Steel and Composite Structures, *Vol. 26, No. 6 (2018) 663-672*
DOI: https://doi.org/10.12989/scs.2018.26.6.663

Buckling analysis of FGM Euler-Bernoulli nano-beams with 3D-varying properties based on consistent couple-stress theory

Amin Hadi [1], Mohammad Zamani Nejad [*2], Abbas Rastgoo [1], and Mohammad Hosseini [3]

[1] *School of Mechanical Engineering, University of Tehran, Tehran, Iran*
[2] *Department of Mechanical Engineering, Yasouj University, P.O. Box: 75914-353, Yasouj, Iran*
[3] *Department of Mechanical Engineering, Shahid Chamran University of Ahvaz, Ahvaz, Iran*

(*Received August 7, 2017, Revised December 21, 2017, Accepted December 28, 2017*)

Abstract. This paper contains a consistent couple-stress theory to capture size effects in Euler-Bernoulli nano-beams made of three-directional functionally graded materials (TDFGMs). These models can degenerate into the classical models if the material length scale parameter is taken to be zero. In this theory, the couple-stress tensor is skew-symmetric and energy conjugate to the skew-symmetric part of the rotation gradients as the curvature tensor. The material properties except Poisson's ratio are assumed to be graded in all three axial, thickness and width directions, which it can vary according to an arbitrary function. The governing equations are obtained using the concept of minimum potential energy. Generalized differential quadrature method (GDQM) is used to solve the governing equations for various boundary conditions to obtain the natural frequencies of TDFG nano-beam. At the end, some numerical results are performed to investigate some effective parameter on buckling load. In this theory the couple-stress tensor is skew-symmetric and energy conjugate to the skew-symmetric part of the rotation gradients as the curvature tensor.

Keywords: Euler-Bernoulli nano-beams; buckling analysis; consistent couple-stress theory; Three-directional functionally graded materials (TDFGMs); size effect; generalized differential quadrature method (GDQM)

1. Introduction

Functionally graded material (FGM) is one of latest concept in the composite material design field. The material properties of functionally graded material continuously vary from point to another point. In other words, material properties are functions of location. The use of FG materials reduces the weight and increases the strength of structures. A number of papers considering various aspects of FGM have been published in recent years (Nejad and Rahimi 2009, Nejad *et al*. 2009, 2014a, b, 2015a, b, 2016a, b, 2017a, b, c, Nejad and Rahimi 2010, Ghannad *et al*. 2012, 2013, Nejad *et al*. 2013, Fatehi and Nejad 2014, Nejad and Fatehi 2015, Jabbari *et al*. 2015, Mazarei *et al*. 2016a, Nejad and Hadi 2016b, Jabbari *et al*. 2016, Afshin *et al*. 2017, Sadrabadi *et al*. 2017, Burlayenko *et al*. 2017, Civalek 2017, Gharibi *et al*. 2017, Kashkoli *et al*. 2017, Najibi and Talebitooti 2017, Şimşek and Al-shujairi 2017, Taczała *et al*. 2017, Wang and Zu 2017). It should be noted that most of the above-mentioned analyses are related to FGMs with material properties varying in one direction only. However, there are practical occasions which require tailored grading of properties in two or even three directions. As reported by Steinberg (Steinberg 1986), the fuselage of an aerospace craft undergoes an extremely high temperature field with excessive temperature gradient on the surface and through the thickness, when the plane sustains flight at a speed of Mach 8 and at an altitude of 29 km. In this circumstance, the conventional unidirectional FGMs may not be so appropriate to resist multi-directional severe variations of temperature. Therefore, it is of great significance to develop novel FGMs with properties varying in two or three directions (2D or 3D FGMs) to withstand a more general temperature field (Lü *et al*. 2008). Thanks to the advances in technology, FGMs have started to find their ways into micro-nano-electro-mechanical systems (MEMS/NEMS) (Apuzzo *et al*. 2017, Belkorissat *et al*. 2015, Bounouara *et al*. 2016, Ebrahimi and Barati 2016a, c, 2017, Ebrahimi *et al*. 2017, Goodarzi *et al*. 2017, Hosseini *et al*. 2016, Li and Hu 2016, 2017a, Li *et al*. 2016, 2017, Nguyen *et al*. 2014, Rahmani *et al*. 2017, Sahmani and Aghdam 2017, Shishesaz *et al*. 2017).

At nano and micro meter scales, size effects often become important. Both experimental and Molecular dynamics simulation results have shown that the small-scale effects in the analysis of mechanical properties of nano and micro structures cannot be neglected and classical continuum theories is not usable. Molecular dynamics simulation is convenient method to simulate the mechanical behavior of small size structures but it is computationally expensive for structures with large number of atoms (Gopalakrishnan and Narendar 2013, Keivani *et al*. 2016). Thus researchers stimulated to develop several higher-order continuum theories (Eringen 1972a, b, 1983, 2002, Lam *et al*. 2003, Mindlin and Tiersten 1962, Toupin 1962, Yang *et al*. 2002). Recently, by considering true continuum

∗Corresponding author, Ph.D., Associate Professor,
E-mail: m_zamani@yu.ac.ir; m.zamani.n@gmail.com

http://www.techno-press.org/?journal=scs&subpage=6
ISSN: 1229-9367 (Print), 1598-6233 (Online)

kinematical displacement and rotation, Hajesfandiari and Dargush (2011) demonstrate that the couple-stress tensor is skew-symmetric and is energy conjugate to the skew-symmetric part of the rotation gradients as the curvature tensor.

Shafiei *et al*. (2017) investigate the buckling behavior of axially functionally grade nanobeams. Nonlocal elasticity and Euler-Bernoulli theories are used. Different boundary conditions are considered for micro/nanobeam. Nonlinear equations of nanobeam are solved by using of generalized differential quadrature method. The effect of some parameters such as temperature and nonlocal parameter on the nonlinear buckling of FG nanobeam are investigated. Ebrahimi and Barati (Ebrahimi and Barati 2017) suggested a model for buckling behavior of curved functionally graded nanobeams. Their model is based on the nonlocal strain gradient theory. Material properties of nanomeans vary according to power-law distribution. In other work (Ebrahimi and Barati 2016b), on the basis of Third-Order shear deformable beam, they investigated buckling behaviors of size-dependent through the thickness functionally graded nanobeams under thermal loading. Hamilton's principle is used to drive the governing equation of nanomeams. Post-buckling behavior of through the thickness functionally graded nanobeams based on the nonlocal strain gradient theory is studied by Li and Hu (2017a). They considered the von Karman geometry nonlinearity. It is found that the scaling parameters have a significant effect on the stiffness-hardening or stiffness-softening behavior. Ansari *et al*. (2016) investigate size-dependent free vibration of post buckled functionally graded micro/nanobeams based on the strain gradient and shear deformation beam theories. Hamilton's principle was used to obtain the nonlinear governing equation and associated boundary conditions. These equations were solved using generalized differential quadrature method. Results show the effect of small-scale parameters, material gradient parameter and boundary conditions on the frequency response and post-buckling behavior of functionally graded nanobeams. Adeli *et al*. (2017) studied torsional vibration of nano-cone. They used nonlocal strain gradient theory to capture size-dependent behavior of this nano-structure. Hosseini *et al*. (2017) investigated stress distribution of nanotubes under internal pressure using strain gradient theory. Yu *et al*. (2016) obtained the critical buckling loads of Euler-Bernoulli nanobeams under thermal load with different boundary condition based on the nonlocal elasticity theory. They discussed about the effects of nonlocal parameter on the buckling behavior. Nejad *et al*. (Nejad and Hadi 2016a, b, Nejad *et al*. 2016a) analyzed the buckling, vibration and bending behavior of two directional functionally graded nanobeams, respectively. Minimum potential energy principle, Euler-Bernoulli beam and nonlocal elasticity theories are adopted to drive the fundamental equations. It is considered that the material properties of functionally graded nanobeams vary in the length and thickness direction. Differential equations were solved using generalized differential quadrature method. Also, the effects of length scale parameter and gradient index were shown. Ebrahimi and Salari (2015) presented a new solution path in order to explain the effect of temperature on the free vibration and buckling behavior of through the thickness functionally graded Timoshenko nanobeams based on the nonlocal elasticity theory. Material changes in accordance power-law distribution. Moreover, Temperature-dependent were considered for nanobeams. Bahrami and Teimourian (2015) were employed nonlocal elasticity theory to investigate the buckling and free vibration behaviors of Euler-Bernoulli nanobeams with different boundary conditions. The influence of nonlocal parameter and preload on the natural frequency and buckling load are explained. Eltaher *et al*. (2014) used Eringen nonlocal elasticity theory to study buckling behavior of nanobeams. Material properties of nanobeams vary in the thickness direction. Minimum energy principle was employed to obtain the equilibrium equations. Chen *et al*. (2014) consider the effect of van der Waals forces on the stability and buckling behavior of a piezoelectric viscoelastic nanobeam. Results indicate the effects of van der Waals force, inner damping and electrostatic load. Sismek and Yurtcu (2013) employed nonlocal elasticity, Timoshenko and Euler-Bernoulli beam theories to analyze static bending and buckling behaviors of through-thickness functionally graded nanobeams. Emam (2013) offered a general model based on the nonlocal elasticity theory to investigate the buckling and post-buckling behavior of Euler-Bernoulli and Timoshenko nanobeams. Results show the reverse relation between critical buckling load and nonlocal parameter. Thai and Vo (2012) introduce a new model on the basis of Eringen nonlocal elasticity theory for bending, buckling and vibration of nanobeams that is able to examine both small scale and shear deformation effects. Results indicate that the effects of small scale parameter on the bending, buckling and vibration behavior of nanobeams are significant. Thai (2012) analyzed the bending, buckling and vibration of nanobeams by introducing a new beam theory based on the nonlocal elasticity theory. His model is capable to consider the shear deformations, while it does not require shear correction factor. Ansari and Sahmani (2011) consider the surface stress effects to analyze bending and buckling behaviors of nanobeams. Aydogdu (2009) studied the buckling, bending and vibration of nanobeams based on the a generalized nonlocal beam theory. Nejad *et al*. (2017a) employed consistent couple-stress theory for free vibration analysis of Euler-Bernoulli nano-beams made of arbitrary bi-directional functionally graded materials. Nonlinear bending of a two-dimensionally functionally graded beam are presented by Li *et al*. (2018). Li and Yu (2017b) presented torsional vibration of bi-directional functionally graded nanotubes based on nonlocal elasticity theory.

To the best of the researchers' knowledge, in this article the buckling analysis of TDFGMs Euler-Bernoulli nano-beams has been presented based on the consistent couple-stress theory for the first time. The effects of changes of some important parameters such as material length scale, FG index on the values of buckling load in different modes are studied. The results of this study can be a reference for designing the elastic types three-directional FGM Euler-Bernoulli nano-beams.

2. Analysis

Consider a nano-beam of length *L*, width *b*, and thickness *h* made of three-directional functionally graded materials (Fig. 1). Cartesian coordinates (*x*, *y*, *z*) are considered.

The modulus of elasticity *E* and density *ρ* are assumed to vary as arbitrary functions in axial, thickness and width directions, as indicated below

$$E(x,y,z)=f(x)g(y)k(z) \tag{1}$$

where *f*(*x*), *g*(*y*) and *k*(*z*) are arbitrary functions.

In the consistent couple-stress theory, the equations of equilibrium of the linear isotropic materials are formulated as Hadjesfandiari and Dargush (2011).

$$\sigma_{ji,j}+f_i=0 \tag{2}$$

$$m_{ji,j}+\varepsilon_{ijk}\sigma_{jk}=0 \tag{3}$$

where σ_{ij} and m_{ji} represent the non-symmetric force-stress and couple-stress tensors, respectively. In addition, f_i and ε_{ijk} denote the body force per unit volume and permutation or Levi-Civita symbol, respectively. As mentioned before, Hadjesfandiari and Dargush (2011) proved that in the couple-stress theory, the body force and body couple are not distinguishable from each other and the body couple transform to the equivalent body force (Hadjesfandiari and Dargush 2011). Moreover, in the couple-stress theory, unlike the classical elasticity, the stress tensor is generally non-symmetric. Thus, it can be decomposed to the symmetric and skew-symmetric components as following

$$\sigma_{ji}=\sigma_{(ji)}+\sigma_{[ji]} \tag{4}$$

where $\sigma_{(ji)}$ is the symmetric part and $\sigma_{[ji]}$ is the skew-symmetric part of the force-stress tensor. In order to define the elements of Eqs. (2)-(4) required in the couple-stress theory, the kinematic parameters should be utilized. The displacement gradient can be decomposed into two distinct parts

$$u_{ij}=e_{ij}+\omega_{ij} \tag{5}$$

where

$$e_{ij}=u_{(i,j)}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right) \tag{6}$$

$$\omega_{ij}=u_{[i,j]}=\frac{1}{2}\left(u_{i,j}-u_{j,i}\right) \tag{7}$$

In the above relations, e_{ij} and ω_{ij} are strain and rotations tensors, respectively. Similar to the couple-stress tensor, the rotation tensor is skew-symmetrical and a vector can be defined dual to it as

$$\omega_i=\frac{1}{2}\varepsilon_{ijk}\omega_{kj} \tag{8}$$

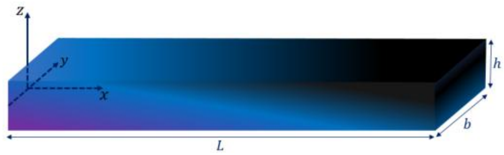

Fig. 1 Geometry of the TDFG Euler-Bernoulli nano-beam

The gradient of rotation tensor can be decomposed into two sub-tensors as

$$\omega_{i,j}=\chi_{ij}+\kappa_{ij} \tag{9}$$

where

$$\chi_{ij}=\omega_{(i,j)}=\frac{1}{2}\left(\omega_{i,j}+\omega_{j,i}\right) \tag{10}$$

$$\kappa_{ij}=\omega_{[i,j]}=\frac{1}{2}\left(\omega_{i,j}-\omega_{j,i}\right) \tag{11}$$

The diagonal arrays of the former known as the torsion tensor show the pure torsion of the element about the coordinate axis and the off-diagonal terms are deviations from sphericity. It does not contribute as a fundamental measure of deformation and will not be included in the strain energy. On the other hand, in the couple-stress theory, the curvature tensor (κ_{ij}) plays a crucial role in the strain energy. The corresponding dual vector of the skew - symmetric curvature tensor can be formulated as

$$\kappa_i=\frac{1}{2}\varepsilon_{ijk}\kappa_{kj} \tag{12}$$

It is now the time of formulating the force and couple-stresses corresponding to the above kinematic parameters. The symmetrical part of the force-stress tensor in Eq. (4) is same as the force-stress tensor in classical elasticity and can be obtained from Eq. (13)

$$\sigma_{(ji)}=\lambda e_{kk}\delta_{ij}+2\mu e_{ij} \tag{13}$$

where λ and μ are the Lame's constants. The couple-stress tensor is skew-symmetrical ($m_{ij}=-m_{ji}$) and a vector m_i can be introduced dual to the tensor.

$$m_i=\frac{1}{2}\varepsilon_{ijk}m_{kj} \tag{14}$$

For the isotropic linear materials, Hadjesfandiari and Dargush (2011) proved that the couple-stress can be computed from Eq. (15).

$$m_i=-8\eta\kappa_i \tag{15}$$

The above relation shows that the couple-stress theory for the isotropic linear materials has only one extra size-dependent parameter. The ratio $\eta=\mu l^2$ is the constant makes difference between the classical and consistent couple-stress theories. The size-dependent parameter, *l*, varies from one material to another or from one scale to another scale. For the zero value of this parameter, the latter reduces to the former.

In addition, Hadjesfandiari and Dargush (2011) showed that the skew-symmetric component of the stress tensor can be obtained from Eq. (16).

$$\sigma_{[ji]} = -m_{[i,j]} \tag{16}$$

According to the consistent couple-stress developed by Hadjesfandiari and Dargush (2011), the strain energy density of an isotropic linear elastic material with volume Ω experiencing an infinitesimal displacement is defined as

$$U = \frac{1}{2}\int_{\Omega}\left(\sigma_{(ji)}e_{ij} + m_{ji}\kappa_{ij}\right)dv \tag{17}$$

Components of displacement vector (u_1, u_2, u_3) for Nano-beams based on Euler-Bernoulli beam theories can be expressed as

$$\begin{cases} u_1 = -z\,(dw/dx) \\ u_2 = 0 \\ u_3 = w\,(x) \end{cases} \tag{18}$$

The underlying assumption in our theory will be that plane sections initially perpendicular to the midsurface will remain plane and perpendicular. Moreover the cross-section is infinitely rigid in its own plane.

Substitution of Eq. (18) into the Eq. (11), the skew-symmetric curvature tensor is expressed as

$$\kappa = \frac{1}{2}\frac{\partial^2 w}{\partial x^2}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{19}$$

From Eq. (15), the couple-stress tensor is defined as follows

$$m = 4\mu l^2 \frac{\partial^2 w}{\partial x^2}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{20}$$

From the displacement field, the strain components can be calculated by substituting Eq. (18) into Eq. (6).

$$e = -z\frac{\partial^2 w}{\partial x^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{21}$$

For a slender beam with a large aspect ratio, the Poisson effect is secondary and can be disregarded to simplify the formulation of the beam theory. Hence, the stress component is presented as

$$\sigma = -Ez\frac{\partial^2 w}{\partial x^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{22}$$

Substituting Eq. (1) and Eqs. (19)-(22) into Eq. (18), the variation of strain energy is simplified to

$$\begin{aligned} \delta U = & \int_0^L \left(I_2 + SI_0 l^2\right)\left(f''\frac{\partial^2 w}{\partial x^2} + 2f'\frac{\partial^3 w}{\partial x^3} + \right. \\ & \left. + f\frac{d^4 w}{dx^4}\right)\delta w dx + f\frac{d^2 w}{dx^2}\frac{d\,\delta w}{dx}\Bigg|_0^L \\ & -\left(f'\frac{d^2 w}{dx^2} + f\frac{d^3 w}{dx^3}\right)\delta w\Bigg|_0^L \end{aligned} \tag{23}$$

where

$$\begin{Bmatrix} I_0 \\ I_2 \end{Bmatrix} = \int_A g\,(y)\,k\,(z)\begin{Bmatrix} 1 \\ z^2 \end{Bmatrix} dA \tag{24}$$

$$S = \frac{2}{1+\nu} \tag{25}$$

The first variation of the work due to, P, the axial compressive force is given by

$$\delta V = -\int_0^L P\frac{dw}{dx}\frac{d\,\delta w}{dx}dx = \int_0^L P\frac{d^2 w}{dx^2}\delta w dx \tag{26}$$

The governing equations of the FGM Euler-Bernoulli beam can be obtained, using the concept of minimum total potential energy principle. According to the minimum total potential energy principle, the first variation of the total potential energy must be zero. That is

$$\delta\Pi = \delta U - \delta V = 0 \tag{27}$$

Substituting Eqs. (23), (26) and $(x) = e^{\frac{n_1}{L}x}$ (n_1: material constant) into Eq. (28), the Navier equation is expressed as

$$\begin{aligned} \left(I_2 + SI_0 l^2\right)e^{\frac{n_1}{L}x}\left(\left(\frac{n_1}{L}\right)^2\frac{\partial^2 w}{\partial x^2} + 2\frac{n_1}{L}\frac{\partial^3 w}{\partial x^3} + \frac{\partial^4 w}{\partial x^4}\right) \\ = P\frac{d^2 w}{dx^2} \end{aligned} \tag{28}$$

For convenience, the following nondimensionalizations are used

$$\bar{w} = \frac{w}{L}, \quad \bar{x} = \frac{x}{L}, \quad \bar{P} = \frac{PL^2}{I_2}, \quad \alpha = \frac{I_0 l^2}{I_2} \tag{29}$$

The non-dimensional governing equation expression can be obtained as

$$\left(1 + S\,\alpha\right)e^{n_1\bar{x}}\left(\frac{d^4\bar{w}}{d\bar{x}^4} + 2n_1\frac{d^3\bar{w}}{d\bar{x}^3} + n_1^2\frac{d^2\bar{w}}{d\bar{x}^2}\right) = \bar{P}\frac{d^2\bar{w}}{d\bar{x}^2} \tag{30}$$

3. Generalized differential quadrature method

In the case of the general boundary conditions, the analytical solution of Eq. (30) is difficult to obtain, so a generalized differential quadrature approach has been adopted for the solution of Eq. (30). The GDQ approach may be an easy and useful tool for the purpose of analyzing more complex problems. The generalized differential quadrature method is an efficient numerical method for the solution of differential equations. It is assumed that the grid points are located on the zeros of the Chebyshev polynomials (Shu and Chew 1998) and to discretize the solution domain, one can assume a set of N grid points in the x-direction

$$X_{\mathrm{i}}=\frac{L}{2}\left\{1-\cos\left(\frac{\mathrm{i}-1}{\mathrm{N}-1}\right)\pi\right\},\quad i=1,\dots,N \tag{31}$$

In this method, the derivatives of a function $h(x)$, at a point x_i are expressed as

$$h_x^{(n)}\left(x_i\right)=\sum_{j=1}^{N}C_{ij}^{(n)}h\left(x_j\right),\qquad n=1,\dots,N-1 \tag{32}$$

where N is the number of the grid points over the x direction. $C_{ij}^{(n)}$ is the respective weighting coefficients through the x direction obtained through the following equations:

If n = 1, i.e., for the first order derivative, then

$$C_{ij}^{(1)}=\frac{M\left(X_i\right)}{\left(X_i-X_j\right)M\left(X_j\right)},\quad i,j=1,\dots,N\qquad j\neq i \tag{33}$$

where

$$M\left(X_i\right)=\prod_{\substack{j=1\\ j\neq i}}^{N_x}\left(X_i-X_j\right) \tag{34}$$

To obtain the weighting coefficients for the second-order or higher-order derivatives, the matrix multiplication procedure is implemented

$$C_{ij}^{(n)}=n\left(C_{ii}^{(n-1)}C_{ij}^{(1)}-\frac{C_{ij}^{(n-1)}}{X_i-X_j}\right),\qquad i,j=1,\dots,N \tag{35}$$

$$C_{ii}^{(n)}=-\sum_{\substack{j=1\\ j\neq i}}^{N_x}C_{ij}^{(n)},\qquad \begin{cases}i=1,\dots,N\\ n=1,2,\dots,N-1\end{cases} \tag{36}$$

Substituting Eq. (32) into the first governing equations (Eq. (30)), the following equation is obtained

$$\begin{aligned}&\left(1+S\alpha\right)e^{n_1\bar{x}_i}\left(\sum_{j=1}^{N}C_{ij}^{(4)}\bar{W}\left(\bar{x}_j\right)+2n_1\sum_{j=1}^{N}C_{ij}^{(3)}\bar{W}\left(\bar{x}_j\right)\right.\\&\left.+n_1^{\,2}\sum_{j=1}^{N}C_{ij}^{(2)}\bar{W}\left(\bar{x}_j\right)\right)=\bar{P}\left(\sum_{j=1}^{N}C_{ij}^{(2)}\bar{W}\left(\bar{x}_j\right)\right)\end{aligned} \tag{37}$$

Then arranging the displacement variable and corresponding coefficient, the governing equations can be obtained in the following form

$$\begin{bmatrix}A_{bb} & A_{bd}\\ A_{db} & A_{dd}\end{bmatrix}\begin{bmatrix}X_b\\ X_d\end{bmatrix}=\left[\bar{P}\right]\begin{bmatrix}0 & 0\\ B_{db} & B_{dd}\end{bmatrix}\begin{bmatrix}X_b\\ X_d\end{bmatrix} \tag{38}$$

in which subscripts b and d denote boundary and domain sample points, respectively. In addition, coefficients A and B are matrices and their dimensions depend on the number of domain and boundary sample points. After eliminating boundary nodes X_b in Eq. (3) by using the boundary conditions, the dimension of the coefficient matrices reduces. Finally, Eq. (38) can be rewritten to give an eigenvalue problem as

$$\left[K\right]\left[X_d\right]=\left[\bar{P}\right]\left[I\right]\left[X_d\right] \tag{39}$$

Solving the obtained eigenvalue problem gives the critical buckling load $(\bar{P})$ of the TDFGM Euler-Bernoulli nano-beams based on consistent couple-stress theory.

4. Results and discussion

In this section based on consistent couple-stress theory, buckling analysis of TDFG Euler-Bernoulli nano-beams are presented for different parameters. In order to illustrate the size effect on behavior of TDFG Euler-Bernoulli nano-beams, several numerical examples have been performed. To validate the validity and reliability of present work, when n_1 and l are neglected, results of this paper with various boundary conditions (S-S: simply supported-simply supported, C-C: clamped-clamped and C-S: clamped-simply supported) at two ends are compared with (Ghannadpour *et al*. 2013, Nejad *et al*. 2016a, Pradhan and Phadikar 2009, Wang *et al*. 2006), as shown in Table 1. It can be seen from this table that, results of this paper can well agree with those obtained using other methods. The material properties of TDFG Euler-Bernoulli nano-beam are shown in Table 2.

It is proposed that the modulus of elasticity of the nano-beam material vary in the x, y and z directions, as follows

$$\begin{aligned}E\left(x,y,z\right)=e^{\frac{n_1}{L}x}&\left[\left(\frac{2y+b}{2b}\right)^{n_2}+k\left(1-\left(\frac{2y+b}{2b}\right)^{n_2}\right)\right]\\&\left[E_c\left(\frac{2z+h}{2h}\right)^{n_3}+E_m\left(1-\left(\frac{2z+h}{2h}\right)^{n_3}\right)\right]\end{aligned} \tag{40}$$

where, k, n_1, n_2, and n_3 are constant material parameters. Fig. 2 illustrate the variation of the modulus of elasticity at (a) plane $\bar{x}=0$; (b) plane $\bar{y}=0.5$ for k = 0.5, $n_1=n_2=n_3$ = 0.5, E_c = 69 GPa, E_m = 339 GPa.

Fig. 3 illustrates the convergence of the GDQM in obtaining the non-dimensional buckling load. It is observed that considering more than 14 sample points does not affect the accuracy of the results significantly. In this figure, the non-dimensional critical buckling load error is defined as

$$e=\left|\frac{\bar{P}_{N+1}-\bar{P}_N}{\bar{P}_N}\right|\times 100 \tag{41}$$

Table 1 Comparison of non-dimensional critical buckling load for a clamped-clamped nano-beam with (Ghannadpour *et al.* 2013, Nejad *et al.* 2016a, Pradhan and Phadikar 2009, Wang *et al.* 2006)

	Boundary conditions		
	S-S	C-S	C-C
Present work	9.8696	20.1907	39.4784
Nejad *et al.* 2016a	9.8696	20.1907	39.4784
Ghannadpour *et al.* 2013	9.8696	20.1907	39.4784
Wang *et al.* 2006	9.8695	20.1907	39.4786
Pradhan and Phadikar 2009	9.8696	20.1907	39.4784

Table 2 Material properties used in the numerical study

Materials	Properties		
	E (GPa)	ρ (kg/m^3)	υ
Ceramic: C	69	2700	0.292
Metal: M	339	3800	0.292

where e is a small value number and in this analysis, it is taken to be 10^{-2}.

Critical buckling ratio is defined as follows

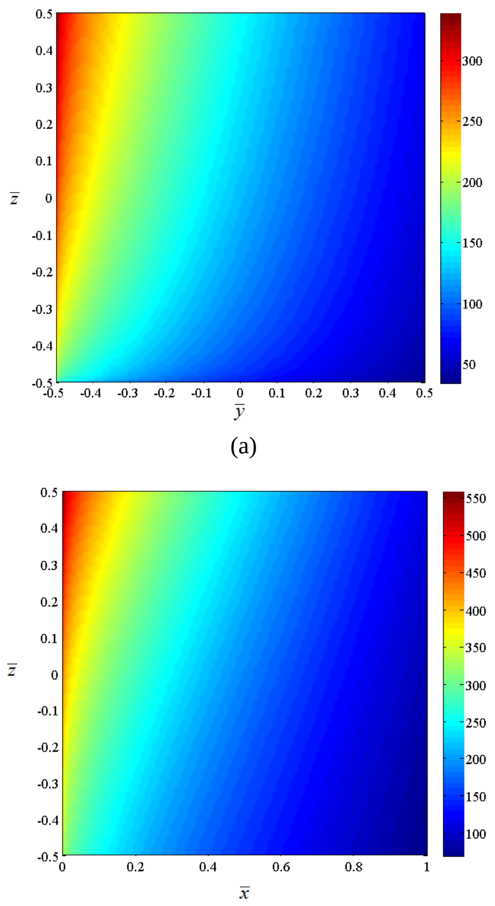

Fig. 2 Distribution of modulus of elasticity for k = 0.5, $n_1 = n_2 = n_3 = 0.5$, E_c = 69 GPa, E_m = 339 GPa at (a) plane $\bar{x} = 0$; (b) plane $\bar{y} = 0.5$

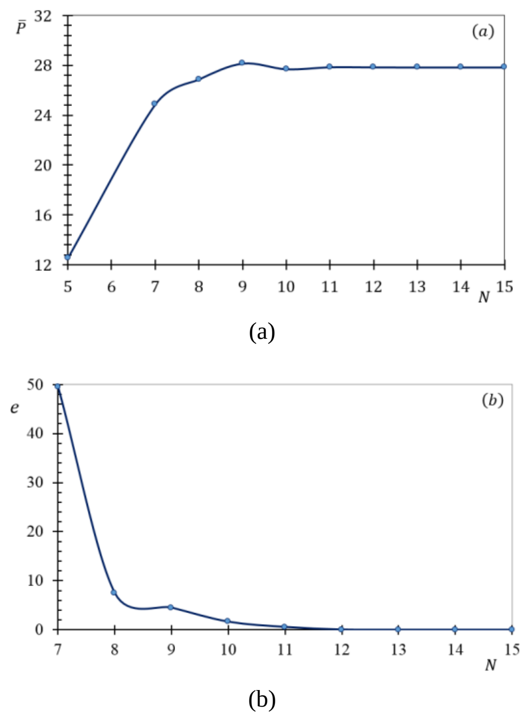

Fig. 3 Convergence of dimensionless buckling load and error in mode 1 S-S TDFG (n_1 = 2, v = 0.292, α = 0.01): (a) Convergence of dimensionless buckling load; (b) error

$$Br = \frac{P}{P_O} \tag{42}$$

In the above relation, P_o is the buckling load when the size scale parameter is taken to be zero. Also it should be noted that when the buckling load ratio approaches 1, size effects are negligible.

Fig. 4 shows the ratio of buckling load in the case of considering couple-stress effect to the classic case in terms of dimensionless thickness, h/l. It can be seen, with increasing the dimensionless thickness, the buckling load ratio tend to 1 which shows that with increasing the thickness against size scale parameter, couple-stress effect decreases. For the dimensionless thickness equal to 1, relative buckling load ratio is equal to 21.0233 which shows the difference between classic and couple-stress theory in small sizes.

Fig. 5 illustrates the dimensionless buckling load against α for the first five modes. This figure shows that as α increases, the dimensionless buckling load increases too. In other words, this figure shows that for higher values of α, size effect increases. Fig. 6 shows the changes of dimension less buckling load versus n_1 with various boundary conditions. Dimension less buckling load increases in all boundary conditions by increasing n_1.

Figs. 7 and 8 illustrate the buckling load of the nano-beam against the n_2 and n_3, respectively. This figure shows that with increases in n_2 and n_3, the buckling load increases for all boundary conditions. Also, it can be conclude that graded in material properties in y and z-directions are a significant effect on buckling analysis of nano structure.

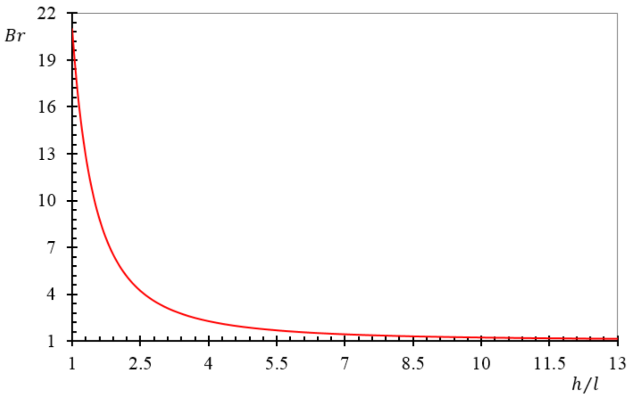

Fig. 4 The buckling load ratio of S-S TDFG nano beam versus to dimensionless thickness (L = 100 nm, $b = h$, $n_1 = 0$, $n_2 = n_3 = 2$, $k = 0.5$, and l = 1 nm)

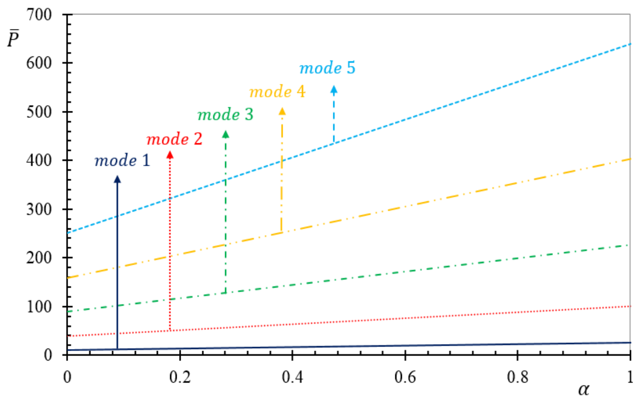

Fig. 5 Dimensionless buckling load of S-S TDFG nano-beam versus α in different modes ($n_1 = 0$)

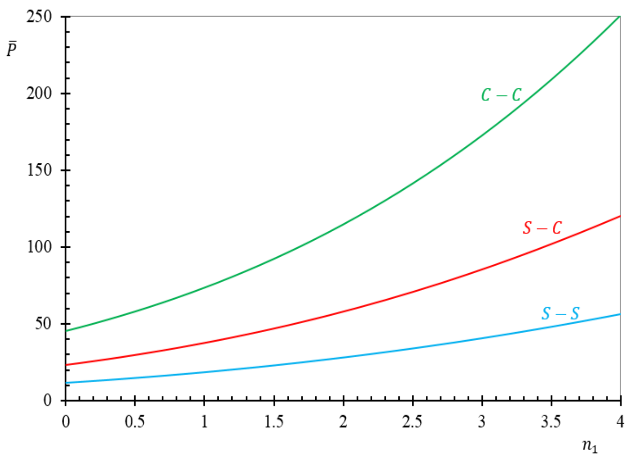

Fig. 6 Dimensionless buckling load TDFG nano-beam versus n_1 with various boundary conditions ($\alpha = 0.1$)

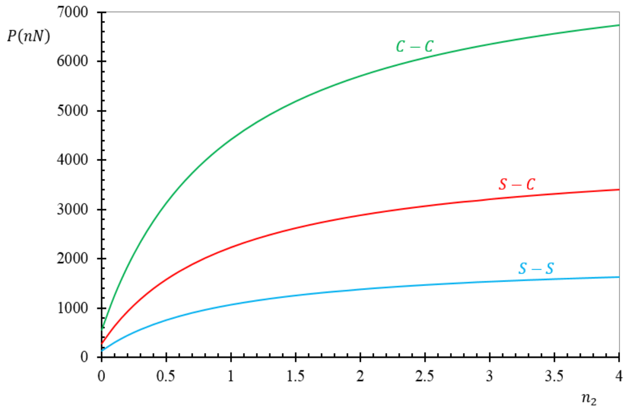

Fig. 7 Buckling load TDFG nano-beam versus n_2 with various boundary conditions ($n_1 = n_3 = 2$, $b = h$ = 5 nm, L = 50 nm, $k = 0.5$, $\alpha = 0.1$)

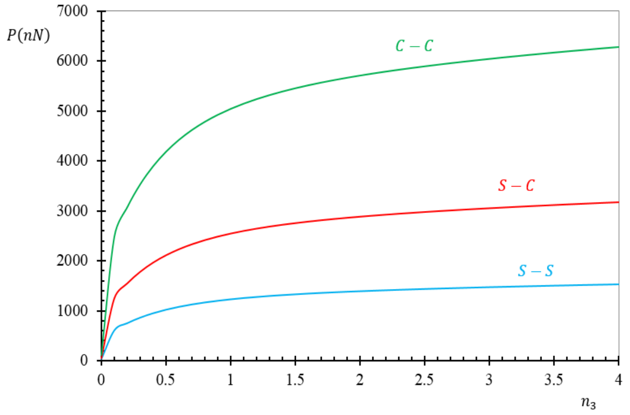

Fig. 8 Buckling load TDFG nano-beam versus n_3 with various boundary conditions ($n_1 = n_3 = 2$, $b = h$ = 5 nm, L = 50 nm, $k = 0.5$, $\alpha = 0.1$)

5. Conclusions

In this article we have investigated the size dependent buckling behavior of three dimensional functionally graded Euler-Bernoulli nanobeams by using consistent couple-stress theory. The TDFG Euler-Bernoulli nano-beam is assumed to be graded through thickness, width and length directions, following the arbitrary material distribution. Minimum total potential energy principle was used to drive the governing differential equation and associated boundary conditions. After wards, GDQM is applied to solve the equations to obtain the critical buckling loads of FG nano-beam. Results show that small scale effects significantly contribute to the mechanical behavior of TDFG nano-beam under compressive loads, a significant fact which cannot be neglected. It is observed that by increasing α, buckling loads increase. Also, the effects of inhomogeneity materials parameters in the thickness, width and length directions was investigated. To show the effect of inhomogeneity on the buckling properties of FG nano-beam, different values were considered for material inhomogeneity parameters n_1, n_2 and n_3. Results show that by increasing n_1, n_2 and n_3 the buckling loads increased. Finally, the comparison between the results obtained from the classical and consistent couple-stress theory reveals that application of the latter leads to a model of the nano-beam with higher stiffness and larger buckling loads.

References

Adeli, M.M., Hadi, A., Hosseini, M. and Gorgani, H.H. (2017), "Torsional vibration of nano-cone based on nonlocal strain gradient elasticity theory", *Eur. Phys. J. Plus*, **132**(9), 393.

Afshin, A., Nejad, M.Z. and Dastani, K. (2017), "Transient thermoelastic analysis of FGM rotating thick cylindrical pressure vessels under arbitrary boundary and initial conditions", *J. Computat. Appl. Mech.*, **48**(1), 15-26.

Ansari, R. and Sahmani, S. (2011), "Bending behavior and buckling of nanobeams including surface stress effects corresponding to different beam theories", *Int. J. Eng. Sci.*, **49**(11), 1244-1255.

Ansari, R., Gholami, R., Faghih Shojaei, M., Mohammadi, V. and Darabi, M.A. (2016), "Coupled longitudinal-transverse-rotational free vibration of post-buckled functionally graded first-order shear deformable micro- and nano-beams based on the Mindlin's strain gradient theory", *Appl. Math. Model.*, **40**(23-24), 9872-9891.

Sadrabadi, S.A., Rahimi, G.H., Citarella, R., Shahbazi Karami, J., Sepe, R. and Esposito, R. (2017), "Analytical solutions for yield onset achievement in FGM thick walled cylindrical tubes undergoing thermomechanical loads", *Compos. Part B: Eng.*, **116**, 211-223.

Apuzzo, A., Barretta, R., Luciano, R., Marotti de Sciarra, F. and Penna, R. (2017), "Free vibrations of Bernoulli-Euler nano-beams by the stress-driven nonlocal integral model", *Compos. Part B: Eng.*, **123**, 105-111.

Aydogdu, M. (2009), "A general nonlocal beam theory: Its application to nanobeam bending, buckling and vibration", *Physica E: Low-dimens. Syst. Nanostruct.*, **41**(9), 1651-1655.

Bahrami, A. and Teimourian, A. (2015), "Nonlocal scale effects on buckling, vibration and wave reflection in nanobeams via wave propagation approach", *Compos. Struct.*, **134**, 1061-1075.

Belkorissat, I., Houari, M.S.A., Tounsi, A., Bedia, E. and Mahmoud, S. (2015), "On vibration properties of functionally graded nano-plate using a new nonlocal refined four variable model", *Steel Compos. Struct., Int. J.*, **18**(4), 1063-1081.

Bounouara, F., Benrahou, K.H., Belkorissat, I. and Tounsi, A. (2016), "A nonlocal zeroth-order shear deformation theory for free vibration of functionally graded nanoscale plates resting on elastic foundation", *Steel Compos. Struct., Int. J.*, **20**(2), 227-249.

Burlayenko, V.N., Altenbach, H., Sadowski, T., Dimitrova, S.D. and Bhaskar, A. (2017), "Modelling functionally graded materials in heat transfer and thermal stress analysis by means of graded finite elements", *Appl. Math. Model.*, **45**, 422-438.

Chen, C., Li, S., Dai, L. and Qian, C. (2014), "Buckling and stability analysis of a piezoelectric viscoelastic nanobeam subjected to van der Waals forces", *Commun. Nonlinear Sci. Numer. Simul.*, **19**(5), 1626-1637.

Civalek, Ö. (2017), "Vibration of laminated composite panels and curved plates with different types of FGM composite constituent", *Compos. Part B: Eng.*, **122**, 89-108.

Ebrahimi, F. and Barati, M.R. (2016a), "Buckling analysis of nonlocal third-order shear deformable functionally graded piezoelectric nanobeams embedded in elastic medium", *J. Brazil. Soc. Mech. Sci. Eng.*, **39**(3), 937-952.

Ebrahimi, F. and Barati, M.R. (2016b), "Thermal Buckling Analysis of Size-Dependent FG Nanobeams Based on the Third-Order Shear Deformation Beam Theory", *Acta Mechanica Solida Sinica*, **29**(5), 547-554.

Ebrahimi, F. and Barati, M.R. (2016c), "Vibration analysis of nonlocal beams made of functionally graded material in thermal environment", *Eur. Phys. J. Plus*, **131**(8), 279.

Ebrahimi, F. and Barati, M.R. (2017), "A nonlocal strain gradient refined beam model for buckling analysis of size-dependent shear-deformable curved FG nanobeams", *Compos. Struct.*, **159**, 174-182.

Ebrahimi, F. and Salari, E. (2015), "Thermal buckling and free vibration analysis of size dependent Timoshenko FG nanobeams in thermal environments", *Compos. Struct.*, **128**, 363-380.

Ebrahimi, F., Barati, M.R. and Zenkour, A.M. (2017), "A new nonlocal elasticity theory with graded nonlocality for thermo-mechanical vibration of FG nanobeams via a nonlocal third-order shear deformation theory", *Mech. Adv. Mater. Struct.*, **25**(6), 512-522.

Eltaher, M.A., Khairy, A., Sadoun, A.M. and Omar, F.-A. (2014), "Static and buckling analysis of functionally graded Timoshenko nanobeams", *Appl. Math. Computat.*, **229**, 283-295.

Emam, S.A. (2013), "A general nonlocal nonlinear model for buckling of nanobeams", *Appl. Math. Model.*, **37**(10-11), 6929-6939.

Eringen, A.C. (1972a), "Nonlocal polar elastic continua", *Int. J. Eng. Sci.*, **10**(1), 1-16.

Eringen, A.C. (1972b), "Theory of micromorphic materials with memory", *Int. J. Eng. Sci*, **10**(7), 623-641.

Eringen, A.C. (1983), "On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves", *J. Appl. Phys.*, **54**(9), 4703-4710.

Eringen, A.C. (2002), *Nonlocal Continuum Field Theories*, Springer Science & Business Media.

Fatehi, P. and Nejad, M.Z. (2014), "Effects of material gradients on onset of yield in FGM rotating thick cylindrical shells", *Int. J. Appl. Mech.*, **6**(4), Article Number: 1450038.

Ghannad, M., Nejad, M.Z., Rahimi, G.H. and Sabouri, H. (2012), "Elastic analysis of pressurized thick truncated conical shells made of functionally graded materials", *Struct. Eng. Mech., Int. J.*, **43**(1), 105-126.

Ghannad, M., Rahimi, G.H. and Nejad, M.Z. (2013), "Elastic analysis of pressurized thick cylindrical shells with variable thickness made of functionally graded materials", *Compos. Part B-Eng.*, **45**(1), 388-396.

Ghannadpour, S., Mohammadi, B. and Fazilati, J. (2013), "Bending, buckling and vibration problems of nonlocal Euler beams using Ritz method", *Compos. Struct.*, **96**, 584-589.

Gharibi, M., Nejad, M.Z. and Hadi, A. (2017), "Elastic analysis of functionally graded rotating thick cylindrical pressure vessels with exponentially-varying properties using power series method of Frobenius", *J. Computat. Appl. Mech.*, **48**(1), 89-98.

Goodarzi, M., Nikkhah Bahrami, M. and Tavaf, V. (2017), "Refined plate theory for free vibration analysis of FG nanoplates using the nonlocal continuum plate model", *J. Computat. Appl. Mech.*, **48**(1), 123-136.

Gopalakrishnan, S. and Narendar, S. (2013), *Wave Propagation in Nanostructures: Nonlocal Continuum Mechanics Formulations*, Springer Science & Business Media.

Hadi, A., Rastgoo, A., Daneshmehr, A. and Ehsani, F. (2013), "Stress and strain analysis of functionally graded rectangular plate with exponentially varying properties", *Indian J. Mater. Sci*.

Hadjesfandiari, A.R. and Dargush, G.F. (2011), "Couple stress theory for solids", *Int. J. Solids Struct.*, **48**(18), 2496-2510.

Hosseini, M., Shishesaz, M., Tahan, K.N. and Hadi, A. (2016), "Stress analysis of rotating nano-disks of variable thickness made of functionally graded materials", *Int. J. Eng. Sci.*, **109**, 29-53.

Hosseini, M., Gorgani, H.H., Shishesaz, M. and Hadi, A. (2017), "Size-Dependent Stress Analysis of Single-Wall Carbon Nanotube Based on Strain Gradient Theory", *Int. J. Appl. Mech.*, **9**(6), 1750087.

Jabbari, M., Nejad, M.Z. and Ghannad, M. (2015), "Thermo-elastic analysis of axially functionally graded rotating thick cylindrical pressure vessels with variable thickness under mechanical loading", *Int. J. Eng. Sci.*, **96**, 1-18.

Jabbari, M., Nejad, M.Z. and Ghannad, M. (2016), "Thermo-elastic analysis of axially functionally graded rotating thick truncated conical shells with varying thickness", *Compos. Part B-Eng.*, **96**, 20-34.

Jeyakarthikeyan, P.V., Subramanian, G. and Yogeshwaran, R. (2017), "An alternate stable midpoint quadrature to improve the element stiffness matrix of quadrilaterals for application of functionally graded materials (FGM)", *Comput. Struct.*, **178**, 71-87.

Kashkoli, M.D., Tahan, K.N. and Nejad, M.Z. (2017), "Time-Dependent Thermomechanical Creep Behavior of FGM Thick Hollow Cylindrical Shells Under Non-Uniform Internal Pressure", *Int. J. Appl. Mech.*, **9**(6), Article Number: 1750086.

Keivani, M., Koochi, A. and Abadyan, M. (2016), "Coupled effects of surface energy and size dependency on the stability of nanotweezers using GDQ method", *Microsyst. Technol.*, **23**(5), 1295-1308.

Lam, D., Yang, F., Chong, A., Wang, J. and Tong, P. (2003), "Experiments and theory in strain gradient elasticity", *J. Mech. Phys. Solids*, **51**(8), 1477-1508.

Li, L. and Hu, Y. (2016), "Nonlinear bending and free vibration analyses of nonlocal strain gradient beams made of functionally graded material", *Int. J. Eng. Sci.*, **107**, 77-97.

Li, L. and Hu, Y. (2017a), "Post-buckling analysis of functionally graded nanobeams incorporating nonlocal stress and microstructure-dependent strain gradient effects", *Int. J. Mech. Sci.*, **120**, 159-170.

Li, L. and Hu, Y. (2017b), "Torsional vibration of bi-directional functionally graded nanotubes based on nonlocal elasticity theory", *Compos. Struct.*, **172**, 242-250.

Li, L., Li, X. and Hu, Y. (2016), "Free vibration analysis of nonlocal strain gradient beams made of functionally graded material", *Int. J. Eng. Sci.*, **102**, 77-92.

Li, X., Li, L., Hu, Y., Ding, Z. and Deng, W. (2017), "Bending, buckling and vibration of axially functionally graded beams based on nonlocal strain gradient theory", *Compos. Struct.*, **165**, 250-265.

Li, L., Li, X. and Hu, Y. (2018), "Nonlinear bending of a two-dimensionally functionally graded beam", *Compos. Struct.*, **184**, 1049-1061.

Lü, C., Chen, W., Xu, R. and Lim, C.W. (2008), "Semi-analytical elasticity solutions for bi-directional functionally graded beams", *Int. J. Solids Struct.*, **45**(1), 258-275.

Mazarei, Z., Nejad, M.Z. and Hadi, A. (2016), "Thermo-elasto-plastic analysis of thick-walled spherical pressure vessels made of functionally graded materials", *Int. J. Appl. Mech.*, **8**(4), Article Number: 1650054.

Mindlin, R. and Tiersten, H. (1962), "Effects of couple-stresses in linear elasticity", *Arch. Rational Mech. Anal*, **11**(1), 415-448.

Najibi, A. and Talebitooti, R. (2017), "Nonlinear transient thermo-elastic analysis of a 2D-FGM thick hollow finite length cylinder", *Compos. Part B: Eng.*, **111**, 211-227.

Nejad, M.Z. and Fatehi, P. (2015), "Exact elasto-plastic analysis of rotating thick-walled cylindrical pressure vessels made of functionally graded materials", *Int. J. Eng. Sci.*, **86**, 26-43.

Nejad, M.Z. and Hadi, A. (2016a), "Eringen's non-local elasticity theory for bending analysis of bi-directional functionally graded Euler-Bernoulli nano-beams", *Int. J. Eng. Sci.*, **106**, 1-9.

Nejad, M.Z. and Hadi, A. (2016b), "Non-local analysis of free vibration of bi-directional functionally graded Euler-Bernoulli nano-beams", *Int. J. Eng. Sci.*, **105**, 1-11.

Nejad, M.Z. and Rahimi, G.H. (2009), "Deformations and stresses in rotating FGM pressurized thick hollow cylinder under thermal load", *Sci. Res. Essays*, **4**(3), 131-140.

Nejad, M.Z. and Rahimi, G.H. (2010), "Elastic analysis of FGM rotating cylindrical pressure vessels", *J. Chinese Inst. Engr.*, **33**(4), 525-530.

Nejad, M.Z., Rahimi, G.H. and Ghannad, M. (2009), "Set of field equations for thick shell of revolution made of functionally graded materials in curvilinear coordinate system", *Mechanika*, **77**(3), 18-26.

Nejad, M., Rastgoo, A. and Hadi, A. (2014a), "Effect of Exponentially-Varying Properties on Displacements and Stresses in Pressurized Functionally Graded Thick Spherical Shells with Using Iterative Technique", *J. Solid Mech.*, **6**(4), 366-377.

Nejad, M.Z., Rastgoo, A. and Hadi, A. (2014b), "Exact elasto-plastic analysis of rotating disks made of functionally graded materials", *Int. J. Eng. Sci.*, **85**, 47-57.

Nejad, M.Z., Jabbari, M. and Ghannad M. (2015a), "Elastic analysis of FGM rotating thick truncated conical shells with axially-varying properties under non-uniform pressure loading", *Compos. Struct.*, **122**, 561-569.

Nejad, M.Z., Jabbari, M. and Ghannad, M. (2015b), "Elastic analysis of axially functionally graded rotating thick cylinder with variable thickness under non-uniform arbitrarily pressure loading", *Int. J. Eng. Sci.*, **89**, 86-99.

Nejad, M.Z., Hadi, A. and Rastgoo, A. (2016a), "Buckling analysis of arbitrary two-directional functionally graded Euler-Bernoulli nano-beams based on nonlocal elasticity theory", *Int. J. Eng. Sci.*, **103**, 1-10.

Nejad, M.Z., Abedi, M., Lotfian, M.H. and Ghannad, M. (2016b), "Exact and numerical elastic analysis for the FGM thick-walled cylindrical pressure vessels with exponentially-varying properties", *Arch. Metal. Mater.*, **61**(3), 1303-1308.

Nejad, M.Z., Jabbari, M. and Ghannad, M. (2017a), "A general disk form formulation for thermo-elastic analysis of functionally graded thick shells of revolution with arbitrary curvature and variable thickness", *Acta Mechanica*, **228**(1), 215-231.

Nejad, M.Z., Jabbari, M. and Hadi, A. (2017b), "A review of functionally graded piezoelectric thick shells", *J. Computat. Appl. Mech*. DOI: 10.22059/JCAMECH.2017.247963.220

Nejad, M.Z., Hadi, A. and Farajpour, A. (2017c), "Consistent couple-stress theory for free vibration analysis of Euler-Bernoulli nano-beams made of arbitrary bi-directional functionally graded materials", *Struct. Eng. Mech., Int. J.*, **63**(2), 161-169.

Nguyen, N.-T., Kim, N.-I. and Lee, J. (2014), "Analytical solutions for bending of transversely or axially FG nonlocal beams", *Steel Compos. Struct., Int. J.*, **17**(5), 641-665.

Pradhan, S. and Phadikar, J. (2009), "Bending, buckling and vibration analyses of nonhomogeneous nanotubes using GDQ and nonlocal elasticity theory", *Struct. Eng. Mech., Int. J.*, **33**(2), 193-213.

Rahmani, O., Refaeinejad, V. and Hosseini, S. (2017), "Assessment of various nonlocal higher order theories for the bending and buckling behavior of functionally graded nanobeams", *Steel Compos. Struct., Int. J.*, **23**(3), 339-350.

Sahmani, S. and Aghdam, M.M. (2017), "Nonlinear instability of hydrostatic pressurized hybrid FGM exponential shear deformable nanoshells based on nonlocal continuum elasticity", *Compos. Part B: Eng.*, **114**, 404-417.

Shafiei, N., Mirjavadi, S.S., Afshari, B.M., Rabby, S. and Hamouda, A. (2017), "Nonlinear thermal buckling of axially functionally graded micro and nanobeams", *Compos. Struct.*, **168**, 428-439.

Shishesaz, M., Hosseini, M., Tahan, K.N. and Hadi, A. (2017), "Analysis of functionally graded nanodisks under thermoelastic loading based on the strain gradient theory", *Acta Mechanica*, **228**(12), 4141-4168.

Shu, C. and Chew, Y. (1998), "On the equivalence of generalized differential quadrature and highest order finite difference scheme", *Comput*. *Methods Appl*. *Mech*. *Eng*,, **155**(3), 249-260.

Şimşek, M. and Al-shujairi, M. (2017), "Static, free and forced vibration of functionally graded (FG) sandwich beams excited by two successive moving harmonic loads", *Compos*. *Part B*: *Eng*., **108**, 18-34.

Şimşek, M. and Yurtcu, H.H. (2013), "Analytical solutions for bending and buckling of functionally graded nanobeams based on the nonlocal Timoshenko beam theory", *Compos*. *Struct*., **97**, 378-386.

Steinberg, M.A. (1986), "Materials for aerospace", *Sci*. *Am*., (*United States*), **255**(4), 66-73.

Taczała, M., Buczkowski, R. and Kleiber, M. (2017), "Nonlinear buckling and post-buckling response of stiffened FGM plates in thermal environments", *Compos*. *Part B*: *Eng*., **109**, 238-247.

Thai, H.-T. (2012), "A nonlocal beam theory for bending, buckling, and vibration of nanobeams", *Int*. *J*. *Eng*. *Sci*., **52**, 56-64.

Thai, H.-T. and Vo, T.P. (2012), "A nonlocal sinusoidal shear deformation beam theory with application to bending, buckling, and vibration of nanobeams", *Int*. *J*. *Eng*. *Sci*., **54**, 58-66.

Toupin, R.A. (1962), "Elastic materials with couple-stresses", *Arch*. *Rational Mech*. *Anal*., **11**(1), 385-414.

Wang, Y.Q. and Zu, J.W. (2017), "Nonlinear dynamic thermoelastic response of rectangular FGM plates with longitudinal velocity", *Compos*. *Part B*: *Eng*., **117**, 74-88.

Wang, C., Zhang, Y., Ramesh, S.S. and Kitipornchai, S. (2006), "Buckling analysis of micro-and nano-rods/tubes based on nonlocal Timoshenko beam theory", *J*. *Phys*. *D*: *Appl*. *Phys*., **39**(17), 3904.

Yang, F., Chong, A., Lam, D.C.C. and Tong, P. (2002), "Couple stress based strain gradient theory for elasticity", *Int*. *J*. *Solids Struct*., **39**(10), 2731-2743.

Yu, Y. J., Xue, Z.-N., Li, C.-L. and Tian, X.-G. (2016), "Buckling of nanobeams under nonuniform temperature based on nonlocal thermoelasticity", *Compos*. *Struct*., **146**, 108-113.

CC