_m)\left(\frac{z}{h} + \frac{1}{2}\right)^k + P_m \tag{1}$$

The effective Young modulus $E(z)$ and length scale parameter $l(z)$ are given by the rule of mixtures as

$$E(z) = (E_c - E_m)\left(\frac{z}{h} + \frac{1}{2}\right)^k + E_m \tag{2}$$

$$l(z) = (l_c - l_m)\left(\frac{z}{h} + \frac{1}{2}\right)^k + l_m \tag{3}$$

k is the material distribution parameter which takes the value greater or equal to zero.

2.2 Kinematics

In order to incorporate both shear deformation and thickness stretching effects, the axial and transverse displacements are supposed to follow a hyperbolic variation through the thickness. Based on the assumptions made above, the displacement field of the present theory can be obtained as

$$u(x,z) = u_0(x) - z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x} \tag{4a}$$

$$w(x,z) = w_b(x) + w_s(x) + w_{st}(x,z) \tag{4b}$$

where u_0 is the axial displacement along the midplane of the nanoscale beam; w_b, w_s and w_{st} are the bending, shear and stretching components of the transverse displacement along the midplane of the beam. A new hyperbolic shear deformation beam function is fitted and used

$$f(z) = z - \frac{h\tanh\left(\frac{2z}{h}\right) + 2z\left(\tanh(1)^2 - 1\right)}{2\tanh(1)^2} \quad (5)$$

The component due to the stretching effect w_{st} can be given as

$$w_{st}(x,z) = g(z)\,\varphi(x) \quad (6)$$

The additional displacement φ accounting for the effect of normal stress is included and $g(z)$ is given as follows

$$g(z) = 1 - f'(z) \quad (7)$$

The nonzero strains of the considered beam theory are

$$\varepsilon_x = \varepsilon_x^0 + z\,k_x^b + f(z)\,k_x^s, \qquad \gamma_{xz} = g(z)\,\gamma_{xz}^0 \quad \text{and} \quad \varepsilon_z = g'(z)\,\varepsilon_z^0 \quad (8)$$

where

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x}, \quad k_x^b = -\frac{\partial^2 w_b}{\partial x^2}, \quad k_x^s = -\frac{\partial^2 w_s}{\partial x^2}, \quad \gamma_{xz}^0 = \frac{\partial w_s}{\partial x} + \frac{\partial \varphi}{\partial x}, \quad \varepsilon_z^0 = \varphi \quad (9)$$

2.3 The nonlocal FG nanobeam strain gradient model

Nonlocal strain gradient elasticity (or the high-order nonlocal strain gradient theory) exposed in Lim *et al*. (2015) gives the stress for both nonlocal stress and strain fields. Consequently, the stress can be expressed by

$$\sigma_{ij} = \sigma_{ij}^{(0)} - \frac{\partial \sigma_{ij}^{(1)}}{\partial x} \quad (10)$$

where the stresses $\sigma_{xx}^{(0)}$ and $\sigma_{xx}^{(1)}$ are related to strain ε_{xx} and strain gradient $\varepsilon_{xx,x}$, respectively, and are defined as

$$\sigma_{ij}^{(0)} = \int_0^L C_{ijkl}\,\alpha_0(x,x',e_0a)\varepsilon'_{kl}(x')dx' \quad (11)$$

$$\sigma_{ij}^{(1)} = l^2\int_0^L C_{ijkl}\,\alpha_1(x,x',e_1a)\varepsilon'_{kl,x}(x')dx' \quad (12)$$

in which C_{ijkl} are the elastic constants and e_0a and e_1a take into account the effect of nonlocal stress field, and l is the length scale parameter and introduces the influence of higher-order strain gradient stress field. When the nonlocal functions α_0 (x, x', e_0a) and α_1 (x, x', e_1a) satisfy the developed conditions by Eringen (1983), the constitutive relation for a FGM nanobeam can be stated as

$$\left(1-(e_1a)^2\nabla^2\right)\left(1-(e_0a)^2\nabla^2\right)\sigma_{ij} = C_{ijkl}\left(\left(1-(e_1a)^2\nabla^2\right)\varepsilon_{kl} - l^2\left(1-(e_0a)^2\nabla^2\right)\nabla^2\varepsilon_{kl}\right) \quad (13)$$

in which ∇^2 denotes the Laplacian operator. Supposing $e_1 = e_0 = e$ and discarding terms of order $O(\nabla^2)$, the general constitutive relation in Eq. (33) can be rewritten as

$$\left(1-(ea)^2\nabla^2\right)\sigma_{ij} = C_{ijkl}\left(1-l^2\nabla^2\right)\varepsilon_{kl} \quad (14)$$

Thus, the constitutive relations for a nonlocal refined shear deformable FG nanobeam can be stated as

$$\sigma_{xx} - \mu\frac{\partial\sigma_{xx}}{\partial x^2} = Q_{11}\left(\varepsilon_{xx} - \lambda\frac{\partial\varepsilon_{xx}}{\partial x^2}\right) + Q_{13}\left(\varepsilon_{zz} - \lambda\frac{\partial\varepsilon_{zz}}{\partial x^2}\right) \quad (15a)$$

$$\tau_{xz} - \mu\frac{\partial\tau_{xz}}{\partial x^2} = Q_{55}\left(\gamma_{xz} - \lambda\frac{\partial\gamma_{xz}}{\partial x^2}\right) \quad (15b)$$

$$\sigma_z - \mu\frac{d^2\sigma_z}{dx^2} = Q_{13}\left(\varepsilon_{xx} - \lambda\frac{\partial\varepsilon_{xx}}{\partial x^2}\right) + Q_{33}\left(\varepsilon_{zz} - \lambda\frac{\partial\varepsilon_{zz}}{\partial x^2}\right) \quad (15c)$$

where $\mu = (ea)^2$ and $\lambda = l^2$.

It is of interest that Eq. (15) can be simplified to some interested cases:

(a) (Nonlocal continuum theory): The constitutive equation of the nonlocal continuum theory can be easily obtained by setting $l = 0$ in Eq. (15), that is

$$\sigma_{xx} - \mu\frac{\partial\sigma_{xx}}{\partial x^2} = Q_{11}\varepsilon_{xx} + Q_{13}\varepsilon_{zz}$$
$$\tau_{xz} - \mu\frac{\partial\tau_{xz}}{\partial x^2} = Q_{55}\gamma_{xz} \quad (16)$$
$$\sigma_z - \mu\frac{d^2\sigma_z}{dx^2} = Q_{13}\varepsilon_{xx} + Q_{33}\varepsilon_{zz}$$

which are identical to Eringen (1983).

(b) (Strain gradient theory): The constitutive equation of the strain gradient theory can be easily obtained by setting $ea = 0$ in Eq. (15), that is

$$\sigma_{xx} = Q_{11}\left(\varepsilon_{xx} - \lambda\frac{\partial\varepsilon_{xx}}{\partial x^2}\right) + Q_{13}\left(\varepsilon_{zz} - \lambda\frac{\partial\varepsilon_{zz}}{\partial x^2}\right)$$
$$\tau_{xz} = Q_{55}\left(\gamma_{xz} - \lambda\frac{\partial\gamma_{xz}}{\partial x^2}\right) \quad (17)$$
$$\sigma_z = Q_{13}\left(\varepsilon_{xx} - \lambda\frac{\partial\varepsilon_{xx}}{\partial x^2}\right) + Q_{33}\left(\varepsilon_{zz} - \lambda\frac{\partial\varepsilon_{zz}}{\partial x^2}\right)$$

which are identical to Aifantis (1992).

In this section, the equations of motion for size-dependent FG quasi-3D hyperbolic shear deformation beam theory considering shear and normal deformations will be formed based on the general constitutive equation of nonlocal strain gradient theory.

The governing equations will be derived by using principal of the minimum total potential energy as follows

$$\delta\Pi = \delta U_{\text{int}} + \delta W_{ext} \tag{18}$$

where $\delta\Pi$ is the total potential energy. δU_{int} is the virtual variation of the strain energy; and δW_{ext} is the variation of work done by external forces. The first variation of the strain energy is given as

$$\begin{aligned}\delta U_{\text{int}} &= \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_x \delta\, \varepsilon_x + \sigma_z \delta\, \varepsilon_z + \tau_{xz} \delta\, \gamma_{xz}) dz_{ns} dx \\ &= \int_0^L \left(N \frac{d\delta\, u_0}{dx} + N_z \delta\, \varphi - M_b \frac{d^2 \delta\, w_b}{dx^2} + \right. \\ &\left. - M_s \frac{d^2 \delta\, w_s}{dx^2} + Q\left[\frac{d\delta\, w_s}{dx} + \frac{d\delta\, \varphi}{dx} \right] \right) dx\end{aligned} \tag{19}$$

where N, M_b, M_s, N_z and Q are the stress resultants defined as

$$\begin{gathered}(N, M_b, M_s) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1, z, f(z))\, \sigma_x dz, \\ N_z = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_z g'(z) dz, \quad \text{and} \quad Q = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} g(z) dz\end{gathered} \tag{20}$$

The variation of potential energy of external force can be expressed as

$$\delta V = \int_0^L N_0 \frac{dw}{dx} \frac{d\delta w}{dx} dx \tag{21}$$

where N_0 is the axial loads.

Substituting the relations for δU_{int}, δW_{ext}, and δV from Eqs. (19), (20), and (21) into Eq. (18) and integrating by parts, and collecting the coefficients of δu_0, δw_b, δw_s and $\delta\varphi$, the following equations of motion of the FG beam are found

$$\delta\, u_0 : \ \frac{dN}{dx} = 0 \tag{22a}$$

$$\delta\, w_b : \ \frac{d^2 M_b}{dx^2} - N_0 \frac{d^2 w}{dx^2} = 0 \tag{22b}$$

$$\delta\, w_s : \ \frac{d^2 M_s}{dx^2} + \frac{dQ}{\partial x} - N_0 \frac{d^2 w}{dx^2} = 0 \tag{22c}$$

$$\delta\, \varphi : \ \frac{dQ}{dx} - N_z - N_0 \frac{d^2 w}{dx^2} = 0 \tag{22d}$$

By integrating Eq. (20) over the cross-sectional area of nanobeam, the following nonlocal relations for FG nanosbeam model are deduced

$$\begin{aligned}N = A_{11} \frac{du_0}{dx} - A_{11}^{\lambda} \frac{d^3 u_0}{dx^3} - B_{11} \frac{d^2 w_b}{dx^2} + B_{11}^{\lambda} \frac{d^4 w_b}{dx^4} \\ - B_{11}^s \frac{d^2 w_s}{dx^2} + B_{11}^{s\lambda} \frac{d^4 w_s}{dx^4} + X_{13}\, \varphi - X_{13}^{\lambda} \frac{d^2 \varphi}{dx^2}\end{aligned} \tag{23a}$$

$$\begin{aligned}M_b = B_{11} \frac{du_0}{dx} - B_{11}^{\lambda} \frac{d^3 u_0}{dx^3} - D_{11} \frac{d^2 w_b}{dx^2} + D_{11}^{\lambda} \frac{d^4 w_b}{dx^4} \\ - D_{11}^s \frac{d^2 w_s}{dx^2} + D_{11}^{s\lambda} \frac{d^4 w_s}{dx^4} + Y_{13} \varphi - Y_{13}^{\lambda} \frac{d^2 \varphi}{dx^2}\end{aligned} \tag{23b}$$

$$\begin{aligned}M_s = B_{11}^s \frac{du_0}{dx} - D_{11}^{s\lambda} \frac{d^3 u_0}{dx^3} - D_{11}^s \frac{d^2 w_b}{dx^2} + D_{11}^{s\lambda} \frac{d^4 w_b}{dx^4} \\ - H_{11}^s \frac{d^2 w_s}{dx^2} + H_{11}^{s\lambda} \frac{d^4 w_s}{dx^4} + Y_{13}^s \varphi - Y_{13}^{s\lambda} \frac{d^2 \varphi}{dx^2}\end{aligned} \tag{23c}$$

$$Q = A_{55}^s \left(\frac{dw_s}{dx} + \frac{d\,\varphi}{dx} \right) - A_{55}^{s\lambda} \left(\frac{d^3 w_s}{dx^3} + \frac{d^3 \varphi}{dx^3} \right) \tag{23d}$$

$$\begin{aligned}N_z = X_{13} \frac{du_0}{dx} - X_{13}^{\lambda} \frac{d^3 u_0}{dx^3} - Y_{13} \frac{d^2 w_b}{dx^2} + Y_{13}^{\lambda} \frac{d^4 w_b}{dx^4} \\ - Y_{13}^s \frac{d^2 w_s}{dx^2} + Y_{13}^{s\lambda} \frac{d^4 w_s}{dx^4} + Z_{33}\, \varphi - Z_{33}^{\lambda} \frac{d^2 \varphi}{dx^2}\end{aligned} \tag{23e}$$

where the cross-sectional rigidities are expressed as

$$\begin{aligned}&\left(A_{11}, B_{11}, D_{11}, B_{11}^s, D_{11}^s, H_{11}^s\right) \\ &= \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11} \left(1, z, z^2, f(z), z\, f(z), f^2(z)\right) dz\end{aligned} \tag{24a}$$

$$\begin{aligned}&\left(A_{11}^{\lambda}, B_{11}^{\lambda}, D_{11}^{\lambda}, B_{11}^{s\lambda}, D_{11}^{s\lambda}, H_{11}^{s\lambda}\right) \\ &= \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11} \lambda(z) \left(1, z, z^2, f(z), z\, f(z), f^2(z)\right) dz\end{aligned} \tag{24b}$$

$$A_{55}^s = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{55} g(z)^2 dz, \quad A_{55}^{s\lambda} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{55} \lambda(z) g(z)^2 dz \tag{24c}$$

$$\left(X_{13}, Y_{13}, Y_{13}^s\right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{13} \left(1, z, f(z)\right) g'(z) dz \tag{24d}$$

$$\left(X_{13}^{\lambda}, Y_{13}^{\lambda}, Y_{13}^{s\lambda}\right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{13} \lambda(z) \left(1, z, f(z)\right) g'(z) dz \tag{24e}$$

$$Z_{33} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{33} g'(z)^2 \, dz, Z_{33}^{\lambda} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{33} \lambda(z) g'(z)^2 \, dz \tag{24f}$$

The governing equations of a quasi-3D FG nanobeam in terms of displacements are obtained by inserting N, M_b, M_s, N_z and Q from Eq. (23), respectively, into Eq. (22) as follows

$$\begin{aligned} & A_{11} \frac{d^2 u_0}{dx^2} - A_{11}^{\lambda} \frac{d^4 u_0}{dx^4} - B_{11} \frac{d^3 w_b}{dx^3} + B_{11}^{\lambda} \frac{d^5 w_b}{dx^5} \\ & - B_{11}^{s} \frac{d^3 w_s}{dx^3} + B_{11}^{s\lambda} \frac{d^5 w_s}{dx^5} + X_{13} \frac{d\varphi}{dx} - X_{13}^{\lambda} \frac{d^3 \varphi}{dx^3} = 0 \end{aligned} \tag{25a}$$

$$\begin{aligned} & B_{11} \frac{d^3 u_0}{dx^3} - B_{11}^{\lambda} \frac{d^5 u_0}{dx^5} - D_{11} \frac{d^4 w_b}{dx^4} + D_{11}^{\lambda} \frac{d^6 w_b}{dx^6} \\ & - D_{11}^{s} \frac{d^4 w_s}{dx^4} + D_{11}^{s\lambda} \frac{d^6 w_s}{dx^6} \\ & + Y_{13} \frac{d^2 \varphi}{dx^2} - Y_{13}^{\lambda} \frac{d^4 \varphi}{dx^4} = N_0 \frac{d^2 w}{dx^2} \end{aligned} \tag{25b}$$

$$\begin{aligned} & B_{11}^{s} \frac{d^3 u_0}{dx^3} - B_{11}^{s\lambda} \frac{d^5 u_0}{dx^5} - D_{11}^{s} \frac{d^4 w_b}{dx^4} + D_{11}^{s\lambda} \frac{d^6 w_b}{dx^6} \\ & - H_{11}^{s} \frac{d^4 w_s}{dx^4} + H_{11}^{s\lambda} \frac{d^6 w_s}{dx^6} + A_{55}^{s} \frac{d^2 w_s}{dx^2} - A_{55}^{s\lambda} \frac{d^4 w_s}{dx^4} \\ & + \left(A_{55}^{s} + Y_{13}^{s}\right) \frac{d^2 \varphi}{dx^2} - \left(A_{55}^{s\lambda} + Y_{13}^{s\lambda}\right) \frac{d^4 \varphi}{dx^4} = N_0 \frac{d^2 w}{dx^2} \end{aligned} \tag{25c}$$

$$\begin{aligned} & - X_{13} \frac{du_0}{dx} + X_{13}^{s\lambda} \frac{d^3 u_0}{dx^3} + Y_{13} \frac{d^2 w_b}{dx^2} - Y_{13}^{\lambda} \frac{d^4 w_b}{dx^4} \\ & + \left(A_{55}^{s} + Y_{13}^{s}\right) \frac{d^2 w_s}{dx^2} - \left(A_{55}^{s\lambda} + Y_{13}^{s\lambda}\right) \frac{d^4 w_s}{dx^4} \\ & + A_{55}^{s} \frac{d^2 \varphi}{\partial x^2} - A_{55}^{s\lambda} \frac{d^4 \varphi}{\partial x^4} - Z_{33} \varphi + Z_{33}^{\lambda} \frac{d^2 \varphi}{dx^2} = N_0 \frac{d^2 w}{dx^2} \end{aligned} \tag{25d}$$

3. Analytical solution

he above equations of motion are analytically solved for buckling problem. The Navier solution technique is employed to obtain the analytical solutions for a simply supported FG nanobeam. The solution is assumed to be of the form

$$\begin{Bmatrix} u_0 \\ w_b \\ w_s \\ \varphi \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} U_m \cos(\alpha\, x) \\ W_{bm} \sin(\alpha\, x) \\ W_{sm} \sin(\alpha\, x) \\ \Phi_{stm} \sin(\alpha\, x) \end{Bmatrix} \tag{26}$$

where U_m, W_{bm}, W_{sm} and Φ_{stm} are arbitrary parameters to be determined and $\alpha = m\pi/L$.

Substituting Eq. (26) into Eq. (25), the analytical solutions can be obtained by

$$\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} - \zeta & a_{23} - \zeta & a_{24} - \zeta \\ a_{13} & a_{23} - \zeta & a_{33} - \zeta & a_{34} - \zeta \\ a_{14} & a_{24} - \zeta & a_{34} - \zeta & a_{44} - \zeta \end{bmatrix} \right) \begin{Bmatrix} U_m \\ W_{bm} \\ W_{sm} \\ \Phi_{stm} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{27}$$

where

$$\begin{gathered} a_{11} = -\alpha^2 \left(A_{11} + A_{11}^{\lambda} \alpha^2\right), \quad a_{12} = \alpha^3 \left(B_{11} + B_{11}^{\lambda} \alpha^2\right), \\ a_{13} = \alpha^3 \left(B_{11}^{s} + B_{11}^{s\lambda} \alpha^2\right), \quad a_{14} = \alpha \left(X_{13} + X_{13}^{\lambda} \alpha^2\right), \\ a_{22} = -D_{11} \alpha^4 - D_{11}^{\lambda} \alpha^6, \quad a_{23} = -D_{11}^{s} \alpha^4 - D_{11}^{s\lambda} \alpha^6, \\ a_{24} = -\alpha^2 \left(Y_{13} + Y_{13}^{s\lambda} \alpha^2\right), \\ a_{33} = -\alpha^2 \left(A_{55} + \alpha^2 H_{11}^{s} + A_{55}^{\lambda} \alpha^2 + H_{11}^{s\lambda} \alpha^4\right), \\ a_{34} = -\alpha^2 \left(A_{55} + Y_{13}^{s} + \alpha^2 A_{55}^{\lambda} + \alpha^2 Y_{13}^{s\lambda}\right), \\ a_{44} = -\left(\alpha^2 A_{55} + Z_{33} + \alpha^4 A_{55}^{\lambda} + \alpha^2 Z_{33}^{\lambda}\right), \\ \beta = 1 + \mu \lambda^2, \quad \xi = \beta\, N_0 \alpha^2 \end{gathered} \tag{28}$$

4. Numerical results and discussions

In order to check the validity and the accuracy of the presented new quasi-3D hyperbolic beam theory, comparison studies, as possible, for buckling behaviors using the present model in the framework of the nonlocal strain gradient theory, have been performed out with the results of the available published works. In addition, selected numerical results are presented to indicate the effects of the power-law index, nonlocal parameter, slenderness ratio, shear deformation and thickness stretching on the buckling of FG nanobeam with graded nonlocality.

As a first example, Fig. 1 shows the variation of the first non-dimensional critical buckling load versus the length-to-side ratio, L/h, for new hyperbolic shear deformation beam models by the present approach with and without stretching effect and the results are compared to those reported by

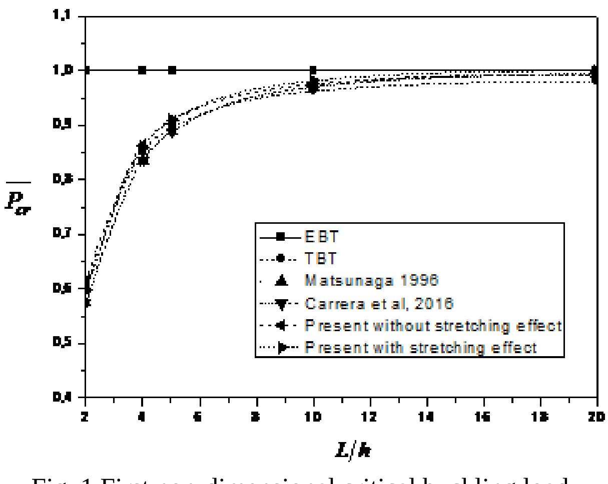

Fig. 1 First non-dimensional critical buckling load $\overline{P_{cr}} = \frac{N_0 L^2}{\pi^2 EI}$ versus length-to-height ratio, L/h, for the rectangular isotropic beam

Matsunaga (1996) based on the power series expansion of displacement components and the Carrera Unified Formulation theories (Carrera *et al*. 2016) (CUF) achieved through the application of the Dynamic Stiffness Method (DSM). The following parameters are employed in calculating the numerical results: E = 1 TPa, v = 0.3, h = 10 nm, b = h (where b and h are the width and the thickness of the beam, respectively). The shear correction factor is taken as 5/6 for Timoshenko beam theory. The results obtained using Timoshenko beam theory (TBT) and classical Euler Bernoulli beam theory (EBT), are also presented in Fig. 1. The present results are in excellent agreement with other results for all aspect ratios.

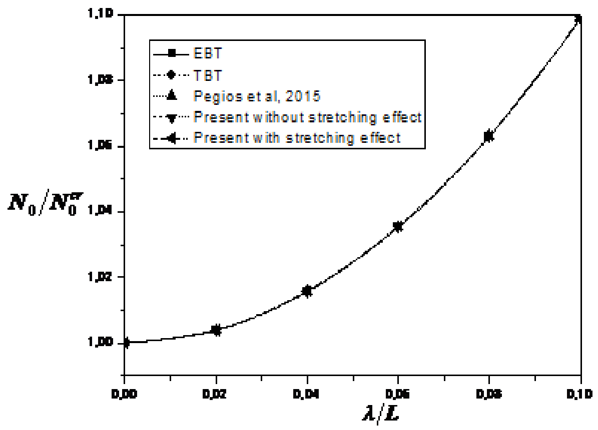

Fig. 2 Variation of the dimensionless buckling load N_0/N_0^{cr} versus length scale parameter λ / L of isotropic rectangular beams

In the second example, the validation of the solution of the proposed new hyperbolic shear deformation beam model is carried out by comparing the obtained results with those computed via finite element stiffness matrices of a strain gradient elastic Euler–Bernoulli beam of Pegios *et al*. (2015), and the results obtained by Timoshenko beam theory (TBT) and Euler Bernoulli beam theory (EBT).

The comparison of the first non-dimensional critical buckling load N_0/N_0^{cr} versus length scale parameter λ of isotropic rectangular beams is depicted in Fig. 2, where $N_0^{cr} = \pi^2 EI/L^2$ is the classical value of N_0. It can be seen that the dimensionless critical buckling load predicted by the new hyperbolic theory with and without the stretching effect are almost identical with those generated by the Euler–Bernoulli beam of Pegios *et al*. (2015). Also, it is clearly shows an increase of this dimensionless buckling ratio for increasing values of the dimensionless gradient coefficient λ/L with a constantly increasing rate of increase.

In the third example of the present section, analytical solutions of the buckling responses obtained in the previous sections are exploited for numerical examples and compared with those obtained using Timoshenko beam theory (TBT) and classical Euler Bernoulli beam theory (EBT) for a wide range of length scale parameter (l), the material distribution parameter (k) and thickness ratio (L/h).

Table 1 Dimensionless critical buckling load ($\bar{N}$) of the FG nanobeam

l_m/l_c	Beam theory	L/h = 10				L/h = 100			
		k = 0.3	k = 1	k = 3	k = 10	k = 0.3	k = 1	k = 3	k = 10
Classic	CBT	4.0944	5.4283	6.8176	8.3176	4.0944	5.4283	6.8176	8.3176
	FSBT	3.9939	5.3084	6.6721	8.1290	4.0934	5.4271	6.8161	8.3157
	Present $\varepsilon_z = 0$	3.9914	5.3096	6.6794	8.1350	4.0933	5.4271	6.8162	8.3158
	Present $\varepsilon_z \neq 0$	4.0464	5.3981	6.7562	8.1929	4.1587	5.5288	6.9077	8.3906
1/3	CBT	6.7980	7.9706	9.0553	10.0587	4.1220	5.4555	6.8414	8.3354
	FSBT	6.6311	7.7880	8.8481	9.8181	4.1209	5.4542	6.8400	8.3335
	Present $\varepsilon_z = 0$	6.6305	7.7938	8.8561	9.8221	4.1290	5.4542	6.8400	8.3336
	Present $\varepsilon_z \neq 0$	6.6998	7.8738	8.9237	9.8865	4.1864	5.5558	6.9314	8.4085
1	CBT	4.4985	5.9640	7.4905	9.1386	4.0984	5.4336	6.8243	8.3259
	FSBT	4.3881	5.8323	7.3306	8.9313	4.0974	5.4324	6.8229	8.3239
	Present $\varepsilon_z = 0$	4.3854	5.8337	7.3386	8.9379	4.0974	5.4324	6.8229	8.3240
	Present $\varepsilon_z \neq 0$	4.4457	5.9309	7.4231	9.0015	4.1628	5.5342	6.9145	8.3989
3/2	CBT	6.2368	8.8290	11.7643	15.0983	4.1160	5.4629	6.8675	8.3855
	FSBT	6.0852	8.6413	11.5235	14.7637	4.1150	5.4616	6.8660	8.3836
	Present $\varepsilon_z = 0$	6.0791	8.6410	11.5368	14.7761	4.1150	5.4616	6.8661	8.3837
	Present $\varepsilon_z \neq 0$	6.1797	8.8251	11.6993	14.8859	4.1808	5.5644	6.9585	8.4592
2	CBT	4.8172	6.7257	8.8482	11.2378	4.1017	5.4415	6.8381	8.3469
	FSBT	4.6997	6.5814	8.6655	10.9876	4.1007	5.4403	6.8366	8.3449
	Present $\varepsilon_z = 0$	4.6954	6.5816	8.6754	10.9967	4.1007	5.4403	6.8367	8.3450
	Present $\varepsilon_z \neq 0$	4.7701	6.7150	8.7929	11.0776	4.1662	5.5425	6.9286	8.4202

The beam geometry has the following dimensions: (length) L = 10 nm, (width) b = 1 nm. The FG nanobeam has the following material properties:

E_c = 0.25 TPa, E_m = 1 TPa, $v_c = v_m$ = 0.3 (Larbi Chaht *et al*. 2015). It is assumed that length scale parameter as well as material properties of nanobeam are grade through the thickness. The following dimensionless relation is defined in order to calculate the dimensionless buckling load

$$\delta V = \int_0^L N_0 \frac{d\,w}{d\,x}\frac{d\,\delta w}{d\,x}dx \tag{29}$$

Table 1 tabulate nondimensionalized critical buckling load of the FG nanobeam based on the present new quasi-3D hyperbolic beam theory for various values of the volume fraction exponent k, the ratio (l_m/l_c = 1/3, 1, 3/2, 2), and two different values of the aspect ratio (L/h = 10 and 100). The results obtained using the present quasi-3D hyperbolic beam theory show that the inclusion of the thickness stretching effect manifests in an enhancement in the critical buckling loads. According to this table, buckling loads increase with increasing the material distribution parameter (k). Thus, the ratio l_m/l_c indicates the degree of the length scale parameter variation across the beam. However, the increase of v ratio leads to a veritable change of critical buckling loads and the results are significantly different to the case where the length scale parameter is assumed to be a constant (l_m/l_c = 1) at a fixed aspect ratio (L/h).

Fig. 3 shows the variation of the non-dimensional buckling load of the FGM nanobeam with geometrical aspect ratio. The local and nonlocal results are given for $l_c = l_m = 0$ and $l_m/l_c = 2$, respectively. The material distribution parameter is assumed to be constant i.e., k = 2. It is clear that critical buckling load predicted by the strain gradient theory is higher in magnitude than the local buckling load due to the small scale effects. Also, it can be observed that the inclusion of the thickness stretching effect leads to increase in critical buckling load values for FG nanobeam.

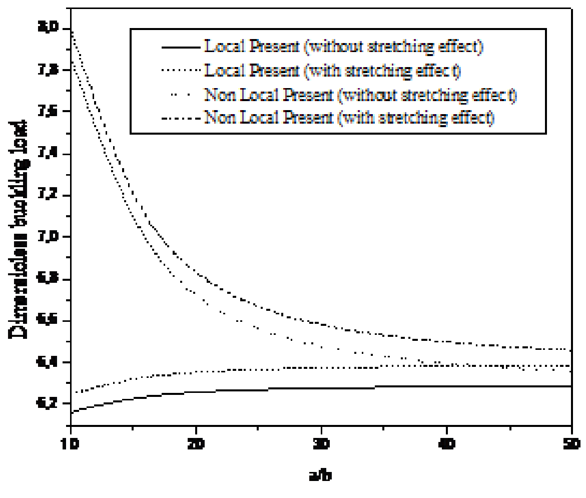

Fig. 3 Effect of the aspect ratio (a/h) on the dimensionless buckling load for k = 2, l_m/l_c = 2

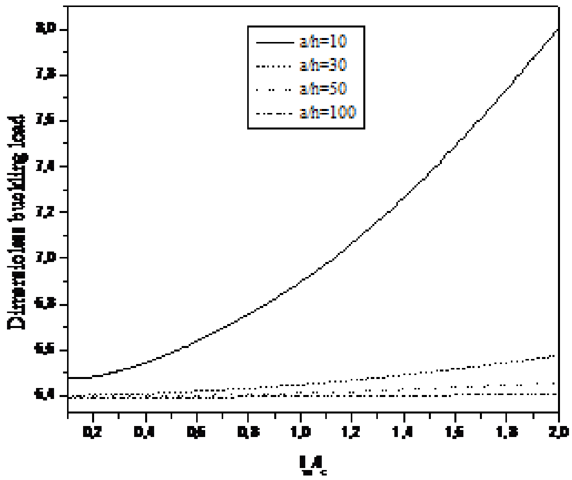

Fig. 4 Effect of nonlocal parameter ratio (l_m/l_c) on the dimensionless buckling load for k = 2

These results effectively demonstrate that the inclusion of small scale parameter softens the nanobeam (reduces stiffness), whereas the inclusion of thickness stretching effect makes it stiffer. As such both small scale and thickness stretching effects exert a significant influence on nanobeam structural performance.

Fig. 4 shows the effect of the length scale parameter ratio l_m/l_c on dimensionless critical buckling loads. The results in this figure are obtained by using the present nonlocal shear deformation beam theory including the thickness stretching effect. The material distribution parameter is assumed to be constant i.e., k = 2. It is observed that the buckling responses vary in a nonlinear fashion with the length scale parameter ratio l_m/l_c. It can be seen that the effect of length scale parameter ratio l_m/l_c on critical buckling loads of FG nanobeam is very important, especially at relatively higher aspect ratios. Therefore, it can be concluded that FGM nanobeam responses are aspect ratio-dependent based on strain gradient elasticity.

5. Conclusions

By introducing the concept of graded length scale parameter, buckling characteristics of FG nanobeams are investigated based on a size-dependent Quasi-3D hyperbolic shear deformation theory in the frame work of the non-local strain gradient theory. Elastic properties and length scale parameter of the FG nanobeams are position-dependent. The nonlocal governing differential equations are derived by implementing Hamilton's principle and using nonlocal strain gradient constitutive equations. It is observed that the critical buckling load change significantly with the increase of length scale parameter ratio and the critical buckling load predicted by the strain gradient theory is higher in magnitude than the local buckling load. Also, the inclusion of the thickness stretching effect leads to increase in critical buckling load. The formulation lend sit self particularly well to study several problems related to the hygro-thermomechanical deformation of laminated and FG structures (Bouderba *et al*. 2016, Beldjelili *et al*. 2016, Bousahla *et al*. 2016, Chikh *et al*. 2017, Menasria *et al*. 2017, Mouffoki *et al*. 2017), also by using the nonlocal strain gradient model for analysis of mechanical behaviour of nanostructures reinforced with nanoparticles and carbon nanotubes (Arani and Kolahchi 2016, Zamanian *et al*. 2017, Zarei *et al*. 2017, Shokravi 2017d), which will be considered in the near future. The present computations also provide a solid benchmark for verification of finite element and other numerical simulations of FGM nanobeam mechanics.

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