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Post-buckling responses of a laminated composite beam

Şeref D. Akbaş*

Department of Civil Engineering, Bursa Technical University, Yıldırım Campus, Yıldırım, Bursa 16330, Turkey

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Abstract. This paper presents post-buckling responses of a simply supported laminated composite beam subjected to a non-follower axially compression loads. In the nonlinear kinematic model of the laminated beam, total Lagrangian approach is used in conjunction with the Timoshenko beam theory. In the solution of the nonlinear problem, incremental displacement-based finite element method is used with Newton-Raphson iteration method. There is no restriction on the magnitudes of deflections and rotations in contradistinction to von-Karman strain displacement relations of the beam. The distinctive feature of this study is post-buckling analysis of Timoshenko Laminated beams full geometric non-linearity and by using finite element method. The effects of the fibber orientation angles and the stacking sequence of laminates on the post-buckling deflections, configurations and stresses of the composite laminated beam are illustrated and discussed in the numerical results. Numerical results show that the above-mentioned effects play a very important role on the post-buckling responses of the laminated composite beams.

Keywords: composite laminated beams; post-buckling analysis; Timoshenko Beam Theory; total lagragian; Finite Element Method

1. Introduction

Laminated composite structures have been used many engineering applications, such as aircrafts, space vehicles, automotive industries, defence industries and civil engineering applications because these structures have higher strength-weight ratios, more lightweight and ductile properties than classical materials. With the great advances in technology, the using of the laminated composite structures is growing in applications.

Buckling or post-buckling is occurred by a sudden failure of a structural member subjected to high compressive loads. Understanding the buckling and post-buckling mechanism of laminated composites is very important. It is known that post-buckling problems are geometric nonlinear problems. In the literature, much more attention has been given to the linear analysis of laminated composite beam structures. However, nonlinear and post-buckling studies of Laminated composite beams are has not been investigated broadly.

In the open literature, studies of the post-buckling and nonlinear behavior of laminated composite beams are as follows; Sheinman and Adan (1987) investigated effect of shear deformation on the post-buckling of laminated beams. Ghazavi and Gordaninejad (1989) studied geometrically nonliner static of laminated bimodular composite beams by using mixed finite element model. Singh *et al*. (1992) investigated nonlinear static responses of laminated composite beam based on higher shear deformation theory and von Karman's nonlinear type. Pai and Nayfeh (1992) presented three-dimensional nonlinear dynamics of anisotropic composite beams with von Karman nonlinear type. Di Sciuva and Icardi (1995) investigated larfe deflection of anisotropic laminated composite beams with Timoshenko beam theory and von Karman nonlinear strain-displacement relations by using Euler method. Donthireddy and Chandrashekhara (1997) investigated thermoelastic nonlinear static and dynamic analysis of laminated beams by using finite element method. Fraternali and Bilotti (1997) analyzed nonlinear stress of laminated composite curved beams. Ganapathi *et al*. (1998) studied nonlinear vibration analysis of laminated composite curved beams. Patel *et al*. (1999) examined nonlinear post-buckling and vibration of laminated composite orthotropic beams/ columns resting on elastic foundation with Von-Karman's strain-displacement relations. Oliveira and Creus (2003) investigated flexure and buckling behaviors of thin-walled composite beams with nonlinear viscoelastic model. Valido and Cardoso (2003) developed a finite element model for optimal desing of laminated composite thin-eslled beams with geometrically nonlinear effects. Machado (2007) studied nonlinear buckling and vibration of thin-walled composite beams. Cardoso *et al*. (2009) investigated geometrically nonlinear behavior of the laminated composite thin-walled beam structures with finite element solution. Kumar and Singh (2009) studied buckling and poat-buckling of laminated composite plates SMA fibers under thermal loading. Emam and Nayfeh (2009) investigated post-buckling of the laminated composite beams with different boundary conditions. Malekzadeh and Vosoughi (2009) studied large amplitude free vibration of laminated composite beams resting on elastic foundation by using differential quadrature method. Gupta *et al*. (2010) studied post-buckling analysis of composite beams with

*Corresponding author, Associate Professor,
E-mail: serefda@yahoo.com

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different boundary conditions by using Ritz method. Chang *et al*. (2011) investigated thermal buckling and post-buckling of laminated composite beams with higher order beam theories. Kocatürk and Akbaş (2011, 2012, 2013) investigated thermal post-buckling behavior of beams. Akgöz and Civalek (2011) and Civalek (2013) examined nonlinear vibration laminated plates resting on nonlinear-elastic foundation. Baghani *et al*. (2011) and Jafari-Talookolaei *et al*. (2011) examined large amplitude free vibration and post-buckling of laminated beams resting on elastic foundation by using the variational iteration method. Youzera *et al*. (2012) presented nonlinear dynamics of laminated composite beams with damping effect. Gunda and Rao (2013) investigated post-buckling analysis of composite beams based on von-Karman nonlinear type. Patel (2014) examined nonlinear static of laminated composite plates with the Green-Lagrange nonlinearity. Akbaş (2013, 2014, 2015a, b, 2017a), Akbaş and Kocatürk (2011, 2012, 2013) studied post-buckling and nonlinear analysis of homogeneous and non-homogeneous beams. Stoykov and Margenov (2014) studied Nonlinear vibrations of 3D laminated composite Timoshenko beams. Cunedioğlu and Beylergil (2014) examined vibration of laminated composite beams under thermal loading. Li and Qiao (2015a, b), Shen *et al*. (2016, 2017), Li and Yang (2016) investigated nonlinear postbuckling analysis of composite laminated beams. Mahi and Tounsi (2015) investigated static and bending of isotropic, functionally graded, sandwich and laminated composite plates. Draiche *et al*. (2016) examined flexure analysis of laminated composite plates with stretching effect. Chikh *et al*. (2017) studied thermal buckling ross-ply laminated plates by using a simplified HSDT. Fouda *et al*. (2017) investigated buckling, bendind and vibration of functionally graded beams with porosity effcet by using finite element methods. Almitani (2017) studied buckling of symmetric and antisymmetric functionally graded beams. Bessaim *et al*. (2013, 2017), Belabed *et al*. (2014), Bousahla *et al*. (2014), Hebali *et al*. (2014), Bourada *et al*. (2015), Hamidi *et al*. (2015), Bennoun *et al*. (2016), Bouafia *et al*. (2017), Abualnour *et al*. (2018) investigated the effects of the shear deformations on the mechanical behavior of composite structures. Houari *et al*. (2016) investigated three-unknown sinusoidal shear deformation theory for functionally graded plates. Asadi and Aghdam (2014), Mareishi *et al*. (2014), Kurtaran (2015), Mororó *et al*. (2015), Pagani and Carrera (2017) analyzed large deflections of laminated composite beams. Benselama *et al*. (2015), Liu and Shu (2015), Topal (2017) investigated buckling behavior of composite laminate beams. Latifi *et al*. (2016), Ebrahimi and Hosseini (2017) presented nonlinear dynamics of laminated composite structures. Meziane *et al*. (2014), Bouderba *et al*. (2016), Bousahla *et al*. (2016), Abdelaziz *et al*. (2017), Bellifa *et al*. (2017a, b), Akbaş (2017a, b, c), El-Haina *et al*. (2017), Menasria *et al*. (2017) investigated stability of the non-homogeneous plates. Chaht *et al*. (2015), Zemri *et al*. (2015), Ahouel *et al*. (2016), Khetir *et al*. (2017) examined buckling of nanoscale beams.

In the most of the post-buckling studies of laminated composite beams, the von-Karman strain displacement approximation is used. In the von-Karman strain, full geometric non-linearity cannot be considered because of neglect of some components of strain, satisfactory results can be obtained only for large displacements but moderate rotations. In the open literature, post-buckling studies of laminated composite beams with considering full geometric nonlinearity has not been investigated broadly.

In the present study, the post-buckling analysis of a laminated Timoshenko beams is considered by using total Lagrangian finite element method in which full geometric nonlinearity which can be considered as distinct from the studies by using von-Karman nonlinearity. The main purpose of this paper is to fill this gap for laminated composite beams for post-buckling behavior. The effects of the fibber orientation angles and the stacking sequence of laminates on the post-buckling deflections, configurations and stresses of the composite laminated beam are examined and discussed.

2. Theory and formulation

A simply supported laminated composite beam with three layers of length L, width b and height h with material or Lagrangian coordinate system (X, Y) and with spatial or Euler coordinate system (X, Y) as shown in Fig. 1. The beam is subjected to a non-follower compressive point load (F) at the end of the beam as seen from Fig. 1. It is assumed that the layers are located as symmetry according to mid-plane axis. The height of each layer is equal to each other.

It is known that the post-buckling case is a geometrically nonlinear problem. In the nonlinear kinematic model of the beam, the total Lagrangian approach is used within Timoshenko beam theory. The Lagrangian formulations of the problem are developed for laminated

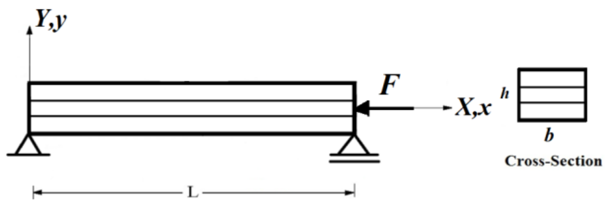


Fig. 1 A simply supported laminated beam subjected to a non-follower compressive point load (F) at the end of the beam and cross-section.

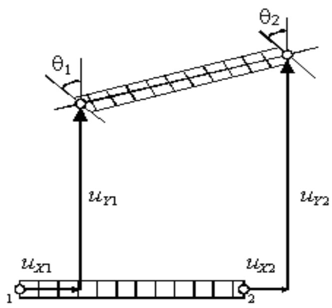
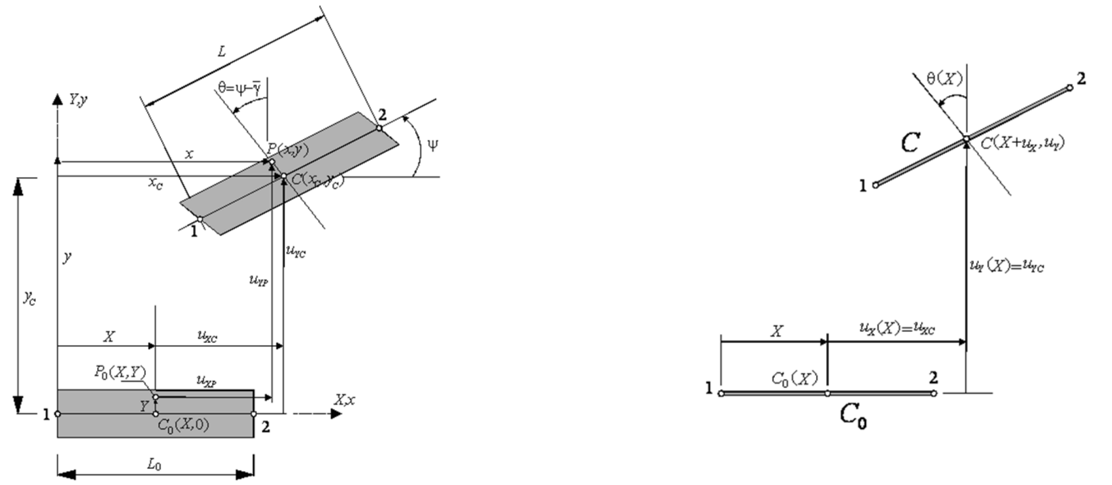


Fig. 2 Two-node C^0 beam element



(a) Motion of plane beam (b) Reduction to one-dimensional element

Fig. 3 Lagrangian kinematics of the C^0 beam element with X-aligned reference configuration Felippa (2017)

composite beam by using the formulations given by Felippa (2017) for isotropic and homogeneous beam material. The finite beam element of the problem is derived by using a two-node beam element shown in Fig. 2, of which each node has three degrees of freedom, i.e., two displacements u_{xi} and u_{yi} and one rotation θ_i about the Z axis.

In the deformation process, a generic point of the beam located at P_0 (X, Y) in the previous configuration C_0 moves to $P(x, y)$ in the current configuration C, as shown in Fig. 3. The projections of P_0 and P along the cross sections at C_0 and C upon the neutral axis are called C_0 (X, 0) and C (x_c, y_c), respectively. It is assumed that the cross section of the beam remains unchanged, such that the shear distortion g << 1 and cos g can be replaced by 1 Felippa (2017). The coordinates of the beam at the current C configuration are

$$\begin{aligned} x &= x_c - Y(\sin\psi + \sin\gamma\,\cos\psi) \\ &= x_c - Y[\sin(\psi+\gamma) + (1-\cos\psi)\sin\psi] \\ &= x_c - Y\sin\theta \end{aligned} \tag{1}$$

$$\begin{aligned} y &= y_c + Y(\cos\psi - \sin\gamma\,\sin\psi) \\ &= y_c + Y[\cos(\psi+\gamma) + (1-\cos\gamma)\cos\psi] \\ &= y_c + Y\cos\theta \end{aligned} \tag{2}$$

where, ψ is total rotation of the cross section, q is the rotation of the cross section, γ is the shear distortion, x_c, y_c coordinates of C point, $x_c = X + u_{XC}$ and $y_c = u_{XC}$. Consequently, $x = X + u_{XC} - Y \sin q$ and $y = u_{YC} + Y \cos q$. From now on, we shall call u_{XC} and u_{YC} simply u_X and u_Y, respectively. Thus the Lagrangian representation of the coordinates of the generic point at C is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X + u_X - Y\sin\theta \\ u_Y + Y\cos\theta \end{bmatrix} \tag{3}$$

in which u_X, u_Y and θ are functions of X only. This concludes the reduction to a one-dimensional model, as sketched in Fig. 3(b). For a two-node C_0 element, it is natural to express the displacements and rotation as linear functions of the node degrees

$$\begin{aligned} w &= \begin{bmatrix} u_X(X) \\ u_Y(X) \\ \theta(X) \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1-\xi & 0 & 0 & 1+\xi & 0 & 0 \\ 0 & 1-\xi & 0 & 0 & 1+\xi & 0 \\ 0 & 0 & 1-\xi & 0 & 0 & 1+\xi \end{bmatrix}\begin{bmatrix} u_{X1} \\ u_{Y1} \\ \theta_1 \\ u_{X2} \\ u_{Y2} \\ \theta_2 \end{bmatrix} \\ &= \mathbf{Nu} \end{aligned} \tag{4}$$

in which $x = (2X/L_0) - 1$ is the isoparametric coordinate that varies from $x = -1$ at node 1 to $x = 1$ at node 2. The Green-Lagrange strains are given as follows Felippa (2017)

$$\begin{aligned} [\boldsymbol{e}] &= \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} e_{XX} \\ 2e_{XY} \end{bmatrix} \\ &= \begin{bmatrix} (1+u_X')\cos\theta + u_Y'\sin\theta - Y\theta' - 1 \\ 2e_{XY} \end{bmatrix} = \begin{bmatrix} e - Y\kappa \\ \gamma \end{bmatrix} \end{aligned} \tag{5}$$

$$e = (1+u_x')\cos\theta + u_y'\sin\theta - 1 \tag{6a}$$

$$\gamma = -1 + u_x'\sin\theta + u_Y'\sin\theta - 1, \qquad \kappa = \theta' \tag{6b}$$

where e is the axial strain, g is the shear strain, and k is curvature of the beam, $u'_X = \mathrm{d}u_X/\mathrm{d}X$, $u'_Y = \mathrm{d}u_Y/\mathrm{d}X$, $q' = \mathrm{d}q/\mathrm{d}X$. The equivalent Young's modulus of kth layer in the x direction (E_x^k) is used the following formulation (Vinson and Sierakowski 2002)

$$\begin{aligned} \frac{1}{E_x^k} = \frac{\cos^4(\theta_k)}{E_{11}} + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_{11}}\right)\cos^2(\theta_k)\sin^2(\theta_k) \\ + \frac{\sin^4(\theta_k)}{E_{22}} \end{aligned} \tag{7}$$

where, E_{11} and E_{22} indicate the Young's modulus in the longitudinal and transverse directions, respectively, G_{12} and ν_{12} are shear modulus and Poisson ratio, respectively. $m = \cos\theta$ and $n = \sin\theta$, θ indicates the fiber orientation angle.

By assuming that the material of the laminated composite beam obeys Hooke's law, the axial force N, shear force V and bending moment M are given as follows

$$N = A_{11}\, e + B_{11}\, \mathrm{k} \tag{8a}$$

$$V = A_{55}\, \mathrm{g} \tag{8b}$$

$$M = B_{11}\, e + D_{11}\, \mathrm{k} \tag{8c}$$

where A_{11}, B_{11}, D_{11} and A_{55} are the extensional, coupling, bending, and transverse shear rigidities respectively, and their expressions are defined as

$$A_{11} = \sum_{k=1}^{n} bE_x^k (z_{k+1} - z_k\ ) \tag{9a}$$

$$B_{11} = \frac{1}{2}\sum_{k=1}^{n} bE_x^k (z_{k+1}^2 - z_k^2) \tag{9b}$$

$$D_{11} = \frac{1}{3}\sum_{k=1}^{n} bE_x^k (z_{k+1}^3 - z_k^3) \tag{9c}$$

Expression of the transverse shear rigidity A_{55} given as follows (Vinson and Sierakowski 2002)

$$A_{55} = \frac{5}{4}\sum_{k=1}^{n} bQ_{55}^k (z_{k+1} - z_k\ - \frac{4}{3h^2}(z_{k+1}^3 - z_k^3)) \tag{10}$$

where Q_{55}^k is given below

$$Q_{55}^k = G_{13}cos^2(\theta_k) + G_{23}sin^2(\theta_k) \tag{11}$$

For the solution of the geometrically nonlinear problem in the total Lagrangian coordinates, a small-step incremental approach based on Newton-Raphson iteration method is used. In the Newton-Raphson solution for the problem, the applied load is divided by a suitable number of increments according to its value. After completing an iteration process, the previous accumulated load is increased by a load increment. The solution for the $n+1$ st load increment and ith iteration is performed using the following relation

$$d\,\mathbf{u}_n^i = (\mathbf{K}_T^i)^{-1}\mathbf{R}_{n+1}^i \tag{12}$$

where $\mathbf{K}_T^i$ is the tangent stiffness matrix of the system at the i th iteration, $d\mathbf{u}_n^i$ is the displacement increment vector at the i th iteration and n+1 st load increment, $(R_{n+1}^i)_S$ is the residual vector of the system at the i th iteration and n+1 st load increment. This iteration procedure is continued until the difference between two successive solution vectors is less than a preset tolerance in the Euclidean norm, given by

$$\sqrt{\frac{[(d\boldsymbol{u}_n^{i+1} - d\boldsymbol{u}_n^i)^T(d\boldsymbol{u}_n^{i+1} - d\boldsymbol{u}_n^i)]^2}{[(d\boldsymbol{u}_n^{i+1})^T(d\boldsymbol{u}_n^{i+1})]^2}} \le \xi_{tol} \tag{13}$$

A series of successive iterations at the n+1 st incremental step gives

$$\boldsymbol{u}_{n+1}^{i+1}\ \boldsymbol{u}_{n+1}^i + d\boldsymbol{u}_{n+1}^i = \boldsymbol{u}_n + \Delta\boldsymbol{u}_n^i \tag{14}$$

where

$$\Delta\boldsymbol{u}_n^i = \sum_{k=1}^{i} d\boldsymbol{u}_n^k \tag{15}$$

The residual vector $\mathbf{R}_{n+1}^i$ for the structural system is given as follows

$$\mathbf{R}_{n+1}^i = \mathbf{f} - \mathbf{p} \tag{16}$$

Where $\mathbf{f}$ is the vector of total external forces and $\mathbf{p}$ is the vector of total internal forces, as given in the appendix. The element tangent stiffness matrix for the total Lagrangian Timoshenko beam element as given (Felippa 2017) is

$$\mathbf{K}_T = \mathbf{K}_M + \mathbf{K}_G \tag{17}$$

where $\mathbf{K}_G$ is the geometric stiffness matrix, and $\mathbf{K}_M$ is the material stiffness matrix given as follows

$$\mathbf{K}_M = \int_{L_0} \mathbf{B}_m^T \mathbf{S} \mathbf{B}_m \,\mathrm{d}X \tag{18}$$

The explicit expressions of the terms in Eq. (17) are given in the appendix. After integration of Eq. (18), the matrix $\mathbf{K}_M$ can be expressed as follows

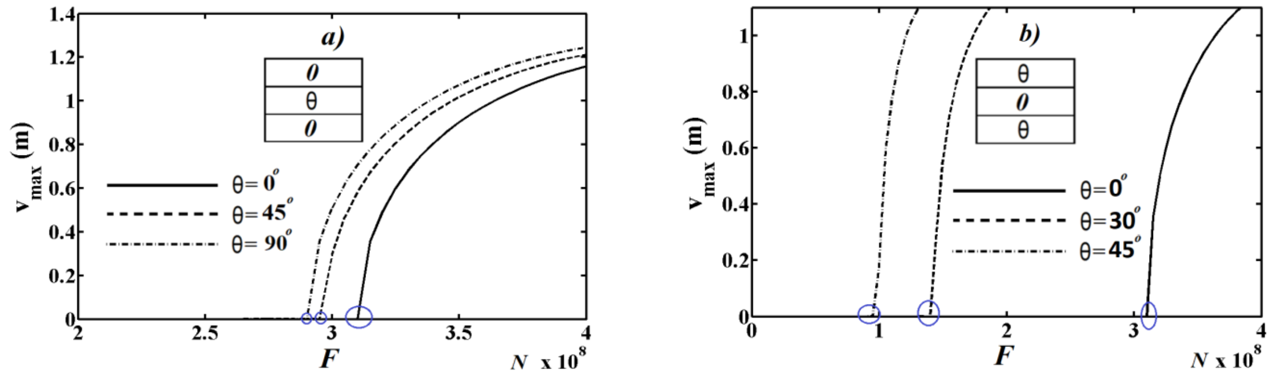
$$\mathbf{K}_M = \mathbf{K}_M^a + \mathbf{K}_M^c + \mathbf{K}_M^b + \mathbf{K}_M^s \tag{19}$$

where $\mathbf{K}_M^a$ is the axial stiffness matrix, $\mathbf{K}_M^c$ the coupling stiffness matrix, $\mathbf{K}_M^b$ the bending stiffness matrix, and $\mathbf{K}_M^s$ the shearing stiffness matrix, of which the explicit expressions are given in the Appendix.

3. Numerical results

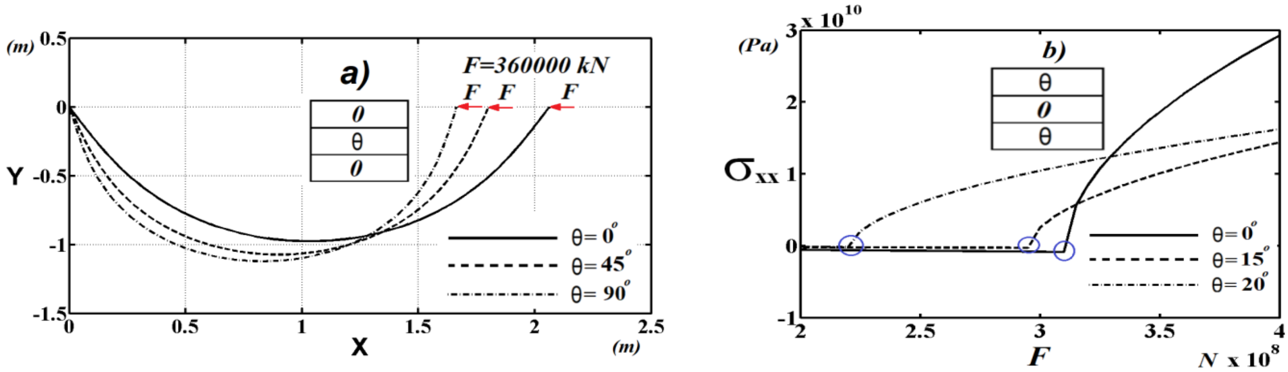
In the numerical study, post-buckling deflections, configurations and stresses of the simply supported laminated beam are calculated and presented for different fibber orientation angles and the stacking sequence of laminates under non-follower compressive point load (F) at the end of the beam (Fig. 1). Using the conventional assembly procedure for the finite elements, the tangent stiffness matrix and the residual vector are obtained from the element stiffness matrices and residual vectors in the total Lagrangian sense for finite element model of the laminated Timoshenko beams. After that, the solution process outlined in the preceding section is used to obtain the solution for the problem of concern. In ob00taining the numerical results, graphs and solution of the nonlinear finite element model, MATLAB program is used.

Numerical calculations of the integrals seen in the rigidity matrices will be performed by using five-point Gauss rule. In the numerical examples, the material



(a) For the stacking sequence [0/θ/0] (b) For the stacking sequence [θ/0/θ]

Fig. 4 Compressive Load (F)- maximum displacements ($v_{\max}$) curves for different values of the fiber orientation angles (θ)



(a) For the stacking sequence [0/θ/0] (b) For the stacking sequence [θ/0/θ]

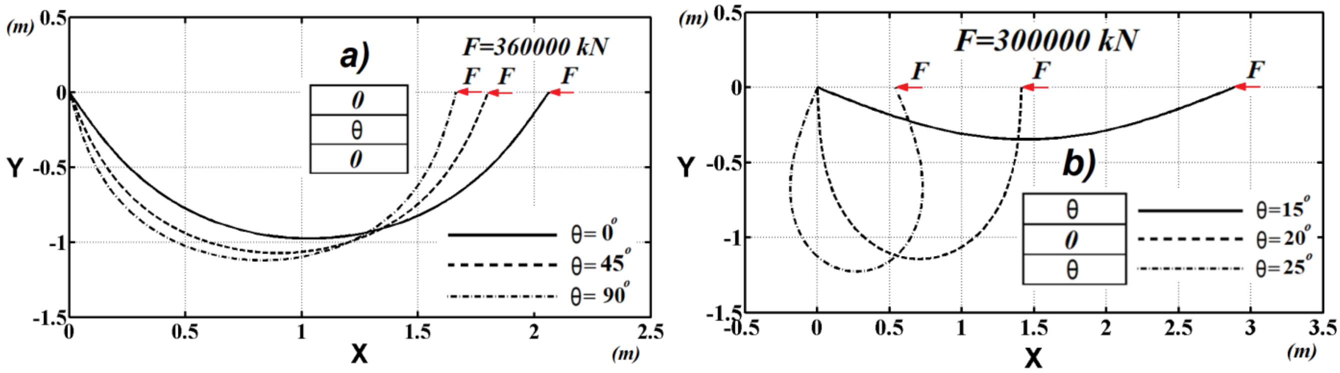
Fig. 5 Compressive Load (F)- Cauchy normal stresses (σ_{xx}) at (X=L/2 and Y = -0.5h) curves for different values of the fiber orientation angles (θ)

properties of the layers are used in Loja *et al*. (2001): E_1 = 129.207 GPa.

E_2 = E_3 = 9.42512 GPa, G_{12} = 5.15658 GPa, G_{13} = 4.3053 GPa, G_{23} = 2.5414 GPa, ν_{12} = ν_{13} = 0.3, ν_{23} = 0.218837. The geometry properties of the beam are considered as follows: b = 0.3 m, h = 0.3 m and L = 3 m. It is mentioned before that the thickness of layers is equal to each other. The number of finite elements is taken as 100 in the numerical calculations.

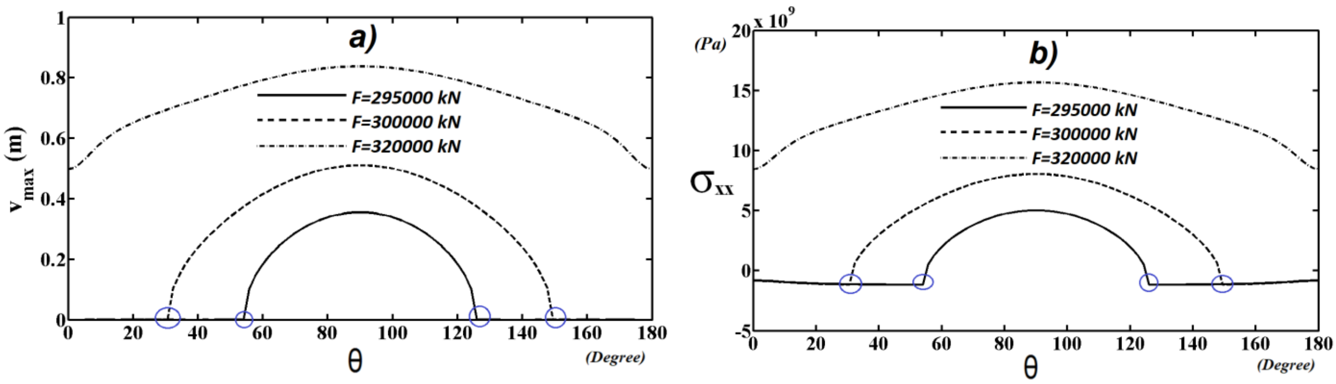
In Fig. 4, the maximum vertical displacements (at the midpoint of the beam) versus compressive load (F) rising are presented for different values of the fiber orientation angles (θ) for the stacking sequences [0/θ/0] and [θ/0/θ]. In Fig. 4, furcation points can be seen (see circle). It is known that buckling case occurs at the furcation points. So, these points give the value of critical buckling loads. It is seen figure 4 that with increase in load, the displacements of the laminated beam converge. It is seen from Fig. 4 that increase in the fiber orientation angle (θ) causes a decrease in the critical buckling loading (see furcation points) in both [0/θ/0] and [θ/0/θ] because the equivalent Young's modulus and bending rigidity decrease according to the Eq. (7). As a result, the strength of the material decreases and the critical buckling load decreases naturally. It is observed from figure 4 that the critical buckling loads in [θ/0/θ] are smaller than [0/θ/0]'s. The post-buckling responses in the [θ/0/θ] are very sensitive and the post-buckling displacements. Critical buckling loads change quickly with increasing the fiber orientation angles in contrast with [θ/0/θ]. It is shows that the stacking sequence plays very important role on the post-buckling responses of the laminated beams.

In order to investigate the effect of the fiber orientation angles on the stresses on the post-bucking case, the Cauchy normal stresses (σ_{xx}) at the midpoint of the beam (X = L/2 and Y = -0.5h) are obtained and illustrated versus compressive load rising (F) in Fig. 5 for the stacking sequences [0/θ/0] and [θ/0/θ]. As seen from Fig. 5 that, the Cauchy normal stresses increase with increasing the fiber orientation angles. Increase in load causes decrease in difference among the fiber orientation angles in stress results for the stacking sequence [0/θ/0]. However, difference among the fiber orientation angles in stress increase considerably with increase the loads for the stacking sequence [θ/0/θ]. It shows that, the stacking sequence is very effective for changes the stresses in the post-buckling stage. Also, it is seen from Fig. 5 that, the normal stresses increase suddenly in the furcation points (see circle), namely critical buckling loads, as like as the load- displacement curve (Fig. 4). Before critical buckling loads, the normal stresses are increase linearly and have



(a) For the stacking sequence [0/θ/0] (b) For the stacking sequence [θ/0/θ]

Fig. 6 Post-buckling configuration of the laminated beam for different values of the fiber orientation angles (θ)



(a) For maximum displacements (b) For Cauchy normal stresses

Fig. 7 The relationship between fiber orientation angles and post-buckling responses of the laminated beam

negative values. However, the normal stresses are increase nonlinearly after critical buckling loads and have great values in the post-buckling stage. After the critical buckling loads, the mechanical responses of the laminated beam seriously change in contrast with pre-buckling stage.

Fig. 6 displays the effect of the fiber orientation angles on the post-buckling configuration of the beam for the constant compressive load (F) for the stacking sequences [0/θ/0] and [θ/0/θ]. It is observed from Fig. 6 that the displacement shapes of the laminated beams change significantly with change the fiber orientation angles. With the increase in the fiber orientation angles, the displacement increase significantly as compressive load is constant.

In Fig. 7, the maximum vertical displacements and Cauchy normal stresses versus the fiber orientation angles rising are presented for different compressive loads values for the stacking sequence [0/θ/0].

As seen from Fig. 7 that the fiber orientation angles have a great influence on the buckling and post-buckling behaviour of the laminated beam. Decreasing the fiber orientation angles to 90° from 0° or to 90° from 180°, the critical buckling loads (see circle) increase seriously. Increasing the fiber orientation angles, the displacements and stresses increase significantly. It can be concluded from here: to choice suitable the fiber orientation angles is very important for safe design of laminated composite structures.

4. Conclusions

Post-buckling responses of a simply supported laminated composite beam is investigated by using total Lagrangian finite element model with the Timoshenko beam theory in conjunction full geometric non-linearity. The considered non-linear problem is solved by using incremental displacement-based finite element method in conjunction with Newton-Raphson iteration method. The effects of the fibber orientation angles and the stacking sequence of laminates on post-buckling responses of the laminated beam are examined and discussed. The shortcomings of this study, the material nonlinearity and elasto-plastic behavior are not considered. It would be interesting to demonstrate the ability of the procedure through a wider campaign of investigations concerning elasto-plastic or material nonlinear analysis of laminated composite beams with geometrically nonlinearity. It is observed from the results that the fibber orientation angles and the stacking sequence have great influences on the post-buckling behaviour of the laminated composite beams. The fibber orientation angles is very effective to change the critical buckling loads and the post-buckling responses. The stacking sequence is very effective to change the stress in the post-buckling stage.

As seen from results that for learn about more realistic post-buckling behaviour of the laminated composite beams in higher loads, full the geometrically non-linear model must be considered. The advantage of the finite element method to the other methods is that in the finite element method, nonlinear post-buckling problems of composite structures can be taken into consideration without any difficulty.

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CC

Appendix

In this appendix, the entries of the following matrices are given: axial stiffness matrix $\mathbf{K}_M^a$, coupling stiffness matrix $\mathbf{K}_M^c$, bending stiffness matrix $\mathbf{K}_M^b$, and shearing stiffness matrix $\mathbf{K}_M^s$ for the laminated composite material Felippa (2017).

$$\boldsymbol{K}_M^a = \frac{\mathrm{A}_{11}}{L_0}\begin{bmatrix} c_m^2 & c_m s_m & \frac{-c_m\gamma_m L_0}{2} & -c_m^2 & -c_m s_m & \frac{-c_m\gamma_m L_0}{2} \\ c_m s_m & s_m^2 & \frac{-s_m\gamma_m L_0}{2} & -c_m s_m & -s_m^2 & \frac{-s_m\gamma_m L_0}{2} \\ \frac{-c_m\gamma_m L_0}{2} & \frac{-s_m\gamma_m L_0}{2} & \frac{\gamma_m^2 L_0^2}{4} & \frac{c_m\gamma_m L_0}{2} & \frac{-s_m\gamma_m L_0}{2} & \frac{\gamma_m^2 L_0^2}{4} \\ -c_m^2 & -c_m s_m & \frac{c_m\gamma_m L_0}{2} & c_m^2 & c_m s_m & \frac{c_m\gamma_m L_0}{2} \\ -c_m s_m & -s_m^2 & \frac{s_m\gamma_m L_0}{2} & c_m s_m & s_m^2 & \frac{s_m\gamma_m L_0}{2} \\ \frac{c_m\gamma_m L_0}{2} & \frac{-s_m\gamma_m L_0}{2} & \frac{\gamma_m^2 L_0^2}{4} & \frac{c_m\gamma_m L_0}{2} & \frac{s_m\gamma_m L_0}{2} & \frac{\gamma_m^2 L_0^2}{4} \end{bmatrix} \tag{A1}$$

$$\boldsymbol{K}_M^c = \frac{\mathrm{B}_{11}}{L_0}\begin{bmatrix} 0 & 0 & -c_m & 0 & 0 & c_m \\ 0 & 0 & -s_m & 0 & 0 & s_m \\ -c_m & -s_m & \gamma_m L_0 & c_m & s_m & 0 \\ 0 & 0 & c_m & 0 & 0 & -c_m \\ 0 & 0 & s_m & 0 & 0 & -s_m \\ c_m & s_m & 0 & -c_m & -s_m & -\gamma_m L_0 \end{bmatrix} \tag{A2}$$

$$\boldsymbol{K}_M^b = \frac{\mathrm{D}_{11}}{L_0}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \tag{A3}$$

$$\boldsymbol{K}_M^a = \frac{\mathrm{A}_{55}}{L_0}\begin{bmatrix} s_m^2 & -c_m s_m & -s_m\alpha_1 L_0/2 & -s_m^2 & c_m s_m & -s_m\alpha_1 L_0/2 \\ -c_m s_m & c_m^2 & c_m\alpha_1 L_0/2 & c_m s_m & -c_m^2 & c_m\alpha_1 L_0/2 \\ -s_m\alpha_1 L_0/2 & c_m\alpha_1 L_0/2 & \alpha_1^2 L_0^2/4 & s_m\alpha_1 L_0/2 & -c_m\alpha_m L_0/2 & \alpha_1^2 L_0^2/4 \\ -s_m^2 & c_m s_m & s_m\alpha_1 L_0/2 & s_m^2 & -c_m s_m & s_m\alpha_1 L_0/2 \\ c_m s_m & -c_m^2 & -c_m\alpha_1 L_0/2 & -c_m s_m & c_m^2 & -c_m\alpha_1 L_0/2 \\ -s_m\alpha_1 L_0/2 & c_m\alpha_1 L_0/2 & \alpha_1^2 L_0^2/4 & s_m\alpha_1 L_0/2 & -c_m\alpha_1 L_0/2 & \alpha_1^2 L_0^2/4 \end{bmatrix} \tag{A4}$$

where m denotes the midpoint of the beam, $x = 0$, and $q_m = (q_1 + q_2)/2$, $w_m = q_m + j$, $c_m = \cos w_m$, $s_m = \sin w_m$, $e_m = L\cos(q_m - y)/L_0 - 1$, $a_1 = 1 + e_m$ and $g_m = L\sin(y - q_m)L_0$ (See Fig. A1 for symbols). The initial axis of the beam considered is taken as horizontal, therefore $j = 0$. The matrix

S is defined as follows

$$\boldsymbol{S} = \begin{bmatrix} \mathrm{A}_{11} & 0 & -\mathrm{B}_{11} \\ 0 & \mathrm{A}_{15} & 0 \\ -\mathrm{B}_{11} & 0 & \mathrm{D}_{11} \end{bmatrix} \tag{A5}$$

The matrix $\mathbf{B}_m$ is given as follows

$$\boldsymbol{B_m} = \boldsymbol{B_m}|_{\xi=0} = \frac{1}{L_0}\begin{bmatrix} -c_m & -s_m & -\frac{1}{2}L_0\gamma_m & c_m & s_m & -\frac{1}{2}L_0\gamma_m \\ s_m & -c_m & \frac{1}{2}L_0(1+e_m) & s_m & -c_m & \frac{1}{2}L_0(1+e_m) \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \tag{A6}$$

The geometric stiffness matrix K_G is given as follows

$$\boldsymbol{K}_G = \frac{\mathrm{N_m}}{2}\begin{bmatrix} 0 & 0 & s_m & 0 & 0 & s_m \\ 0 & 0 & -c_m & 0 & 0 & -c_m \\ s_m & -c_m & -\frac{1}{2}L_0(1+e_m) & -s_m & c_m & -\frac{1}{2}L_0(1+e_m) \\ 0 & 0 & -s_m & 0 & 0 & -s_m \\ 0 & 0 & c_m & 0 & 0 & c_m \\ s_m & -c_m & -\frac{1}{2}L_0(1+e_m) & -s_m & c_m & -\frac{1}{2}L_0(1+e_m) \end{bmatrix} + \frac{\mathrm{N_m}}{2}\begin{bmatrix} 0 & 0 & c_m & 0 & 0 & c_m \\ 0 & 0 & s_m & 0 & 0 & s_m \\ c_m & s_m & -\frac{1}{2}L_0\gamma_m & -c_m & -s_m & -\frac{1}{2}L_0\gamma_m \\ 0 & 0 & -c_m & 0 & 0 & -c_m \\ 0 & 0 & -s_m & 0 & 0 & -s_m \\ c_m & s_m & -\frac{1}{2}L_0\gamma_m & -c_m & -s_m & -\frac{1}{2}L_0\gamma_m \end{bmatrix} \tag{A7}$$

in which N_m and V_m are the axial and shear forces evaluated at the midpoint. The internal nodal force vector is Felippa (2017)

$$\boldsymbol{p} = L_0\boldsymbol{B_m^T z} = \begin{bmatrix} -c_m & -s_m & -\frac{1}{2}L_0\gamma_m & c_m & s_m & \frac{1}{2}L_0\gamma_m \\ s_m & -c_m & -\frac{1}{2}L_0(1+e_m) & s_m & -c_m & -\frac{1}{2}L_0(1+e_m) \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} N \\ V \\ M \end{bmatrix} \tag{A8}$$

where $\boldsymbol{z^T} = [N \quad V \quad M]$. The external nodal force vector is

$$\boldsymbol{z^T} = \mathrm{b}\int_{\mathrm{h}}\int_{L_0}\begin{bmatrix} 1-\xi_1 & 0 & 0 \\ 0 & 1-\xi_1 & 0 \\ 0 & 0 & 1-\xi_1 \\ 1-\xi_2 & 0 & 0 \\ 0 & 1-\xi_2 & 0 \\ 0 & 0 & 1-\xi_2 \end{bmatrix}\begin{bmatrix} f_X \\ f_Y \\ 0 \end{bmatrix} dXdY + b\int_{L_0}\int_{\mathrm{h}}\int_{L_0}\begin{bmatrix} 1-\xi_1 & 0 & 0 \\ 0 & 1-\xi_1 & 0 \\ 0 & 0 & 1-\xi_1 \\ 1-\xi_2 & 0 & 0 \\ 0 & 1-\xi_2 & 0 \\ 0 & 0 & 1-\xi_2 \end{bmatrix}\begin{bmatrix} t_X \\ t_Y \\ mt_Y \end{bmatrix} dX \tag{A9}$$

where f_X, f_Y are the body forces, t_X, t_Y, m_z are the surface loads in the X, Y directions and about the Z axis.

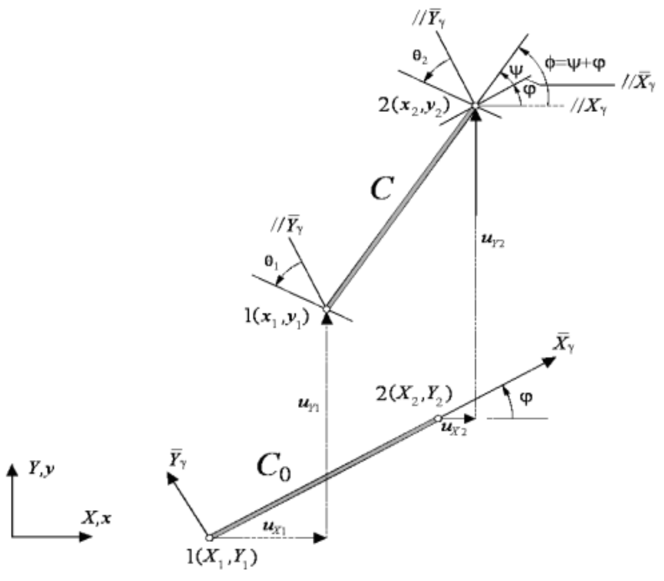

Fig. A1 Plane beam element with arbitrarily oriented reference configuration Felippa (2017)