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Buckling of symmetrically laminated plates using *n*th-order shear deformation theory with curvature effects

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Abstract. In this article, an exact analytical solution for mechanical buckling analysis of symmetrically cross-ply laminated plates including curvature effects is presented. The equilibrium equations are derived according to the refined *n*th-order shear deformation theory. The present refined *n*th-order shear deformation theory is based on assumption that the in-plane and transverse displacements consist of bending and shear components, in which the bending components do not contribute toward shear forces and, likewise, the shear components do not contribute toward bending moments The most interesting feature of this theory is that it accounts for a parabolic variation of the transverse shear strains across the thickness and satisfies the zero traction boundary conditions on the top and bottom surfaces of the plate without using shear correction factors. Buckling of orthotropic laminates subjected to biaxial inplane is investigated. Using the Navier solution method, the differential equations have been solved analytically and the critical buckling loads presented in closed-form solutions. The sensitivity of critical buckling loads to the effects of curvature terms and other factors has been examined. The analysis is validated by comparing results with those in the literature.

Keywords: symmetrically cross-ply laminated; refined *n*th-order shear deformation theory; buckling; curvature terms

1. Introduction

Laminated composite plates are widely used in the aerospace, automotive, marine and other structural applications because of advantageous features such as high ratio of stiffness and strength to weight and low maintenance cost. In company with the increase in the application of laminates in engineering structures. In addition, Plate elements are commonly used in civil, mechanical, aeronautical and marine structures. The considerations of natural frequencies and buckling loads for rectangular plates are essential to have an efficient and reliable design. In the past three decades, researches on composite laminated plates have received great attention, and a variety of

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plate theories has been introduced based on considering the transverse shear deformation effect. The classical plate theory (CPT), which neglects the transverse shear deformation effect, provides reasonable results for thin plate. Mindlin (1951) and Reissner (1945) developed FSDTs which incorporate the effect of shear deformation. In these theories, the transverse shear strain distribution is assumed to be constant through the plate thickness and therefore it requires shear correction factor in order to satisfy traction free boundary conditions at top and bottom surfaces of plates.

Different higher-order shear deformation plate theories (HSDT) were proposed, including the second-order shear deformation formulation of Whitney and Sun (1973) and the third-order shear deformation theory of Lo *et al*. (1977) with 11 unknowns; Kant (1982) with six unknowns; Bhimaraddi and Stevens (1984) with five unknowns; Reddy (Reddy 1984, Reddy and Phan 1985), with five unknowns and Hanna and Leissa (1994) with four unknowns. Ambartsumian (1958), proposed a transverse shear stress function in order to explain plate deformation. A similar method was used later by Soldatos and Timarci (1993), for dynamic analysis of laminated shells. Later some new functions were proposed by (Reddy 1984, Touratier 1991, Karama *et al*. 2003, Soldatos 1992, Aydogdu 2009, Senthilnathan *et al*. 1987, Xiang *et al*. 2011a and Mantari *et al*. 2012). The multiquadrics Radial basis functions (RBFs) method were applied to analyze the laminated composite plates by (Ferreira 2005a, b, Ferreira *et al*. 2003, 2004, 2005, Ferreira and Fasshauer 2006). Inverse multiquadric RBFs were used to analyze composite plates by Xiang (Xiang and Wang 2009). Vel and Batra (2004) presented the three dimensional exact solution for the vibration of functionally graded rectangular plates. Zenkour (2006) proposed a generalized shear deformation theory for bending analysis of functionally graded plates. Whitney (1987) proposed a curvature Effects in the Buckling of Symmetrically-Laminated Rectangular Plates with Transverse Shear Deformation. Whitney and Pagano (1970) presented the shear deformation in heterogeneous anisotropic plates. Liew *et al*. (1996) presented the Navier's solution for laminated plate buckling with prebuckling in-plane deformation.

On the other hand, A two variable refined plate theory (RPT) was first developed for isotropic plates by Shimpi (2002), and was extended to orthotropic plates by (Shimpi and Patel 2006a, b, Kim *et al*. 2009), and Thai and Kim (2010) have studied laminated composite plates using this theory. Ait Amar Meziane *et al*. (2014) studied the buckling and free vibration response of exponentially graded sandwich plates under various boundary conditions. Narendar (2011) studied the mechanical buckling of the nanoplates and Thai (2012) developed a nonlocal refined beam theory for nanobeams based on this theory. Bouazza and Benseddiq (2015) investigated an analytical modeling for the thermoelastic buckling behavior of functionally graded rectangular plates (FGM) under thermal loadings. Bouazza *et al*. (2016) developed an analytical solution of refined hyperbolic shear deformation theory to obtain the critical buckling temperature of cross-ply laminated plates with simply supported edge. Belabed *et al*. (2014) presented an efficient and simple higher order shear and normal deformation theory for functionally graded material (FGM) plates. Thai and Choi (2012) developed the efficient and simple refined theory for buckling analysis of functionally graded plates. Hebali *et al*. (2014) proposed a new quasi-3D hyperbolic shear deformation theory for the static and free vibration analysis of functionally graded plates. Bennoun *et al*. (2016) used a novel five variable refined plate theory for vibration analysis of functionally graded sandwich plates. Mahi *et al*. (2015) used a new hyperbolic shear deformation theory for bending and free vibration analysis of isotropic, functionally graded, sandwich and laminated composite plates. Bourada *et al*. (2015) have developed a new simple shear and normal deformations theory for functionally graded beams. Piscopo (2010) also investigated refined buckling analysis of rectangular plates under uniaxial and biaxial compression. Hassaine Daouadji *et al*. (2012) used a higher order theory which involves only four degrees of freedom for bending analysis of functionally graded plates.

Some researchers have used the four variable refined plate theory for buhafior analysis of the thick functionally graded plates (Ait Yahia 2015, Zidi *et al*. 2014, Attia *et al*. 2015, Boukhari *et al*. 2016). Tounsi *et al*. (2013) studied bending response of FGM sandwich plates by the use of a new four variable refined plate theory under thermal and thermomechanical loading, respectively. Hamidi *et al*. (2015) studied the thermomechanical bending of functionally graded sandwich plates by using a sinusoidal plate theory with 5-unknowns and stretching effect. Bousahla *et al*. (2014) proposed a novel higher order shear including the neutral surface position for the static analysis of advanced composite plates such as functionally graded plates. Bellifa *et al*. (2016) studied the Bending and free vibration analysis of functionally graded plates using a simple shear deformation theory and the concept the neutral surface position. Also in some studies the elastic foundation theory issued in modeling (Bounouara *et al*. 2016, Bakora and Tounsi 2015, Chikh *et al*. 2016, Bouderba *et al*. 2013). Ait Atmane *et al*. (2015) studied a computational shear displacement model for vibrational of functionally graded beams with porosities. Al-Basyouni *et al*. (2015) studied the bending and vibration behaviors of size-dependent nano beams made of functionally graded materials (FGMs) including the thickness stretching effect. The size-dependent FGM nanobeam was investigated on the basis of the nonlocal continuum model. Draiche *et al*. (2014) proposed a trigonometric four variable plate theory to study free vibration of rectangular composite plates with patch mass. Klouche Djedid *et al*. (2014) studied the bending and free vibration of functionally graded plates by using an n-order four variable refined theory.

The present paper deals with the *n*th-order shear deformation theory (Xiang *et al*. 2011a). The effectiveness and accuracy of this theory is demonstrated by (Xiang *et al*. 2011b, 2012, 2013a, b). Moreover, the present paper mainly uses the ideas behind the new refined plate theory (Shimpi 2002) that the authors include w_b and w_s (bending and shear transverse displacement) to model the transverse displacement of the shear deformation theories (in many theories assumed constant and called w0 (Mantari *et al*. 2012, Xiang *et al*. 2011a, 2012, 2013a, b). In the present paper, the authors combine this idea for developing the *n*th-order shear deformation theory with modified displacement field to its optimization. Unlike other theories, there are only four unknown functions involved, as compared to five in other shear deformation theories. The theory presented is variationally consistent and strongly similar to the classical plate theory in many aspects. It does not require the shear correction factor, and gives rise to the transverse shear stress variation so that the transverse shear stresses vary parabolically across the thickness to satisfy free surface conditions for the shear stress. Closed form solutions for mechanical buckling analysis of mechanical buckling analysis of symmetrically cross-ply laminated plates including curvature effects are obtained. Numerical examples are presented to verify the accuracy of the present theory.

2. THEORETICAL FORMULATION

2.1 Kinematics

In this study, further simplifying assumptions are made to the *n*th-order shear deformation theory so that the number of unknowns is reduced. The displacement field of the conventional *n*th-order shear deformation theory is given by Xiang *et al*. 2011a.

$$u_1(x,y,z)=z\phi_x(x,y)-\frac{1}{n}\left(\frac{2}{h}\right)^{n-1}z^n\left(\phi_x(x,y)+\frac{\partial w(x,y)}{\partial x}\right)$$
$$u_2(x,y,z)=z\phi_y(x,y)-\frac{1}{n}\left(\frac{2}{h}\right)^{n-1}z^n\left(\phi_y(x,y)+\frac{\partial w(x,y)}{\partial y}\right) \tag{1}$$
$$n=3,5,7,9,.......$$
$$u_3(x,y,z)=w_0(x,y)$$

where w_0, ϕ_x and ϕ_y are there unknown displacement functions of the mid-plane of the plate; and h is the thickness of the plate. By dividing the transverse displacement w_0 into bending and shear parts (i.e., $w_0 = w_b + w_s$) and making further assumptions given by $\phi_x = -\partial w_b / \partial x$ and $\phi_y = -\partial wb_b / \partial y$, the displacement field of the new refined theory can be rewritten in a simpler form as

$$u_1(x,y,z)=-z\frac{\partial w_b}{\partial x}-\frac{1}{n}\left(\frac{2}{h}\right)^{n-1}z^n\left(\frac{\partial w_s}{\partial x}\right)$$
$$u_2(x,y,z)=-z\frac{\partial w_b}{\partial y}-\frac{1}{n}\left(\frac{2}{h}\right)^{n-1}z^n\left(\frac{\partial w_s}{\partial y}\right) \tag{2}$$
$$n=3,5,7,9,.......$$
$$u_3(x,y,z)=w_b(x,y)+w_s(x,y)$$

Clearly, the displacement field in Eq. (3) contains only two unknowns, w_b and w_s. The nonzero strains associated with the displacement field in Eq. (3) are

$$\{\varepsilon\}=\{k^b\}z+\{k^s\}f(z)$$
$$\begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}=\begin{Bmatrix}\gamma_{yz}^s\\ \gamma_{xz}^s\end{Bmatrix}g(z) \tag{3}$$

where

$$\{\varepsilon\}=\{\varepsilon_x,\varepsilon_y,\gamma_{xy}\}^T$$
$$\{k^b\}=\{k_x^b,k_y^b,k_{xy}^b\}^T=\left\{-\frac{\partial^2 w_b}{\partial x^2},-\frac{\partial^2 w_b}{\partial y^2},-2\frac{\partial^2 w_b}{\partial x\partial y}\right\}^T$$
$$\{k^s\}=\{k_x^s,k_y^s,k_{xy}^s\}^T=\left\{-\frac{\partial^2 w_s}{\partial x^2},-\frac{\partial^2 w_s}{\partial y^2},-2\frac{\partial^2 w_s}{\partial x\partial y}\right\}^T \tag{4}$$
$$\gamma_{xz}^s=\frac{\partial w_s}{\partial x},\ \gamma_{yz}^s=\frac{\partial w_s}{\partial y}$$

$$f(z) = \frac{1}{n}\left(\frac{2}{h}\right)^{n-1} z^n, \; g(z) = 1 - \left(\frac{2z}{h}\right)^{n-1} \tag{4}$$

2.2 Constitutive equations

Under the assumption that each layer possesses a plane of elastic symmetry parallel to the *x–y* plane, the constitutive equations for a layer can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{5}$$

$$\begin{gathered} Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}} \\ Q_{66} = G_{12}, Q_{44} = G_{23}, \quad Q_{55} = G_{13} \end{gathered} \tag{6}$$

Transforming the above equations of an arbitrary *k* layer in local coordinate system into the global coordinate system, the laminate constitutive equations can be expressed as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix}^{(k)} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} & 0 & 0 \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} & 0 & 0 \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} \\ 0 & 0 & 0 & \overline{Q}_{45} & Q_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^{(k)} \tag{7}$$

where usual notations for normal and shear stress components are adopted. $\overline{Q}_{ij}$ are the transformed material constants given as (Reddy 1997, Jones 1975).

$$\begin{aligned} \overline{Q}_{11} &= Q_{11}\cos^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + +Q_{22}\sin^4\theta \\ \overline{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta + \cos^4\theta) \\ \overline{Q}_{22} &= Q_{11}\sin^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta \\ \overline{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})\sin\theta\cos^3\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin^3\theta\cos\theta \\ \overline{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})\sin^3\theta\cos\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin\theta\cos^3\theta \\ \overline{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta + \cos^4\theta) \\ \overline{Q}_{44} &= Q_{44}\cos^2\theta + Q_{55}\sin^2\theta \end{aligned} \tag{8}$$

$$\begin{aligned}\overline{Q}_{45} &= (Q_{55} - Q_{44})\cos\theta\sin\theta \\ \overline{Q}_{55} &= Q_{55}\cos^2\theta + Q_{44}\sin^2\theta\end{aligned} \tag{8}$$

2.3 Governing equation

The strain energy of the plate is calculated by

$$2U = \int_V \sigma_{ij}\varepsilon_{ij}dV = \int_V \left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_{xy}\gamma_{xy} + \sigma_{yz}\gamma_{yz} + \sigma_{xz}\gamma_{xz}\right)dV \tag{9}$$

Substituting Eq. (3) into Eq. (9) and integrating through the thickness of the plate, the strain energy of the plate can be rewritten as

$$2U = \int_A \left[M_x^b\delta k_x^b + M_y^b\delta k_y^b + M_{xy}^b k_{xy}^b + M_x^s k_x^s + M_y^s k_y^s + M_{xy}^s k_{xy}^s + Q_{yz}\gamma_{yz}^s + Q_{xz}\gamma_{xz}^s\right]dxdy \tag{10}$$

where (M_x^b, M_y^b, M_{xy}^b), (M_x^s, M_y^s, M_{xy}^s) denote the total moment resultants and (Q_{xz}, Q_{yz}) are transverse shear stress resultants and they are defined as

$$\begin{aligned}(M_x^b, M_y^b, M_{xy}^b) &= \int_{-h/2}^{h/2}(\sigma_x, \sigma_y, \sigma_{xy})zdz \\ (M_x^s, M_y^s, M_{xy}^s) &= \int_{-h/2}^{h/2}(\sigma_x, \sigma_y, \sigma_{xy})f(z)dz \\ (Q_{xz}, Q_{yz}) &= \int_{-h/2}^{h/2}(\sigma_{xz}, \sigma_{yz})\,g(z)dz\end{aligned} \tag{11}$$

From Eq. (11), one can obtain the following equations

$$\begin{bmatrix} M_x^b \\ M_y^b \\ M_{xy}^b \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} -\dfrac{\partial^2 w_b}{\partial x^2} \\ -\dfrac{\partial^2 w_b}{\partial y^2} \\ -2\dfrac{\partial^2 w_b}{\partial x\partial y} \end{bmatrix} + \begin{bmatrix} D_{11}^s & D_{12}^s & D_{16}^s \\ D_{12}^s & D_{22}^s & D_{26}^s \\ D_{16}^s & D_{26}^s & D_{66}^s \end{bmatrix} \begin{bmatrix} -\dfrac{\partial^2 w_s}{\partial x^2} \\ -\dfrac{\partial^2 w_s}{\partial y^2} \\ -2\dfrac{\partial^2 w_s}{\partial x\partial y} \end{bmatrix} \tag{12}$$

$$\begin{bmatrix} M_x^s \\ M_y^s \\ M_z^s \end{bmatrix} = \begin{bmatrix} D_{11}^s & D_{12}^s & D_{16}^s \\ D_{12}^s & D_{22}^s & D_{26}^s \\ D_{16}^s & D_{26}^s & D_{66}^s \end{bmatrix} \begin{bmatrix} -\dfrac{\partial^2 w_b}{\partial x^2} \\ -\dfrac{\partial^2 w_b}{\partial y^2} \\ -2\dfrac{\partial^2 w_b}{\partial x\partial y} \end{bmatrix} + \begin{bmatrix} H_{11}^s & H_{12}^s & H_{16}^s \\ H_{12}^s & H_{22}^s & H_{26}^s \\ H_{16}^s & H_{26}^s & H_{66}^s \end{bmatrix} \begin{bmatrix} -\dfrac{\partial^2 w_s}{\partial x^2} \\ -\dfrac{\partial^2 w_s}{\partial y^2} \\ -2\dfrac{\partial^2 w_s}{\partial x\partial y} \end{bmatrix} \tag{13}$$

$$\begin{bmatrix} Q_y \\ Q_x \end{bmatrix} = \begin{bmatrix} A_{44}^s & A_{45}^s \\ A_{45}^s & A_{55}^s \end{bmatrix} \begin{bmatrix} \partial w_s / \partial y \\ \partial w_s / \partial x \end{bmatrix} \tag{14}$$

where

$$\begin{gathered} (D_{ij}, D_{ij}^s, H_{ij}^s) = \int_{-h/2}^{h/2} \overline{Q}_{ij} (z^2, zf(z), (f(z))^2) dz \quad (i, j = 1,2,6) \\ A_{ij}^s = \int_{-h/2}^{h/2} \overline{Q}_{ij} (g(z))^2 \, dz \qquad (i, j = 4,5) \end{gathered} \tag{15}$$

Substituting Eqs. (12)-(14) and (4) to Eq. (10), the strain energy per unit area, U, due to the buckling deformation is of the form

$$\begin{aligned} 2U = {} & D_{11} \frac{\partial^4 w_b}{\partial x^4} + D_{22} \frac{\partial^4 w_b}{\partial y^4} + 2\left(D_{12} + 2D_{66}\right) \frac{\partial^4 w_b}{\partial x^2 \partial y^2} + 4D_{16} \frac{\partial^4 w_b}{\partial x^3 \partial y} + 4D_{26} \frac{\partial^4 w_b}{\partial x \partial y^3} \\ & + 2D_{11}^s \frac{\partial^2 w_b}{\partial x^2} \frac{\partial^2 w_s}{\partial x^2} + 2D_{22}^s \frac{\partial^2 w_b}{\partial y^2} \frac{\partial^2 w_s}{\partial y^2} + 2D_{12}^s \frac{\partial^2 w_b}{\partial x^2} \frac{\partial^2 w_s}{\partial y^2} + 2D_{12}^s \frac{\partial^2 w_b}{\partial y^2} \frac{\partial^2 w_s}{\partial x^2} \\ & + 4D_{16}^s \frac{\partial^2 w_b}{\partial x^2} \frac{\partial^2 w_s}{\partial x \partial y} + 4D_{16}^s \frac{\partial^2 w_b}{\partial x \partial y} \frac{\partial^2 w_s}{\partial x^2} + 4D_{26}^s \frac{\partial^2 w_b}{\partial y^2} \frac{\partial^2 w_s}{\partial x \partial y} + 4D_{26}^s \frac{\partial^2 w_b}{\partial x \partial y} \frac{\partial^2 w_s}{\partial y^2} + 8D_{66}^s \frac{\partial^2 w_b}{\partial x \partial y} \frac{\partial^2 w_s}{\partial x \partial y} \\ & H_{11}^s \frac{\partial^4 w_s}{\partial x^4} + H_{22}^s \frac{\partial^4 w_s}{\partial y^4} + 2\left(H_{12}^s + 2H_{66}^s\right) \frac{\partial^4 w_s}{\partial x^2 \partial y^2} + 4H_{16}^s \frac{\partial^4 w_s}{\partial x^3 \partial y} + 4H_{26}^s \frac{\partial^4 w_s}{\partial x \partial y^3} + \\ & A_{44}^s \frac{\partial^2 w_s}{\partial y^2} + A_{55}^s \frac{\partial^2 w_s}{\partial x^2} + 2A_{45}^s \frac{\partial^2 w_s}{\partial x \partial y} \end{aligned} \tag{16}$$

The potential energy of the applied in-plane stresses σ_x^0, σ_y^0 and τ_{xy}^0 arises from the action of the applied d stresses on the corresponding second order strain $\varepsilon_x^{Nl}, \varepsilon_y^{Nl}$ and γ_{xy}^{Nl}. Following the usual procedure (Whitney 1987, Whitney and Pagano 1970, Dawe and Roufaeil 1982), after taking into account the displacement field given by Eq. (1)

$$\begin{aligned} \varepsilon_x^{Nl} = {} & \frac{z^2}{2} \left[\frac{\partial^4 w_b}{\partial x^4} + \frac{\partial^4 w_b}{\partial x^2 \partial y^2} \right] + zf(z) \left[\frac{\partial^2 w_b}{\partial x^2} \frac{\partial^2 w_s}{\partial x^2} + \frac{\partial w_b^2}{\partial x \partial y} \frac{\partial^2 w_s}{\partial x \partial y} \right] \\ & + \frac{f(z)^2}{2} \left[\frac{\partial^4 w_s}{\partial x^4} + \frac{\partial^4 w_s}{\partial x^2 \partial y^2} \right] + \frac{1}{2} \left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_s}{\partial x^2} + 2 \frac{\partial w_b}{\partial x} \frac{\partial w_s}{\partial x} \right) \end{aligned} \tag{17}$$

$$\begin{aligned} \varepsilon_y^{Nl} = {} & \frac{z^2}{2} \left[\frac{\partial^4 w_b}{\partial y^4} + \frac{\partial^4 w_b}{\partial x^2 \partial y^2} \right] + zf(z) \left[\frac{\partial^2 w_b}{\partial y^2} \frac{\partial^2 w_s}{\partial y^2} + \frac{\partial^2 w_b}{\partial x \partial y} \frac{\partial^2 w_s}{\partial x \partial y} \right] \\ & + \frac{f(z)^2}{2} \left[\frac{\partial^4 w_s}{\partial y^4} + \frac{\partial^4 w_s}{\partial x^2 \partial y^2} \right] + \frac{1}{2} \left(\frac{\partial^2 w_b}{\partial y^2} + \frac{\partial^2 w_s}{\partial y^2} + 2 \frac{\partial w_b}{\partial y} \frac{\partial w_s}{\partial y} \right) \end{aligned} \tag{18}$$

$$\begin{aligned}\gamma_{xy}^{Nl} &= z^2\left[\frac{\partial^4 w_b}{\partial x^3 \partial y}+\frac{\partial^4 w_b}{\partial x \partial y^3}\right]+zf(z)\left[\frac{\partial^2 w_b}{\partial x^2}\frac{\partial^2 w_s}{\partial x \partial y}+\frac{\partial^2 w_b}{\partial y^2}\frac{\partial^2 w_s}{\partial x \partial y}+\frac{\partial^2 w_s}{\partial x^2}\frac{\partial^2 w_b}{\partial x \partial y}+\frac{\partial^2 w_s}{\partial y^2}\frac{\partial^2 w_b}{\partial x \partial y}\right] \\ &+ f(z)^2\left[\frac{\partial^4 w_s}{\partial x^3 \partial y}+\frac{\partial^4 w_s}{\partial x \partial y^3}\right]+\left(\frac{\partial^2 w_b}{\partial x \partial y}+\frac{\partial^2 w_s}{\partial x \partial y}+\frac{\partial w_b}{\partial x}\frac{\partial w_s}{\partial y}+\frac{\partial w_b}{\partial y}\frac{\partial w_s}{\partial x}\right)\end{aligned} \tag{19}$$

The potential energy of the plate fiat of volume is

$$V=\int_{-h/2}^{h/2}\left(\sigma_x^0 \varepsilon_x^{Nl}+\sigma_y^0 \varepsilon_y^{Nl}+\tau_{xy}^0 \gamma_{xy}^{Nl}\right)dz \tag{20}$$

Denoting conventional inplane force terms by V_1 and "curvature" terms by V_2, then after combining Eqs. (17)-(19) and (20) we find that

$$V=V_1+V_2 \tag{21}$$

where

$$\begin{aligned}2V_1 &= N_x^0\left(\frac{\partial^2 w_b}{\partial x^2}+\frac{\partial^2 w_s}{\partial x^2}+2\frac{\partial w_b}{\partial x}\frac{\partial w_s}{\partial x}\right)+N_y^0\left(\frac{\partial^2 w_b}{\partial y^2}+\frac{\partial^2 w_s}{\partial y^2}+2\frac{\partial w_b}{\partial y}\frac{\partial w_s}{\partial y}\right) \\ &+ 2N_{xy}^0\left(\frac{\partial^2 w_b}{\partial x \partial y}+\frac{\partial^2 w_s}{\partial x \partial y}+\frac{\partial w_b}{\partial x}\frac{\partial w_s}{\partial y}+\frac{\partial w_b}{\partial y}\frac{\partial w_s}{\partial x}\right)\end{aligned} \tag{22}$$

$$2V_2=\int_{-h/2}^{h/2}\left(\begin{array}{l}\left\{\sigma_x^0\left[\frac{\partial^4 w_b}{\partial x^4}+\frac{\partial^4 w_b}{\partial x^2 \partial y^2}\right]+\sigma_y^0\left[\frac{\partial^4 w_b}{\partial y^4}+\frac{\partial^4 w_b}{\partial x^2 \partial y^2}\right]+2\tau_{xy}^0\left[\frac{\partial^4 w_b}{\partial x^3 \partial y}+\frac{\partial^4 w_b}{\partial x \partial y^3}\right]\right\}z^2 \\ \left\{\begin{array}{l}2\sigma_x^0\left[\frac{\partial^2 w_b}{\partial x^2}\frac{\partial^2 w_s}{\partial x^2}+\frac{\partial w_b^2}{\partial x \partial y}\frac{\partial^2 w_s}{\partial x \partial y}\right]+2\sigma_y^0\left[\frac{\partial^2 w_b}{\partial y^2}\frac{\partial^2 w_s}{\partial y^2}+\frac{\partial^2 w_b}{\partial x \partial y}\frac{\partial^2 w_s}{\partial x \partial y}\right] \\ 2\tau_{xy}^0\left[\frac{\partial^2 w_b}{\partial x^2}\frac{\partial^2 w_s}{\partial x \partial y}+\frac{\partial^2 w_b}{\partial y^2}\frac{\partial^2 w_s}{\partial x \partial y}+\frac{\partial^2 w_s}{\partial x^2}\frac{\partial^2 w_b}{\partial x \partial y}+\frac{\partial^2 w_s}{\partial y^2}\frac{\partial^2 w_b}{\partial x \partial y}\right]\end{array}\right\}zf(z) \\ \left\{\sigma_x^0\left[\frac{\partial^4 w_s}{\partial x^4}+\frac{\partial^4 w_s}{\partial x^2 \partial y^2}\right]+\sigma_y^0\left[\frac{\partial^4 w_s}{\partial y^4}+\frac{\partial^4 w_s}{\partial x^2 \partial y^2}\right]+2\tau_{xy}^0\left[\frac{\partial^4 w_s}{\partial x^3 \partial y}+\frac{\partial^4 w_s}{\partial x \partial y^3}\right]\right\}f(z)^2\end{array}\right)dz \tag{23}$$

In addition

$$\left(N_x^0, N_y^0, N_{xy}^0\right)=\int_{-h/2}^{h/2}\left\{\sigma_x^0, \sigma_y^0, \tau_{xy}^0\right\}dz \tag{24}$$

In order to put the integral in Eq. (23) in a useful form for heterogeneous plates, we utilize the constitutive relations for the inplane loading of a symmetrically-laminated plate (Whitney 1987, Whitney and Pagano 1970)

$$\begin{bmatrix} N_x^0 \\ N_y^0 \\ N_{xy}^0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} \tag{25}$$

where

$$A_{ij} = \int_{-h/2}^{h/2} \overline{Q}_{ij} dz \quad (i, j = 1,2,6) \tag{26}$$

Eq. (25) can now be written in the form

$$\varepsilon_{i,j}^0 = A_{jk}^* N_k^0 \qquad (j,k = 1,2,6) \tag{27}$$

where the repeated index denotes summation, and A_{jk}^* represents elements of the inverse matrix of A_{jk}. Denoting σ_x^0, σ_y^0 and τ_{xy}^0 by σ_1^0, σ_2^0 and σ_6^0, respectively, we can write the inplane ply constitutive relations in the form

$$\sigma_i^0 = \overline{Q}_{ij} \varepsilon_j^0 \qquad (i, j = 1,2,6) \tag{28}$$

Thus

$$\begin{aligned} \int_{-h/2}^{h/2} \sigma_i^0 z^2 dz &= \int_{-h/2}^{h/2} \overline{Q}_{ij} \varepsilon_j^0 z^2 dz = D_{ij} \varepsilon_j^0 \\ \int_{-h/2}^{h/2} \sigma_i^0 z f(z) dz &= \int_{-h/2}^{h/2} \overline{Q}_{ij} \varepsilon_j^0 z f(z) dz = D_{ij}^s \varepsilon_j^0 \\ \int_{-h/2}^{h/2} \sigma_i^0 f(z)^2 dz &= \int_{-h/2}^{h/2} \overline{Q}_{ij} \varepsilon_j^0 f(z)^2 dz = H_{ij}^s \varepsilon_j^0 \end{aligned} \tag{29}$$

Combining Eqs. (27) and (29), we find that

$$\begin{aligned} \int_{-h/2}^{h/2} \sigma_x^0 z^2 dz &= D_{ij} A_{jk}^* N_k^0 = F_{jk} N_k^0 \\ \int_{-h/2}^{h/2} \sigma_{i,j}^0 z f(z) dz &= D_{ij}^s A_{jk}^* N_k^0 = F_{jk}^s N_k^0 \\ \int_{-h/2}^{h/2} \sigma_{i,j}^0 f(z)^2 dz &= H_{ij}^s A_{jk}^* N_k^0 = F_{jk}^r N_k^0 \end{aligned} \tag{30}$$

where

$$\begin{aligned} F_{jk} &= D_{ij} A^*_{jk} \\ F^s_{jk} &= D^s_{ij} A^*_{jk} \\ F^r_{jk} &= H^s_{ij} A^*_{jk} \end{aligned} \tag{31}$$

Taking into account Eqs. (30) and (31), the "curvature" terms, Eq. (23), are of the form

$$\begin{aligned} 2V_2 =& \left(F_{11}N_x^0 + F_{12}N_y^0 + F_{16}N_{xy}^0\right)\left[\frac{\partial^4 w_b}{\partial x^4} + \frac{\partial^4 w_b}{\partial x^2 \partial y^2}\right] + \left(F_{12}N_x^0 + F_{22}N_y^0 + F_{26}N_{xy}^0\right)\left[\frac{\partial^4 w_b}{\partial y^4} + \frac{\partial^4 w_b}{\partial x^2 \partial y^2}\right] \\ &+ 2\left(F_{16}N_x^o + F_{26}N_y^0 + F_{66}N_{xy}^0\right)\left[\frac{\partial^4 w_b}{\partial x^3 \partial y} + \frac{\partial^4 w_b}{\partial x \partial y^3}\right] + 2\left(F^s_{11}N_x^0 + F^s_{12}N_y^0 + F^s_{16}N_{xy}^0\right)\left[\frac{\partial^2 w_b}{\partial x^2}\frac{\partial^2 w_s}{\partial x^2} + \frac{\partial w_b^2}{\partial x \partial y}\frac{\partial^2 w_s}{\partial x \partial y}\right] \\ &+ 2\left(F^s_{12}N_x^0 + F^s_{22}N_y^0 + F^s_{26}N_{xy}^0\right)\left[\frac{\partial^2 w_b}{\partial y^2}\frac{\partial^2 w_s}{\partial y^2} + \frac{\partial^2 w_b}{\partial x \partial y}\frac{\partial^2 w_s}{\partial x \partial y}\right] \\ &+ 4\left(F^s_{16}N_x^o + F^s_{26}N_y^0 + F^s_{66}N_{xy}^0\right)\left[\frac{\partial^2 w_b}{\partial x^2}\frac{\partial^2 w_s}{\partial x \partial y} + \frac{\partial^2 w_b}{\partial y^2}\frac{\partial^2 w_s}{\partial x \partial y} + \frac{\partial^2 w_s}{\partial x^2}\frac{\partial^2 w_b}{\partial x \partial y} + \frac{\partial^2 w_s}{\partial y^2}\frac{\partial^2 w_b}{\partial x \partial y}\right] \\ &\left(F^r_{11}N_x^0 + F^r_{12}N_y^0 + F^r_{16}N_{xy}^0\right)\left[\frac{\partial^4 w_s}{\partial x^4} + \frac{\partial^4 w_s}{\partial x^2 \partial y^2}\right] + \left(F^r_{12}N_x^0 + F^r_{22}N_y^0 + F^r_{26}N_{xy}^0\right)\left[\frac{\partial^4 w_s}{\partial y^4} + \frac{\partial^4 w_s}{\partial x^2 \partial y^2}\right] \\ &+ 2\left(F^r_{16}N_x^o + F^r_{26}N_y^0 + F^r_{66}N_{xy}^0\right)\left[\frac{\partial^4 w_s}{\partial x^3 \partial y} + \frac{\partial^4 w_s}{\partial x \partial y^3}\right] \end{aligned} \tag{32}$$

Governing equations can be obtained by applying the variational relationship

$$\delta U + \delta V_1 + \delta V_2 = 0 \tag{33}$$

Substituting Eqs. (16), (22) and (32) into Eq. (33), we obtain the following governing equations in operator form

$$\begin{aligned} L_{11} w_b + L_{12} w_s &= 0 \\ L_{12} w_b + L_{22} w_s &= 0 \end{aligned} \tag{34}$$

The linear operators L_{ij} are defined as follows

$$\begin{aligned} L_{11} =& \left(D_{11} + F_{11}N_x^0 + F_{12}N_y^0 + F_{16}N_{xy}^0\right)\left(\ \right)_{,xxxx} + 2\left(2D_{16} + F_{16}N_x^0 + F_{26}N_y^0 + F_{66}N_{xy}^0\right)\left(\ \right)_{,xxxy} \\ &+ \left(2D_{12} + 4D_{66} + F_{11}N_x^0 + F_{12}N_y^0 + F_{16}N_{xy}^0 + F_{12}N_x^0 + F_{22}N_y^0 + F_{26}N_{xy}^0\right)\left(\ \right)_{,xxyy} \\ &+ \left(D_{22} + F_{12}N_x^0 + F_{22}N_y^0 + F_{26}N_{xy}^0\right)\left(\ \right)_{,yyyy} + 2\left(2D_{26} + F_{16}N_x^0 + F_{26}N_y^0 + F_{66}N_{xy}^0\right)\left(\ \right)_{,xyyy} \\ &+ N_x^0\left(\ \right)_{,xx} + 2N_{xy}^0\left(\ \right)_{,xy} + N_y^0\left(\ \right)_{,yy} \end{aligned} \tag{35a}$$

$$
\begin{aligned}
L_{12} &= \left(D_{11}^{s} + F_{11}^{s}N_{x}^{0} + F_{12}^{s}N_{y}^{0} + F_{16}^{s}N_{xy}^{0}\right)(\ )_{,xxxx} + 2\left(2D_{16}^{s} + 2\left(F_{16}^{s}N_{x}^{0} + F_{26}^{s}N_{y}^{0} + F_{66}^{s}N_{xy}^{0}\right)\right)(\ )_{,xxxy} \\
&+ \left(2D_{12}^{s} + 4D_{66}^{s} + F_{11}^{s}N_{x}^{0} + F_{12}^{s}N_{y}^{0} + F_{16}^{s}N_{xy}^{0} + F_{12}^{s}N_{x}^{0} + F_{22}^{s}N_{y}^{0} + F_{26}^{s}N_{xy}^{0}\right)(\ )_{,xxyy} \\
&+ \left(D_{22}^{s} + F_{12}^{s}N_{x}^{0} + F_{22}^{s}N_{y}^{0} + F_{26}^{s}N_{xy}^{0}\right)(\ )_{,yyyy} + 2\left(2D_{26}^{s} + 2\left(F_{16}^{s}N_{x}^{0} + F_{26}^{s}N_{y}^{0} + F_{66}^{s}N_{xy}^{0}\right)\right)(\ )_{,xyyy} \\
&+ N_{x}^{0}(\ )_{,xx} + 2N_{xy}^{0}(\ )_{,xy} + N_{y}^{0}(\ )_{,yy}
\end{aligned}
\tag{35b}
$$

$$
\begin{aligned}
L_{22} &= \left(H_{11}^{s} + F_{11}^{r}N_{x}^{0} + F_{12}^{r}N_{y}^{0} + F_{16}^{r}N_{xy}^{0}\right)(\ )_{,xxxx} + \left(4H_{16}^{s} + 2\left(F_{16}^{r}N_{x}^{0} + F_{26}^{r}N_{y}^{0} + F_{66}^{r}N_{xy}^{0}\right)\right)(\ )_{,xxxy} \\
&+ \left(2H_{12}^{s} + 4H_{66}^{s} + F_{11}^{r}N_{x}^{0} + F_{12}^{r}N_{y}^{0} + F_{16}^{r}N_{xy}^{0} + F_{12}^{r}N_{x}^{0} + F_{22}^{r}N_{y}^{0} + F_{26}^{r}N_{xy}^{0}\right)(\ )_{,xxyy} \\
&+ \left(H_{22}^{s} + F_{12}^{r}N_{x}^{0} + F_{22}^{r}N_{y}^{0} + F_{26}^{r}N_{xy}^{0}\right)(\ )_{,yyyy} + \left(4H_{26}^{s} + 2\left(F_{16}^{r}N_{x}^{0} + F_{26}^{r}N_{y}^{0} + F_{66}^{r}N_{xy}^{0}\right)\right)(\ )_{,xyyy} \\
&+ A_{55}^{s}(\ )_{,xx} + A_{44}^{s}(\ )_{,yy} + 2A_{45}^{s}(\ )_{,xy} + N_{x}^{0}(\ )_{,xx} + 2N_{xy}^{0}(\ )_{,xy} + N_{y}^{0}(\ )_{,yy}
\end{aligned}
\tag{35c}
$$

2.4 Exact solutions of mechanical buckling for symmetric cross-ply plates

Consider a rectangular plate with the length a and width b which is subjected to in-plane loads. Therefore, the pre-buckling forces can be obtained using the equilibrium conditions as Leissa and Ayoub (1988)

$$
N_{x}^{0} = -N, \; N_{y}^{0} = \xi N, \; N_{xy}^{0} = 0 \qquad (N > 0) \tag{36}
$$

Where N the force per unit length is, ξ is the load parameter which indicates the loading conditions. Negative value for ξ indicate that plate is subjected to biaxial compressive loads while positive values are used for tensile loads. Also, zero value for ξ show uniaxial loading in x directions.

The exact solutions of Eqs. (32) and (33) for simply supported, symmetric cross-ply rectangular plates may be obtained by recognizing the following plate stiffness to have zero values

$$
\begin{aligned}
&A_{16} = A_{26} = D_{16} = D_{26} = D_{16}^{s} = D_{26}^{s} = 0 \\
&H_{16}^{s} = H_{26}^{s} = A_{45}^{s} = 0
\end{aligned}
\tag{37}
$$

By following the Navier solution procedure, the solutions to the problem are assumed to take the following forms

$$
\begin{aligned}
w_b(x,y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{bmn} \sin \lambda x \sin \mu y \\
w_s(x,y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{smn} \sin \lambda x \sin \mu y
\end{aligned}
\tag{38}
$$

where

$$
\lambda = \frac{m\pi x}{a}, \qquad \mu = \frac{m\pi x}{a} \tag{39}
$$

Substituting Eq. (38) into Eq. (34) for an symmetric cross-ply laminate, we obtain the following equations

$$\begin{aligned}
&\left[D_{11} - c_c N(F_{11} - \xi F_{12})\right]\left(w_b\right)_{,xxxx} + \left[2D_{12} + 4D_{66} - c_c N\left(\left(F_{11} + F_{12}\right) - \xi\left(F_{12} + F_{22}\right)\right)\right]\left(w_b\right)_{,xxyy} \\
&+ \left[D_{22} - c_c N(F_{12} - \xi F_{22})\right]\left(w_b\right)_{,yyyy} + \left[D_{11}^s - c_c N(F_{11}^s - \xi F_{12}^s)\right]\left(w_s\right)_{,xxxx} \\
&+ \left[2D_{12}^s + 4D_{66}^s - c_c N\left(\left(F_{11}^s + F_{12}^s\right) - \xi\left(F_{12}^s + F_{22}^s\right)\right)\right]\left(w_s\right)_{,xxyy} + \left[D_{22}^s - c_c N(F_{12}^s - \xi F_{22}^s)\right]\left(w_s\right)_{,yyyy} \\
&+ N_x^0\left(w_b\right)_{,xx} + N_y^0\left(w_b\right)_{,yy} + N_x^0\left(w_s\right)_{,xx} + N_y^0\left(w_s\right)_{,yy} = 0
\end{aligned} \tag{40}$$

$$\begin{aligned}
&\left[D_{11}^s - c_c N(F_{11}^s - \xi F_{12}^s)\right]\left(w_b\right)_{,xxxx} + \left[2D_{12}^s + 4D_{66}^s - c_c N\left(\left(F_{11}^s + F_{12}^s\right) - \xi\left(F_{12}^s + F_{22}^s\right)\right)\right]\left(w_b\right)_{,xxyy} \\
&+ \left[D_{22}^s - c_c N(F_{12}^s - \xi F_{22}^s)\right]\left(w_b\right)_{,yyyy} + \left[H_{11}^r - c_c N(F_{11}^r - \xi F_{12}^r)\right]\left(w_s\right)_{,xxxx} \\
&+ \left[2H_{12}^s + 4H_{66}^s - c_c N\left(\left(F_{11}^r + F_{12}^r\right) - \xi\left(F_{12}^r + F_{22}^r\right)\right)\right]\left(w_s\right)_{,xxyy} + \left[H_{22}^s - c_c N(F_{12}^r - \xi F_{22}^r)\right]\left(w_s\right)_{,yyyy} \\
&+ N_x^0\left(w_b\right)_{,xx} + N_y^0\left(w_b\right)_{,yy} + \left[A_{55}^s + N_x^0\right]\left(w_s\right)_{,xx} + \left[A_{44}^s + N_y^0\right]\left(w_s\right)_{,yy} = 0
\end{aligned} \tag{41}$$

where c_c takes on the value 1 when the "curvature" terms are included in the analysis and is 0 when these terms are neglected.

After substituting the Eq. (38) into Eqs. (40) and (41) we get a systems of two equations for finding the W_{bmn} and W_{smn}. By equaling the determinant of coefficient to zero we have

$$\begin{bmatrix} \left(a_{11} - N\left(c_c b_1 + c_1\right)\right) & \left(a_{12} - N\left(c_c b_2 + c_1\right)\right) \\ \left(a_{12} - N\left(c_c b_2 + c_1\right)\right) & \left(a_{22} - N\left(c_c b_3 + c_1\right)\right) \end{bmatrix} \begin{bmatrix} W_{bmn} \\ W_{smn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{42}$$

where

$$\begin{aligned}
a_{11} &= D_{11}\lambda^4 + D_{22}\mu^4 + \left(2D_{12} + 4D_{66}\right)\lambda^2\mu^2 \\
a_{12} &= D_{11}^s\lambda^4 + D_{22}^s\mu^4 + \left(2D_{12}^s + 4D_{66}^s\right)\lambda^2\mu^2 \\
a_{22} &= H_{11}^s\lambda^4 + H_{22}^s\mu^4 + \left(2H_{12}^s + 4H_{66}^s\right)\lambda^2\mu^2 + A_{55}^s\lambda^2 + A_{44}^s\mu^2 \\
b_1 &= \left(F_{11} - \xi F_{12}\right)\lambda^4 + \left(\left(F_{11} + F_{12}\right) - \xi\left(F_{12} + F_{22}\right)\right)\lambda^2\mu^2 + \left(F_{12} - \xi F_{22}\right)\mu^4 \\
b_2 &= \left(F_{11}^s - \xi F_{12}^s\right)\lambda^4 + \left(\left(F_{11}^s + F_{12}^s\right) - \xi\left(F_{12}^s + F_{22}^s\right)\right)\lambda^2\mu^2 + \left(F_{12} - \xi F_{22}\right)\mu^4 \\
b_3 &= (F_{11}^r - \xi F_{12}^r)\lambda^4 + \left(\left(F_{11}^r + F_{12}^r\right) - \xi\left(F_{12}^r + F_{22}^r\right)\right)\lambda^2\mu^2 + \left(F_{12} - \xi F_{22}\right)\mu^4 \\
c_1 &= \left(-\lambda^2 + \xi\mu^2\right)
\end{aligned} \tag{43}$$

Indicated the critical mechanical buckling load as this result in the following cubic equation in N, the determinant of the coefficient matrix in Eq. (42) must be zero.

$$A_2 N^2 + A_1 N + A_0 \tag{44}$$

where

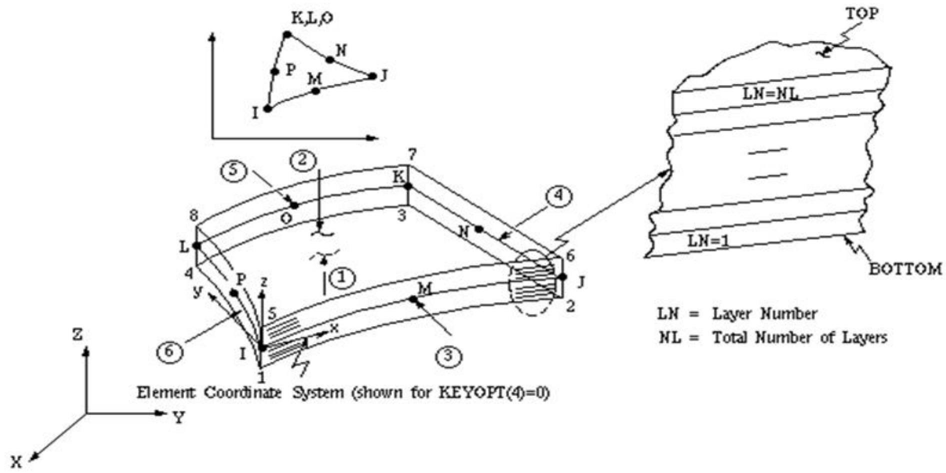

Fig. 1 SHELL99 Linear Layered Structural Shell (Nakasone *et al.* 2006, Madenci and Guven 2007)

$$\begin{aligned} A_2 &= \left[c_c^2\left(b_1 b_3 - b_2^2\right) - c_c c_1\left(b_1 + b_3 - 2b_2\right)\right] \\ A_1 &= \left[c_c\left(a_{11}b_3 + a_{22}b_1 - 2a_{12}b_2\right) - c_1\left(a_{11} + a_{22} - 2a_{12}\right)\right] \\ A_0 &= \left[a_{11}a_{22} - a_{12}^2\right] \end{aligned} \tag{45}$$

The critical buckling load, N_{cr}, corresponds to the value of m and n which yields the lowest value of N.

If curvature terms are neglected, viewing Eq. (44), the buckling load N, can be expressed as

$$N = \frac{\left(a_{11}a_{22} - a_{12}^2\right)}{c_1\left(a_{11} + a_{22} - 2a_{12}\right)} \tag{46}$$

2.5 The finite element method

The present part is a survey of plate buckling of square orthotropic plates by using the finite element method (F.E.M). Using ANSYS, the most known software in the domain for it, type of modeling is chosen shell99 (Fig. 1).

SHELL99 may be used for layered applications of a structural shell model. While SHELL99 does not have some of the nonlinear capabilities of SHELL91, it usually has a smaller element formulation time. SHELL99 allows up to 250 layers. If more than 250 layers are required, a user-input constitutive matrix is available.

3. Numerical results and discussion

In this section, various numerical examples are described and discussed for verifying the accuracy of the present method in predicting the buckling behaviors of simply supported symmetric cross-ply laminates. For the verification purpose, the results obtained by present method are compared with those of the results of previous works in the literature and computer code Ansys. The following lamina propertied is used (Khdeir and Librescu 1988, Fares and Zenkour 1999, Wu and Chen 2007)

$$E_1 / E_2 = open,\ \ E_2 = E_3,\ G_{12} = G_{13} = 0.6E_2,\ \ G_{23} = 0.5E_2,\ \nu_{12} = \nu_{13} = 0.25,\ \ \nu_{23} = 0.49 \qquad (0)$$

The material properties are assumed to be the same for all layers and the critical buckling loads are normalized as

$$\overline{N} = \frac{N_{cr} a^2}{E_2 h^3} \qquad (0)$$

In order to verify the present solutions, the convergence properties of the biaxial critical buckling loads of square cross-ply laminated composite plates are presented in Table 1. As table shows, the present results have a good agreement with Refs. (Khdeir and Librescu 1988, Fares and Zenkour 1999, Wu and Chen 2007) and finite element method using Ansys.

Table 2 presents a comparison of the lowest critical buckling loads of three-layer cross-ply laminated plates with analytical solutions (Fares and Zenkour 1999, Wu and Chen 2007) and finite element method for various values of the degree of orthotropy of the individual layers E_1/E_2. They are in excellent agreement.

Figs. 2-6 display the first mode shapes of a symmetric cross-ply laminated square plates (0°/90°/0°; a/h = 10) for E_1/E_2 = 2, 10, 20, 30, 40 respectively with simply supported (elements Shell 99). The value of the nondimensional critical buckling load of the graphs of first mode exists in Table 1.

Table 1 Comparison of non-dimensional biaxial buckling factors, $\overline{N} = N_{cr} a^2 \big/ E_2 h^3$, for simply supported symmetric cross-ply 0°/90°/0° without consideration of the effects of curvature terms, (a/h = 10)

Theory	E_1/E_2				
	2	10	20*	30*	40*
CPT[b]	2.473	5.746	9.591	12.147	14.704
FSDT[b]	2.344	4.936	7.588	8.575	10.202
HSDT[a]	2.364	4.963	5.516	9.056	10.259
HSDT[b]	2.343	4.916	7.449	8.820	9.975
GLHOT[c]	2.366	4.960	7.493	8.803	8.803
Present ansys	2.128	4.751	7.335	9.376	11.035
Present theory n = 3	2.346	5.107	7.838	9.434	10.882
Present theory n = 5	2.349	5.118	7.910	9.558	11.062
Present theory n = 7	2.351	5.131	7.958	9.634	11.169
Present theory n =9	2.354	5.142	7.990	9.683	11.238

* The lowest critical buckling occurs at mode numbers m = 1, n = 2.
[a] Khdeir and Librescu (1988)
[b] Fares and Zenkour (1999)
[c] Wu and Chen (2007)

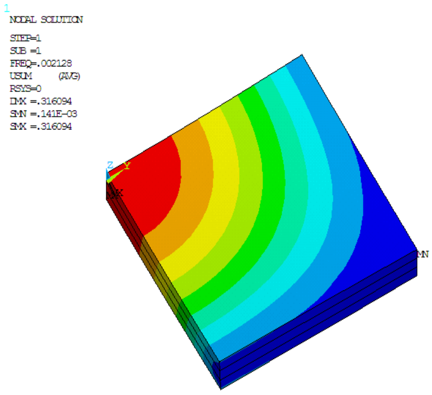

Fig. 2 First buckling mode of symmetric cross-ply laminated square plates subjected to biaxial compression (0°/90°/0°; a/h = 10, E_1/E_2 = 2), Shel99 element

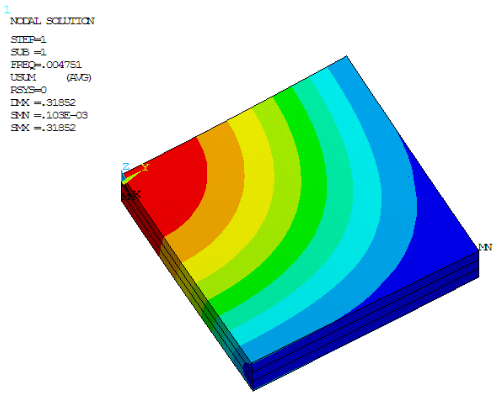

Fig. 3 First buckling mode of symmetric cross-ply laminated square plates subjected to biaxial compression (0°/90°/0°; a/h = 10, E_1/E_2 = 10), Shel99 element

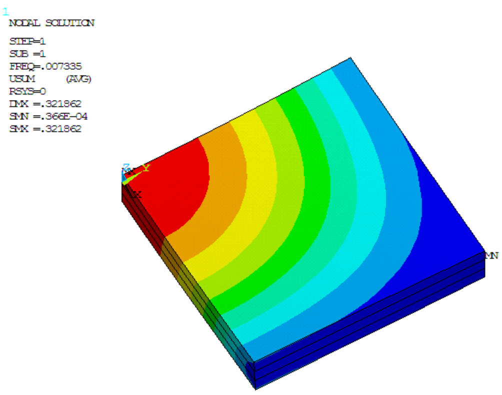

Fig. 4 First buckling mode of symmetric cross-ply laminated square plates subjected to biaxial compression (0°/90°/0°; a/h = 10, E_1/E_2 = 20), Shel99 element

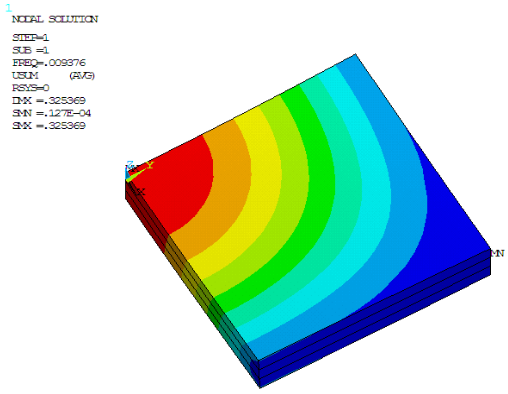

Fig. 5 First buckling mode of symmetric cross-ply laminated square plates subjected to biaxial compression (0°/90°/0°; a/h = 10, E_1/E_2 = 30), Shel99 element

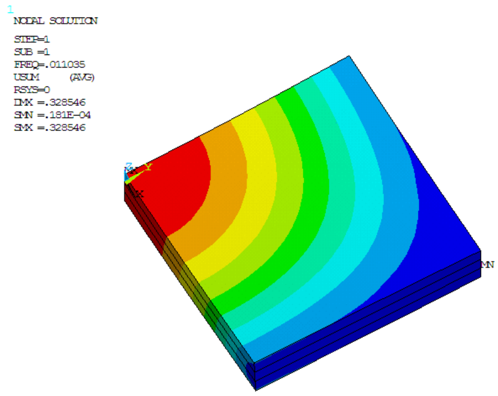

Fig. 6 First buckling mode of symmetric cross-ply laminated square plates subjected to biaxial compression (0°/90°/0°; a/h = 10, E_1/E_2 = 40), Shel99 element

Table 2 Comparison of non-dimensional biaxial buckling factors, $\overline{N} = N_{cr}a^2 / E_2h^3$, for simply supported symmetric cross-ply 0°/90°/0° without consideration of the effects of curvature terms, (a/h = 10, E_1/E_2 = 40)

Theory	(*m*,*n*)					
	(1,1)	(1,2)	(1,3)	(1,5)	(1,7)	(1,9)
HSDT[a]	11.060	9.975	13.627	21.795	27.465	31.280
GLHOT[b]	11.082	9.824	13.050	20.835	27.638	30.529
Present theory n = 3	12.983	10.882	13.891	21.811	27.4698	31.280
Present theory n = 5	13.047	11.062	14.401	23.314	29.695	33.718
Present theory n = 7	13.134	11.169	14.656	24.083	30.935	35.243
Present theory n =9	13.203	11.238	14.808	24.537	31.688	36.199

[a] Fares and Zenkour (1999)
[b] Wu and Chen (2007)

Several examples are solved to demonstrate the accuracy and efficiency of the method. In the examples considered, symmetric cross-ply thick rectangular laminates including curvature effects are considered and the following material properties are assumed, Whitney (1987)

$$E_1 / E_2 = 14, G_{12} / E_2 = 0.533, \quad G_{23} / E_2 = 0.323, \quad \nu_{12} = 0.3\,, \quad \nu_{12} = 0.55 \tag{0}$$

The next two tables are given for cylindrical bending of buckling load of simply supported square cross-ply laminated composite plates [0°/90°/90°/0°] plate. Two different cases have been considered in the numerical study: (1) without the effects of curvature terms; and (2) with the effect of curvature terms. Note that Case 1 is the conventional consideration of thick plate buckling, which forms the basis of comparison for the case (2). The results are presented in Tables 3 and 4 and Figs. 7 and 8. In Tables 3 and 4, the critical buckling factor $(N_{cr}a^2 / h^3 E_2)$ for simply supported square cross-ply laminated composite plates [0°/90°/90°/0°], under biaxial buckling and under in-plane combined tension and compression, respectively for different values of thickness-side ratio (*a*/*h* = 5, 10, 15, 20, 25,30). The material and geometry of the square plate considered here are Whitney (1987). These results are compared with the results found by Whitney (1987) by using first-order shear deformation theory. As seen a very good agreement has been achieved between them. Tables 3 and 4 also show that, the critical buckling factor increases with increase in the thickness-side ratio *a*/*h*. A comparison of Table 3 with Table 4 shows that the critical buckling load for the plate subjected to compression along *x*-direction and tension along *y*-direction, is greater than the corresponding values for the plate under biaxial compression. On the other hand, if the effect of curvature terms is included (Case 2), the buckling factors are always lower than those in Case 1. This appears to be academic, however, as the results in Tables 3 and 4 show that the curvature terms have little practical effect on the critical buckling factor for the laminate geometries considered.

It should be noted that the present theory involves only two independent variables as against

Table 3 Comparisons of critical buckling factor $(N_{cr}a^2 / h^3 E_2)$ for simply supported square cross-ply laminated composite plates [0°/90°/90°/0°], under biaxial buckling

C_c	Theory	*a*/*h*					
		5	10	15	20	25	30
0	FSDT[a]	3.6706	5.8112	6.5605	6.8758	7.0332	7.1221
	Present *n* = 3	4.0417	6.0853	6.7201	6.9752	7.1000	7.1697
	Present *n* = 5	4.0676	6.1049	6.7312	6.9820	7.1046	7.1730
	Present *n* = 7	4.1096	6.1299	6.7448	6.9903	7.1101	7.1769
	Present *n* = 9	4.1440	6.1495	6.7553	6.9967	7.1143	7.1799
1	FSDT[a]	3.5837	5.7459	6.5213	6.8509	7.0163	7.1100
	Present *n* = 3	3.9629	6.0188	6.6801	6.9500	7.0830	7.1576
	Present n = 5	3.9876	6.0378	6.6910	6.9567	7.0875	7.1608
	Present *n* = 7	4.0273	6.0619	6.7044	6.9649	7.0930	7.1647
	Present *n* = 9	4.0596	6.0808	6.7148	6.9712	7.0972	7.1677

[a] Whitney (1987)

Table 4 Comparisons of critical buckling factor $(N_{cr}a^2/h^3E_2)$ for simply supported square cross-ply laminated composite plates [0°/90°/90°/0°], under in-plane combined tension and compression

C_c	Theory	a/h					
		5	10	15	20	25	30
0	FSDT[a]	10.9050	27.4733	38.8042	45.4372	49.3608	51.7962
	Present n = 3	11.7515	28.9051	40.1032	46.4447	50.1229	52.3792
	Present n = 5	11.6513	28.6329	39.7157	46.0800	49.8186	52.1330
	Present n = 7	11.8933	29.0683	40.0991	46.3738	50.0398	52.3018
	Present n = 9	12.0988	29.4029	40.3869	46.5920	50.2032	52.4262
1	FSDT[a]	10.4425	26.7192	38.0402	44.793	48.8479	51.3908
	Present n = 3	11.5465	28.2716	39.3715	45.8088	49.6117	51.9739
	Present n = 5	11.8438	29.2809	40.4623	46.7252	50.3352	52.5415
	Present n =7	12.0901	29.7429	40.8660	47.0306	50.5629	52.7141
	Present n = 9	12.3023	30.0997	41.1699	47.2580	50.7313	52.8413

* Mode (2,1)
[a] Whitney (1987)

three in the case of FSDT (Whitney 1987). Also, the present theory does not required shear correction factors as in the case of FSDT. It can be concluded that the present theory is not only accurate but also efficient in predicting critical buckling load of laminated composite plates.

4. Conclusions

The effect of curvature terms on the buckling response of symmetrically-laminated rectangular plates is studied using the hyperbolic refined shear deformation theory. Using the Navier solution method, the differential equations have been solved analytically and the critical buckling loads presented in closed-form solutions. From this numerical study, the following conclusions may be drawn:

- Laminated composite plates are widely used in civil infrastructure systems and other structural applications because of their advantageous features such as high ratio of stiffness and strength to weight. One of the main failure mechanisms in orthotropic plates is buckling. Unlike any other isotropic plate, the buckling of orthotropic plate is more complicated due to inherently anisotropic. Thus, an accurate buckling analysis of the orthotropic plates is an important part of the structural design.
- By dividing the transverse displacement into the bending shear parts, the number of unknowns of the theory is reduced, thus saving computational time.
- The formulation the theory accounts for the shear deformation effects without requiring a shear correction factor.
- The classical plate theory comes out as a special case of present theory
- The effect of curvature terms on critical buckling load was shown to be negligible for the laminates under consideration by comparing solutions with and without these terms included.

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