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Fractional magneto-thermoelastic materials with phase-lag Green-Naghdi theories

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Abstract. A unified mathematical model of phase-lag Green-Naghdi magneto-thermoelasticty theories based on fractional derivative heat transfer for perfectly conducting media in the presence of a constant magnetic field is given. The GN theories as well as the theories of coupled and of generalized magneto-thermoelasticity with thermal relaxation follow as limit cases. The resulting nondimensional coupled equations together with the Laplace transforms techniques are applied to a half space, which is assumed to be traction free and subjected to a thermal shock that is a function of time. The inverse transforms are obtained by using a numerical method based on Fourier expansion techniques. The predictions of the theory are discussed and compared with those for the generalized theory of magneto-thermoelasticity with one relaxation time. The effects of Alfven velocity and the fractional order parameter on copper-like material are discussed in different types of GN theories.

Keywords: generalized magneto-thermoelasticity; caputo fractional derivatives; phase-lag Green-Naghdi theories; laplace transforms; numerical results

1. Introduction

The Green and Naghdi (GN) thermoelasticity theories (1991, 1992, 1993) are developed to produce a consistent theories, which consider elastic and thermal waves associated with second sound. Many works were devoted to investigate various theoretical and practical aspects in thermoelasticity, in the context of the GN theories of type II or /and of type III. Chandrasekharaiah (1998) has proved uniqueness theorems using energy method. Based on GN theories, the three-phase lag thermoelasticity theory was proposed by Roy Choudhuri (2007). Ciarletta (2009) established a theory of micropolar thermoelasticity without energy dissipation. Chirita and Ciarletta (2010) established the reciprocal and variational principle in linear thermoelasticity without energy dissipation. El-Karamany and Ezzat (2016) proposed three models of generalized thermoelasticity: a single-phase-lag GN theory of type III, a dual-phase-lag GN theory of type II and of type III. The heat conduction law and heat transport equation are given, which consolidate the three theories and also the Lord-Shulman theory (Lord and Shulman 1967) and the GN theories of type II and of type III. Kumar *et al*. (2007, 2008, 2014), Sharma *et al*. (2015) solved some problems in GN theories and Alzahrani and Abbas (2016) studied the effect of the magnetic field on a thermoelastic fiber-reinforced under GN of type III.

Tzou (1995) introduced a generalization of thermo-elasticity theory with dual-phase-lag who proposed two different phase-lags in the Fourier law of heat conduction law in which the first for the heat flux vector and the second for the temperature gradient. One can refer to the references Horgan and Quintanilla (2005), Jou and Criado-Sancho (1998), El-Karamny and Ezzat (2014) for more applications on the dual-phase-lag model Ezzat *et al*. (2012) on three-phase lag heat transfer and Abbas and Kumar (2016) investigated two-dimensional deformation in initially stressed thermoelastic half space problem. Othman *et al*. (2002) constructed thermo-viscoelastic plane waves with two relaxation times in istropic medium. Sharma *et al*. (2013a) studied the propagation of Lamp waves in thermoelastic micropolar solid with two temperature border with layers or half-space of inviscid liquid subjected to stress free boundary conditions and the wave propagation in anisotropic thermoviscoelastic medium in the context Green-Naghdi theories of type-II and type-III examined by Sharma *et al*. (2013). Sharma and Sharma (2014) studied, the temperature fluctuations in tissues based on Penne's bio-heat transfer equation.

An increasing attention is being devoted to the interaction between magnetic and strain field in a thermoelastic solid due to its many applications in the fields of geophysics, plasma physics and related topics. It was assumed that the interactions between the two fields take place by means of the Lorentz forces appearing in the equations of motion and by means of a term entering Ohm's law and describing the electric field produced by the velocity of a material particle moving in a magnetic field.

The inclusion of the electric displacement current in Maxwell's equations modify electromagnetic field equations, changing them from parabolic to hyperbolic type and thereby eliminating the unrealistic result that electro-

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magnetic disturbances are realized instantaneously everywhere within a solid. Solution of boundary value problems for linear thermoelastic perfect conducting materials has made great strides in the last decades, as seen in the works of Mehditabar *et al*. (2014), Zenkour (2014) and Lata *et al*. (2016) and Sharma *et al*. (2014) .

Recently, the fractional order theory of thermoelasticity was derived by Ezzat (2011, 2012). It is a generalization of both the coupled and the generalized theories of thermoelasticity. El-Karamany and Ezzat (2011) introduced two general models of fractional heat conduction law for a non-homogeneous anisotropic elastic solid. Uniqueness and reciprocal theorems are proved and the convolutional variational principle is established and used to prove a uniqueness theorem with no restriction on the elasticity or thermal conductivity tensors except symmetry conditions. One can refer to Ezzat and El-Karamany (2011a, b) and Ezzat and El-Bary (2016) for a survey of applications of fractional calculus.

The aim of the present article is to introduce a unified mathematical model for phase-lag GN magneto-thermoelastticty theories by using the methodology of fractional calculus theory based on Lord-Shulman generalized theory (Lord and Shulman 1967). The resulting formulation is applied to one-dimensional thermal shock problem for a half-space subjected to an arbitrary heating. Laplace transforms techniques are used to get the solution representing the thermal shock. The effect of the different values of Alfven velocity and time fractional derivative parameter is discussed for different types of GN theories. The inversion of the Laplace transforms is carried out using a numerical approach proposed by Honig and Hirdes (1984).

2. Mathematical model

We shall consider a thermoelastic medium of perfect conductivity occupying the half-space and permeated by an initial magnetic field H. Due to the effect of this magnetic field there arises in the conducting medium an induced magnetic field h and induced electric field E (both assumed to be small). We assume that both h and E are small in magnitude in accordance with the assumptions of the linear theory of thermoelasticity. Also, there arises a force F (the Lorentz Force). Due to the effect of the force, points of the medium undergo a displacement vector u, which gives rise to a temperature.

Motivated by this fact, the displacement-temperature formulation is adopted here although in some other practical cases the stress-temperature formulations have a number of advantages (Parkus 1970).

(i) Linearized equations of electromagnetism for slowly moving media (Ezzat 2001)

$$\operatorname{curl} h = J + \varepsilon_o \frac{\partial E}{\partial t} \tag{1}$$

$$\operatorname{curl} E = -\mu_o \frac{\partial h}{\partial t} \tag{2}$$

$$E = -\mu_o \left(\frac{\partial h}{\partial t} \wedge H \right) \tag{3}$$

$$\operatorname{div} h = 0 \tag{4}$$

(ii) Displacement equation, taking into account the Lorentz force is

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \sigma_{ij,j} + \mu_o \left(J \wedge H \right)_i \tag{5}$$

(iii) Constitutive equation

$$\sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij} - \gamma (T - T_o) \delta_{ij} \tag{6}$$

(iv) Strain-displacement relation

$$\sigma_{ij} = \frac{1}{2} \left(\mu_{i,j} + \mu_{j,i} \right) \tag{7}$$

(v) Fractional Fourier law

Introducing the fractional Fourier law in the form

$$q_i + \tau_o \frac{\partial^\upsilon q_i}{\partial t^\upsilon} = -K_{ij} \theta_{,j} \tag{8}$$

We define the operator $K_{ij}\theta_{,j}$ by the relation proposed by El-Karamany and Ezzat (2016)

$$\begin{aligned} & K_{ij} \theta_{,j}(x,t) \\ & = a_1 k_{ij} \theta_{,j}(x,t) + a_2 \tau_\alpha k^*_{ij} \theta_{,j}(x,t) - a_3 k^*_{ij} (\ell * \theta_{,j}(x,t)) \end{aligned} \tag{9}$$

where $(\ell * \theta) = \alpha$ is the thermal displacement (Green and Naghdi 1991), $(\ell * \theta) = \int_0^t \theta(x,\tau) d\tau$, $\frac{d\alpha}{dt} = \theta$ and $\frac{\partial^\upsilon f(x,t)}{\partial t^\upsilon}$ is the Caputo fractional derivative defined by Gorenflo and Mainardi (1997)

$$\frac{\partial^\upsilon f(x,t)}{\partial t^\upsilon} = \begin{cases} \dfrac{1}{\Gamma(1-\upsilon)} \displaystyle\int_0^t (t-\xi)^{-\upsilon} \frac{\partial f(x,\xi)}{\partial \xi} d\xi & 0 < \upsilon < 1 \\ \dfrac{\partial f(x,t)}{\partial t} & \upsilon \to 1 \end{cases} \tag{10}$$

(vi) Heat transport equation

Eqs. (8) and (9) lead to the following fractional order heat transport equation

$$\begin{aligned} & \left(1 + \tau_0 \frac{\partial^\upsilon}{\partial t^\upsilon} \right) (\rho c_E \dot{\theta} + T_0 \gamma_{ij} \dot{\varepsilon}_{ij}) \\ & = a_1 (k_{ij} \theta_{,j}) + a_2 \tau_\alpha (k^*_{ij} \theta_{,j})_{,i} + a_3 \tau_\alpha (k^*_{ij} (\ell * \theta))_{,i} \end{aligned} \tag{11}$$

Limiting cases

Eq. (11) when $\upsilon \to 1$ and $\ell(t) \to 1$ (Chirita and Ciarletta 2010) lead to the Fourier law for the following theories:

(1) The coupled thermoelasticity theory (Biot 1956)

when: $\tau_0 = 0,\ a_1 = 1,\ a_2 = a_3 = 0$

(2) Lord-Shulman theory (LS) (1967), when $\tau_0 = \tau_q$, $a_1 = 1,\ a_2 = a_3 = 0$

(3) The single -phase -lag GN theory of type III (SPLGN-III), when $a_1 = 1,\ a_2 = 0,\ a_3 = 1$

(4) The dual-phase-lag GN theory of type II (DPLGN-II), when $a_1 = 0,\ a_2 = 1,\ a_3 = 1$

(5) The dual-phase-lag GN theory of type III (DPLGN-III), when $a_1 = a_2 = a_3 = 1$

(6) GN theory of type II without energy dissipation (GN-II), when $\tau_0 = \tau_\alpha,\ a_1 = 0,\ a_2 = a_3 = 1$ and we get: $q_i = -k_{ij}^* \alpha_{,j}$

(7) The GN theory of type III (GN-III), when $\tau_0 = 0$, $a_2 = 0,\ a_1 = a_3 = 1$

In the case $0 < \upsilon < 1$, the correspondent heat transfer equations for the fractional phase-lag GN thermoelasticity theories result.

(8) The unified fractional phase-lag model GN magneto-thermoelasticity theories (UFPLGN), when $\tau_0 = \tau_q,\ \tau_\alpha \neq 0,\ a_1 = a_2 = a_3 = 1,\ \ 0 < \upsilon < 1$.

3. One-dimensional formulations

We shall consider a solid occupying the region $x \geq 0$, where x-axis is taken perpendicular to the bounding plane of half-space pointing inwards. Assume also that the initial conditions are homogeneous and the initial magnetic field has components (0, 0, H_o). The induced magnetic field h will have one component h in the z-direction, while the induced electric field E will have one component E in y-direction. For the one-dimensional problems, all the considered functions will depend only on the space variables x and t.

The displacement components

$$u_x = u(x,t), \quad u_y = u_z = 0 \tag{12}$$

The strain-displacement relation

$$e = \frac{\partial u}{\partial x} \tag{13}$$

From Eq. (1), it follows that the electric current density J will have one component only J in y-direction, given by

$$J = -\left(\frac{\partial h}{\partial x} + \varepsilon_o \mu_o H_o \frac{\partial^2 h}{\partial t^2}\right) \tag{14}$$

The vector Eqs. (2) and (3), reduce to the following scalar equations

$$h = -H_o \frac{\partial u}{\partial x}, \tag{15}$$

$$E = \mu_o H_o \frac{\partial u}{\partial t}. \tag{16}$$

Expressing the components of the vector J in terms of displacement, by eliminating from Eq. (16) the quantities h and E and introducing them into the displacement Eq. (5), we arrive as in Ref. Ezzat (2006)

$$a_o \frac{\partial^2 u}{\partial t^2} = C_o^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial \theta}{\partial x}, \tag{17}$$

where, $a_o = 1 + \alpha_o^2 / c^2$.

Eq. (17) is to be supplemented by the fractional phase-lag Green-Naghdi heat conduction equation

$$\begin{gathered}\left(1 + \tau_0 \frac{\partial^\upsilon}{\partial t^\upsilon}\right)\left(\rho C_E \frac{\partial^2 \theta}{\partial t^2} + T_0 \alpha_T \frac{\partial^3 \theta}{\partial x \partial t^2}\right) \\ = (a_1 k + a_2 \tau_\alpha k^*) \frac{\partial^3 \theta}{\partial x \partial t^2} + a_3 k^* \frac{\partial^2 \theta}{\partial x^2}\end{gathered} \tag{18}$$

The constitutive equation

$$\sigma = \sigma_{xx} = (\lambda + 2\mu) \frac{\partial u}{\partial x} - \gamma \theta. \tag{19}$$

Let us introduce the following non-dimensional variables:

$$x^* = C_o \eta x, \quad u^* = C_o \eta u, \quad t^* = C_o^2 \eta t, \quad \sigma^* = \frac{\sigma}{\lambda + 2\mu},$$

$$\theta^* = \frac{\gamma \theta}{\lambda + 2\mu}, \quad h^* = \frac{h}{H_o}, \quad \alpha' = \frac{\gamma \eta \alpha}{\rho}$$

$$k^{*'} = \frac{k^*}{\rho C_E C_0^2}, \quad q_x' = \frac{\gamma q_x}{k \rho C_0^3 \eta}, \quad E^* = \frac{E}{\mu_o H_o C_o},$$

$$J^* = \frac{J}{H_o C_o \eta}, \quad q_i^* = \frac{\gamma}{k \rho C_0^3 \eta_0} q_i.$$

The Eqs. (14)-(19) in non-dimensional form become

$$J = -\left(\frac{\partial h}{\partial x} + V \frac{\partial^2 u}{\partial t^2}\right), \tag{20}$$

$$h = -\frac{\partial u}{\partial x}, \tag{21}$$

$$E = \frac{\partial u}{\partial t}, \tag{22}$$

$$\frac{\partial^2 u}{\partial x^2} = a_o \frac{\partial^2 u}{\partial t^2} + \beta \frac{\partial \theta}{\partial x} \tag{23}$$

$$\left(1 + \tau_0 \frac{\partial^\upsilon}{\partial t^\upsilon}\right)\left(\frac{\partial^2 \theta}{\partial t^2} + \varepsilon \frac{\partial^3 u}{\partial x \partial t^2}\right) = C \frac{\partial^3 \theta}{\partial x \partial t^2} + A \frac{\partial^2 \theta}{\partial x^2} \tag{24}$$

where $C = a_1 k + a_2 \tau_\alpha k^*$ and $A = a_3 k^*$

$$\sigma = \frac{\partial u}{\partial x} - \theta \tag{25}$$

where $V = \dfrac{C_0^2}{c^2}$ and $\beta = \dfrac{C_0^2}{v^2}$.

The heat conduction law can be deduced from Eq. (8) as

$$\left(1+\tau_o \frac{\partial^\upsilon}{\partial t^\upsilon}\right)\frac{\partial q_x}{\partial t} = -\left[C\frac{\partial^2\theta}{\partial x \partial t} + A\frac{\partial \theta}{\partial x}\right] \tag{26}$$

We assume that the boundary conditions have the form

$$\begin{aligned} &\theta(0,t) = f(t), \quad \theta(\infty,t) = 0, \\ &\sigma(0,t) = \sigma(\infty,t) = 0, \qquad\qquad t>0 \end{aligned} \tag{27}$$

where $f(t)$ is a known function of t.

The initial conditions are taken as

$$\begin{aligned} &u(x,0) = \dot{u}(x,0) = \sigma(x,0) = \dot{\sigma}(x,0) \\ &= \theta(x,0) = \dot{\theta}(x,0) = 0 \qquad\qquad t \le 0. \end{aligned}$$

4. Solution in Laplace transform domain

Appling the Laplace transform with parameter s defined by the relation

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt \tag{28}$$

on both sides of Eqs. (20)-(26), we get

$$\bar{J} = -(D\bar{h} + V^2 s^2 \bar{u}) \tag{29}$$

$$\bar{h} = -D\bar{u}, \tag{30}$$

$$\bar{E} = s\bar{u}, \tag{31}$$

$$\bar{e} = D\bar{u} \tag{32}$$

$$(D^2 - a_o s^2)\bar{u} = \beta D\bar{\Theta} \tag{33}$$

$$(D^2 - n)\bar{\theta} = \varepsilon n D\bar{u} \tag{34}$$

$$\bar{\sigma} = D\bar{u} - \bar{\theta} \tag{35}$$

$$s(1+\tau_o s^\upsilon)\bar{q} = -(A+sC)D\bar{\theta} \tag{36}$$

where

$$D = \frac{\partial}{\partial x},\ L\left\{\frac{\partial^\upsilon f(x,t)}{\partial t^v}\right\} = \frac{1}{\Gamma(1-\upsilon)}\cdot\frac{\Gamma(1-\upsilon)}{s^{1-\upsilon}}L\left\{\frac{\partial f}{\partial t}\right\} =$$

$s^\upsilon \bar{f}(s)$ and $n = \dfrac{s^2(1+\tau_0 s^v)}{sC+A}$.

and the boundary conditions (27) become

$$\bar{\theta}(0,s) = f(s), \quad \bar{\sigma}(0,s) = \bar{\sigma}_0 = 0 \tag{37}$$

Eliminating $\bar{u}$ between Eqs. (33) and (34), we obtain

$$\left\{D^4 - [a_o s^2 + n(1+\beta\varepsilon)]D^2 + a_o n s^2\right\}\bar{\theta} = 0 \tag{38}$$

In a similar manner, we can show that $\bar{u}$ satisfies the equation

$$\left\{D^4 - [a_o s^2 + n(1+\beta\varepsilon)]D^2 + a_o n s^2\right\}\bar{u} = 0 \tag{39}$$

The solutions of Eqs. (38) and (39) which are bounded for $x \ge 0$ have the form

$$\bar{\theta}(x,s) = C_1 e^{-k_1 x} + C_2 e^{-k_2 x} \tag{40}$$

$$\bar{u}(x,s) = C_3 e^{-k_1 x} + C_4 e^{-k_2 x} \tag{41}$$

where k_1 and k_2 are the roots with positive real parts of the characteristic equation

$$k^4 - [s^2 + n(1+\beta\varepsilon)]k^2 + ns^2 = 0 \tag{42}$$

satisfying the relations

$$\begin{aligned} &k_1^2 + k_2^2 = a_o s^2 + n(1+\beta\varepsilon) \\ &k_1^2 k_2^2 = a_o n s^2 \end{aligned} \tag{43}$$

and C_i, i = 1, 2, 3, 4 are parameters depending on s to be determined from the boundary conditions of the problem.

Substitution from Eqs. (40) and (41) into Eq. (34), we obtain the following relations

$$\begin{aligned} C_3 &= -\frac{\beta k_1}{k_1^2 - a_o s^2} C_1, \\ C_4 &= -\frac{\beta k_2}{k_2^2 - a_o s^2} C_2 \end{aligned} \tag{44}$$

Substitution from Eq. (44) into Eq. (41), we have

$$\bar{u}(x,s) = \frac{\beta k_1}{k_1^2 - a_o s^2} C_1 e^{-k_1 x} - \frac{\beta k_2}{k_2^2 - a_o s^2} C_2 e^{-k_2 x} \tag{45}$$

Substitution from Eqs. (40) and (45) into Eq. (35), we have

$$\begin{aligned} \bar{\sigma}(x,s) = &\left[\frac{k_1^2(\beta-1) + a_o s^2}{k_1^2 - a_o s^2}\right] C_1 e^{-k_1 x} \\ &+ \left[\frac{k_2^2(\beta-1) + a_o s^2}{k_2^2 - a_o s^2}\right] C_2 e^{-k_2 x} \end{aligned} \tag{46}$$

The strain and induced electric and magnetic fields can be obtained from Eqs. (30)-(32), and (45)

$$\bar{e}(x,s) = \frac{\beta k_1^2}{k_1^2 - a_o s^2} C_1 e^{-k_1 x} - \frac{\beta k_2^2}{k_2^2 - a_o s^2} C_2 e^{-k_2 x} \tag{47}$$

$$\bar{E} = -s\left(\frac{\beta k_1}{k_1^2 - a_o s^2} C_1 e^{-k_1 x} - \frac{\beta k_2}{k_2^2 - a_o s^2} C_2 e^{-k_2 x}\right) \tag{48}$$

$$\bar{h} = \frac{\beta k_1^2}{k_1^2 - a_o s^2} C_1 e^{-k_1 x} - \frac{\beta k_2^2}{k_2^2 - a_o s^2} C_2 e^{-k_2 x} \tag{49}$$

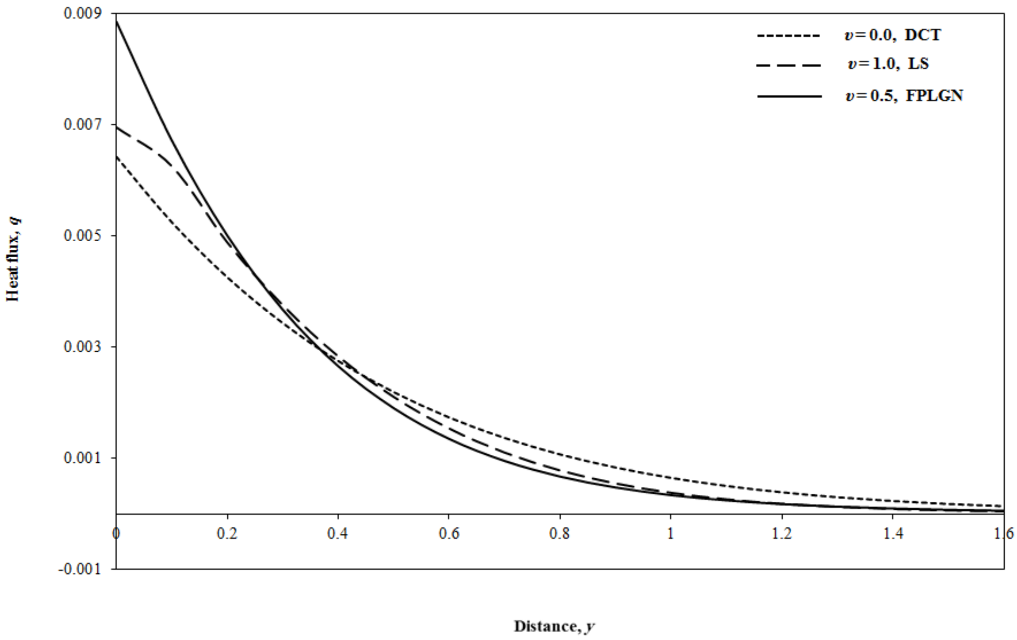

Fig. 1 The variation of heat flux for different theories

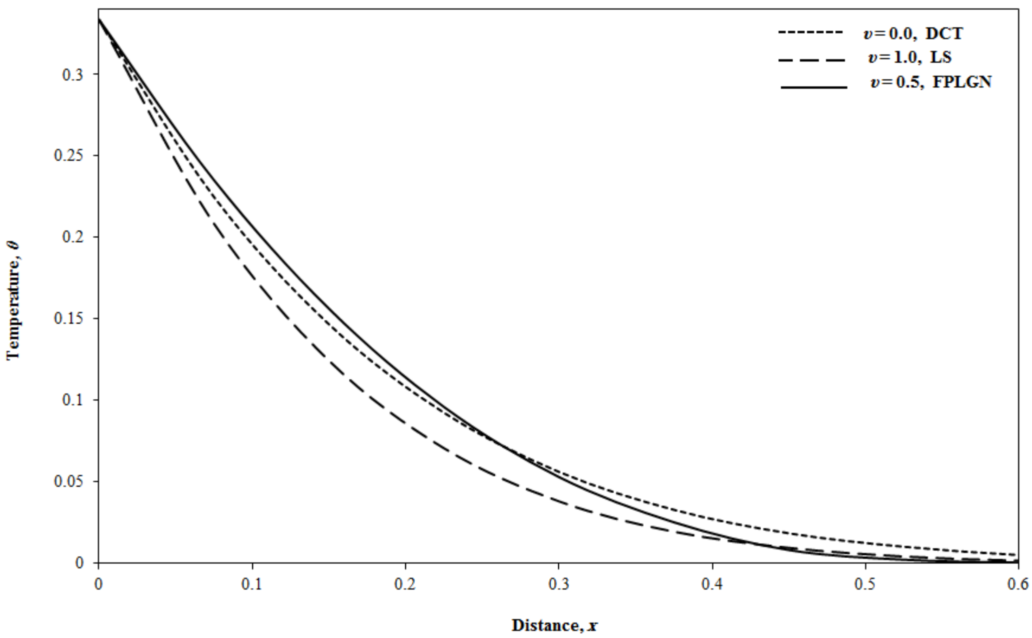

Fig. 2 The variation of temperature for different theories

From Eq. (36) the heat flux component is given by

$$\bar{q} = \frac{s}{n}(C_1 k_1 e^{-k_1 x} + C_2 k_2 e^{-k_2 x}) \tag{50}$$

In order to determine the C_1 and C_2, we shall use the boundary conditions (37), we obtain

$$\begin{aligned} C_1 &= -\frac{(k_1^2 - a_o s^2)(k_2^2[\beta - 1] + a_o \beta s^2)}{a_o s^2 (k_1^2 - k_2^2)} \bar{h}(s), \\ C_2 &= -\frac{(k_2^2 - a_o s^2)(k_1^2[\beta - 1] + a_o \beta s^2)}{a_o s^2 (k_1^2 - k_2^2)} \bar{h}(s) \end{aligned} \tag{51}$$

It should be noted that the corresponding expressions for generalized GN theories in the absence of a magnetic field can be deduced by setting $a_o = \beta = 1.0$.

This completes the solution in the Laplace transform domain.

5. Numerical results and discussion

In this section, we aim to illustrate the numerical results of the analytical expressions obtained in the previous section and explain the influence of fractional orders and phase-lag parameters on the behavior of the field quantities

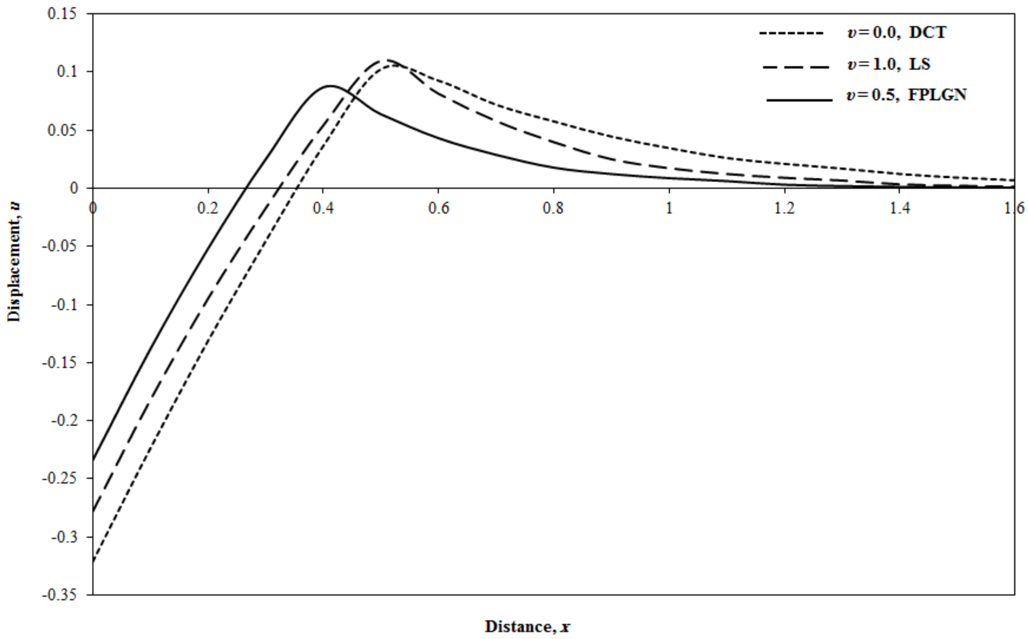

Fig. 3 The variation of displacement for different theories

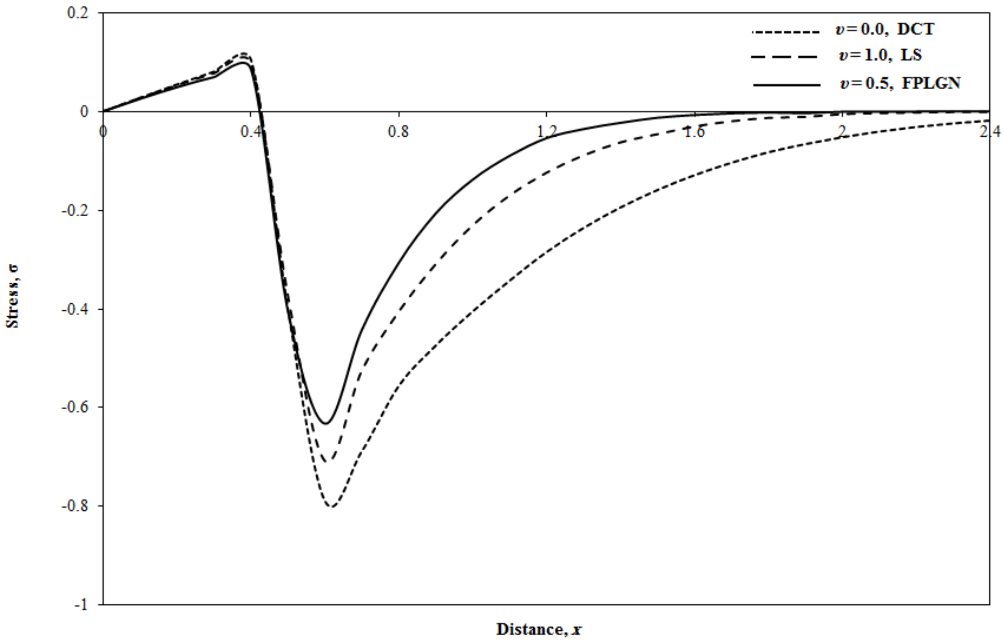

Fig. 4 The variation of stress for different theories

in different types of GN theory.

The method based on a Fourier series expansion proposed by Honig and Hirdes (1984) and developed in detail in many works such as Ezzat (2001) and Ezzat and Abd-Elaal (1997a, b) is adopted to invert the Laplace transform in Eqs. (45)-(51).

In order to interpret the numerical computations, we consider material properties of copper-like material, whose physical data is given in Table 1 as in Refs. (Ezzat and El-Bary 2016).

Considering the above physical data, we have evaluated the numerical values of the field quantities with the help of a computer program developed by using *FORTRAN*

Table 1 Values of the constants

k = 386 N/Ks,	α_T = 1.78 $(10)^{-5}$ K^{-1},	C_E = 383.1 m ^{2}/K,
η = 8886.73 s/m^2,	T_o = 293 K,	μ = 3.86 $(10)^{10}$ N/m^2,
λ = 7.76 $(10)^{10}$ N/m^2,	ρ = 8954 kg/m^3,	C_o = 4158 m/s,
ε = 0.0168,	τ_o = 0.02 sec,	τ_α = 0.03 sec,
c = 415 m/s, k^* = 10,		μ_o = 21.256×10^6 Ns^2/C^2,
ε_o = 8.854×10^{-12} C^2/Nm^2,		$\mu_o H_o$ = 1 Tesla,
a_o = 1.01,	a_o = 1.01,	

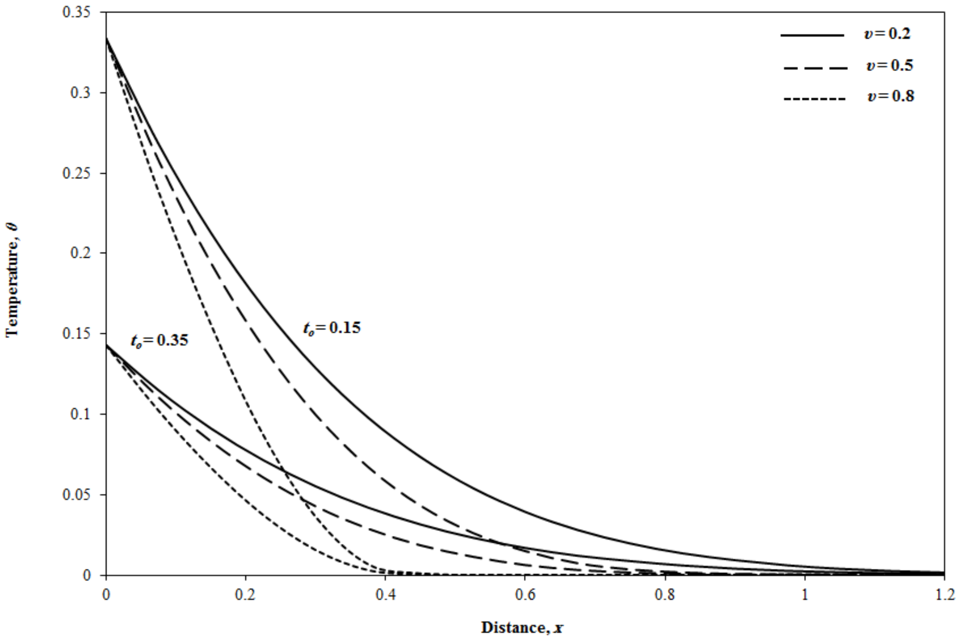

Fig. 5 The variation of temperatures for fractional phase-lag unified GN-theory

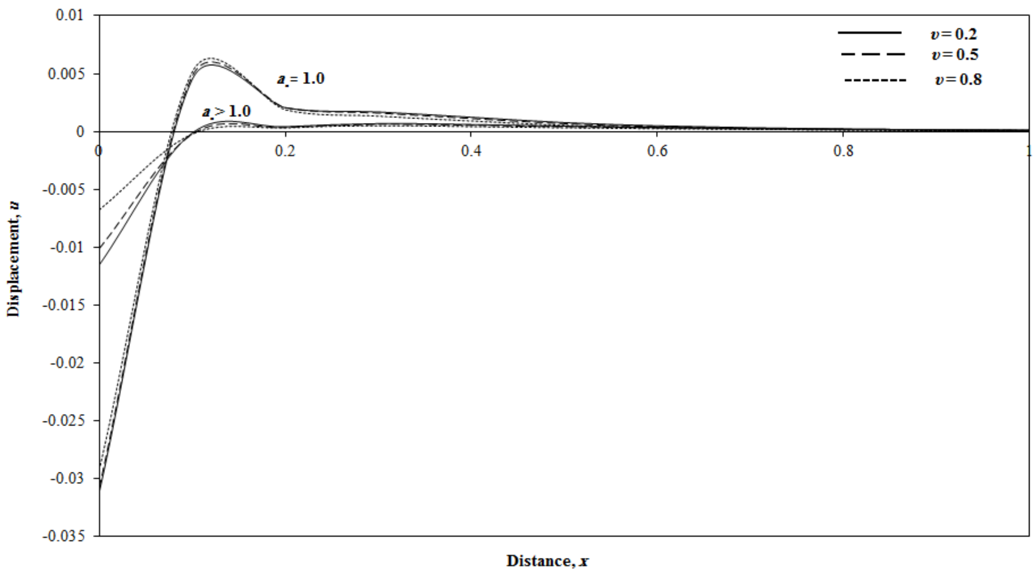

Fig. 6 The stress for different values of fractional order and Alfven velocity

language. The accuracy maintained was seven digits for the numerical program.

The calculations were carried out for a thermal shock problem for half-space subjected to ramp- type heating as

$$h(t)=\begin{cases}0 & 0\le t\\ \theta_1\dfrac{t}{t_o} & 0\le t\le t_o\\ \theta_1 & t>0\end{cases}\quad\text{or}\quad \bar{h}(s)=\frac{\theta_1(1-e^{-st_o})}{t_o s^2}$$

The computations were performed for values of time (t = 0.1), ramping parameter (t_o = 0.15, 0.25, 0.35, 0.45), fractional order (υ = 0.0, 0.2, 0.5, 0.8, 1.0) and Alfven velocity (α_o = 0.0, 5.0). The numerical technique outlined above was used to obtain the heat flux, temperature, stress and displacement distributions as well as the induced electric and magnetic fields. The results are displayed graphically at different positions of x as shown in Figs. 1-9.

The important phenomenon observed in all figures that the solution of any of the considered function in the unified fractional phase-lag model of GN theory is restricted in a bounded region. Beyond this region, the variations of these distributions do not take place. This means that the solutions according the new model exhibit the behavior of finite speeds of wave propagation.

Figs. 1-4 indicate the variation in heat flux, temperature, displacement and stress distributions with distance x for one value of time ($t = 0.1$) for different theories. In these figures, dotted line represents the solution obtained in the frame of

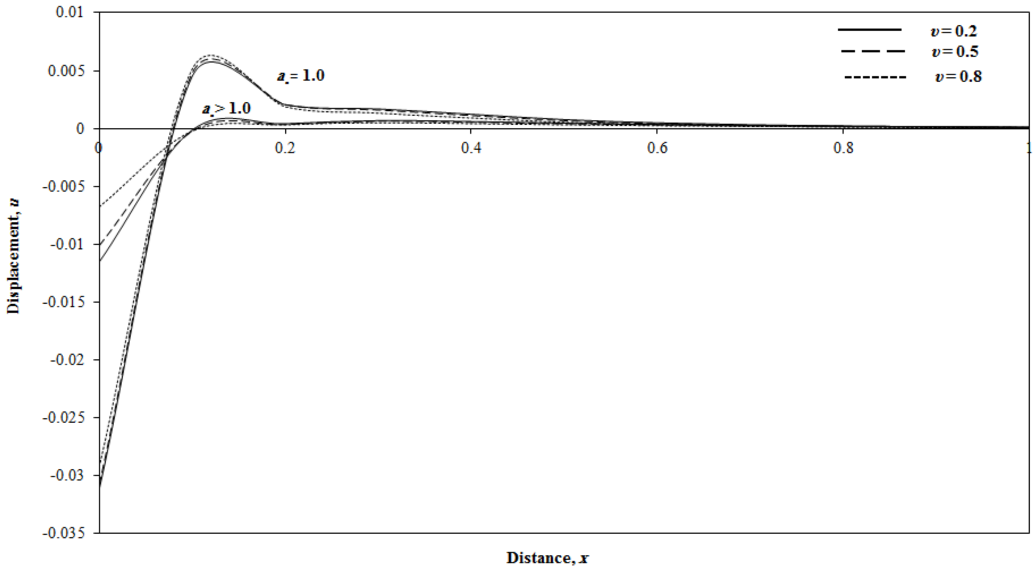

Fig. 6 The stress for different values of fractional order and Alfven velocity

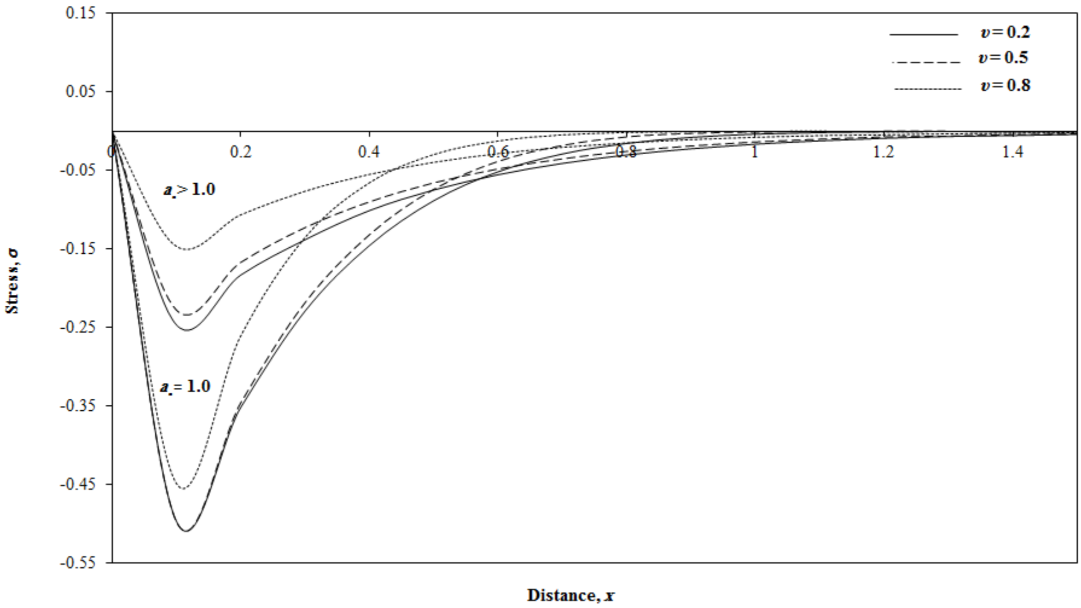

Fig. 7 The stress for different values of fractional order and Alfven velocity

dynamic coupled theory ($\upsilon = 0$, $\tau_o = 0$) (Biot 1956) and dashed line represents the solution obtained in the frame of generalized thermoelasticity with thermal relaxation time ($\upsilon = 1.0$, $\tau_o = 0.02$) (Lord and Shulman 1967), while the solid line represents the solution obtained in the frame of the new unified fractional phase-lag model of GN theory ($0 < \upsilon < 1$, $\tau_q = 0.02$, $\tau_\alpha = 0.03$). A comparison was made between the previous three theories and we observed that the thermal and mechanical waves are continuous functions smooth and reach to steady state depending on the value of fractional parameter υ.

Figs. 5-9 display temperature, displacement and stress distributions as well as induced magnetic and electric fields with distance x for different values of fractional order υ = 0.2, 0.5, 0.8. We noticed from these figures that these fields have been affected by the time-fractional parameter, where the increasing of the value of the parameter υ causes decreasing in temperature. It observed that the displacement, stress, induced magnetic and electric fields have the same behavior as temperature fields except the wide range of x. The effects of ramping parameter on the temperature and on both of magnetic and electric fields are studied in Figs. 5, 8 and 9. It is found that the temperature and the induced magnetic and electric fields increase when the value of the ramping parameter t_o decreases. Also, the effects of Alfven velocity on the displacement and stress distributions are shown in Figs. 6 and 7. One can found that the magnetic field acts to decrease the displacement and stress distributions.

6. Conclusions

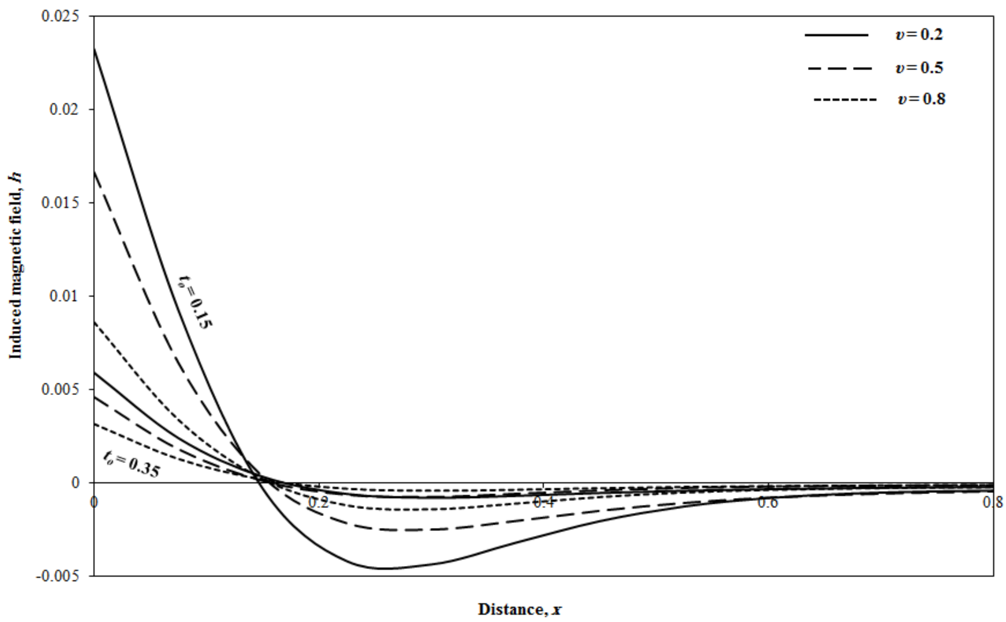

Fig. 8 The variation of induced magnetic field for different values of fractional orders υ and ramping parameter t_o

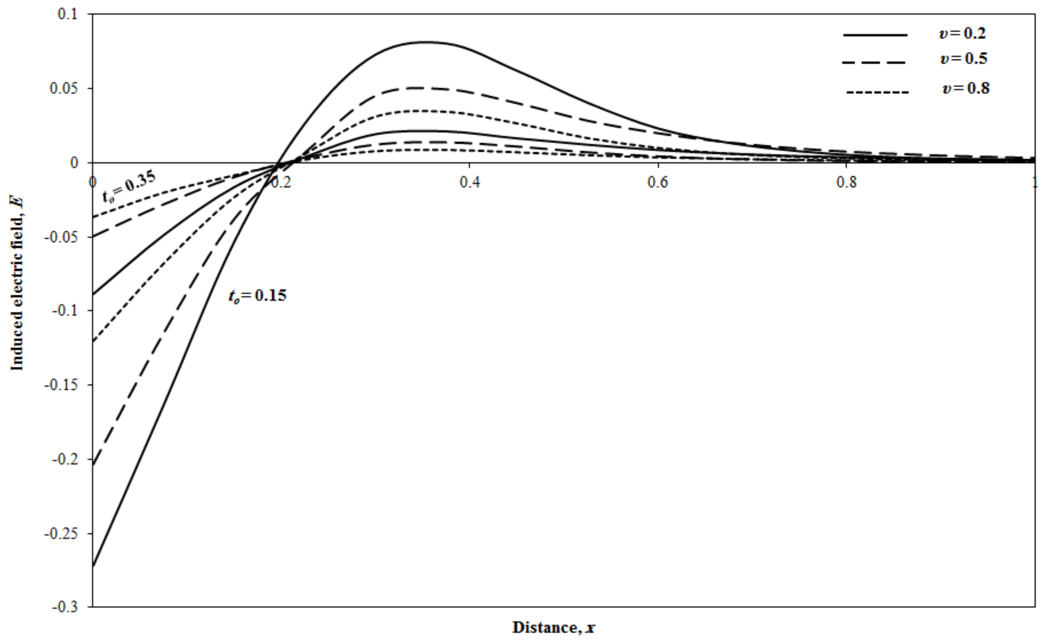

Fig. 7 The variation of induced electric field for different values of fractional orders υ and ramping parameter t_o

(1) The main goal of this work is to introduce a unified mathematical model for GN-thermoelasticity theory with time-derivative fractional order.
(2) Five generalized GN thermoelasticity theories: Single-phase-lag GN theory of type III, Dual-phase-lag GN theory of type II, Dual-phase-lag GN theory of type III, GN theory of type II without energy dissipation and GN theory of type III which admit thermal wave propagation with finite speed are proposed.
(3) For the fractional GN thermoelasticity theory $0 < \upsilon < 1$, the solution seems to behave like the generalized theory of generalized thermoelasticity (LS theory). This result is very important that the new unified model may preserve the advantage of the generalized theory that the velocity of waves is finite. The presence of the magnetic field which acts to the perfect conducting elastic medium raises the velocity of the dilational elastic waves from $v = [(\lambda + 2\mu)/\rho]^{1/2}$, speed of propa-gation of longitudinal waves to $C_o = \sqrt{v^2 + \alpha_o^2}$. The modified electromagnetic elastic wave is propagated with velocity C_o, and that is, with the same velocity as the modified elastic wave that produces a jump in stress. The magnetic field acts to decrease the magnitude of the displacement and thermal stress. This is mainly due to the fact that the magnetic

field corresponds to a term signifying a positive force that tends to accelerate the charge carries.

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Nomenclature

a_1, a_2, a_3	key numbers, each equals to 0 or 1
x	$= (x_1, x_2, x_3)$, position
t	time
H	magnetic field intensity vector
E	electric field intensity vector
J	conduction electric density vector
H_o	constant component of magnetic field
σ_o	electrical conductivity
μ_o	magnetic permeability
ε_o	electric permeability
C_E	specific heat at constant strain
k_{ij}	thermal conductivity tensor
k_{ij}^*	conductivity rate tensor
T	temperature
T_o	reference temperature
u_i	components of displacement vector
v	$= [(\lambda+2\mu)/\rho]^{1/2}$, speed of propagation of longitudinal waves
q_i	components of heat flux vector
e	$= u_{i,i}$, dilatation
c	$= (1/\mu_o\varepsilon_o)^{1/2}$, speed of light
C_o	$= \sqrt{v^2+\alpha_o^2}$

Greek symbols

λ, μ	Lame's constants
ρ	density
α_T	coefficient of linear thermal expansion
α	thermal displacement $\dot{\alpha} = \theta$
ε	thermoelastic coupling parameter
γ	$= (3\lambda = 2\mu)\ \alpha_T$
δ_{ij}	Kronecker delta function
τ_o	relaxation time
τ_α	phase-lag of the thermal displacement gradient
τ_q	phase-lag of the heat flux
υ	fractional derivative order
σ_{ij}	components of stress tensor
η	$= \rho C_E/k$
θ	$= T - T_0$, such that $\lvert\theta / T_0\rvert << 1$
α_o	$= (\mu_o H_o^2/\rho)^{1/2}$, Alfven velocity