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Electro-elastic analysis of a sandwich thick plate considering FG core and composite piezoelectric layers on Pasternak foundation using TSDT

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Abstract. Third order shear deformation theory is used to evaluate electro-elastic solution of a sandwich plate with considering functionally graded (FG) core and composite face sheets made of piezoelectric layers. The plate is resting on the Pasternak foundation and subjected to normal pressure. Short circuited condition is applied on the top and bottom of piezoelectric layers. The governing differential equations of the system can be derived using Hamilton's principle and Maxwell's equation. The Navier's type solution for a sandwich rectangular thick plate with all edges simply supported is used. The numerical results are presented in terms of varying the parameters of the problem such as two elastic foundation parameters, thickness ratio ($h_p/2h$), and power law index on the dimensionless deflection, critical buckling load, electric potential function, and the natural frequency of sandwich rectangular thick plate. The results show that the dimensionless natural frequency and critical buckling load diminish with an increase in the power law index, and vice versa for dimensionless deflection and electrical potential function, because of the sandwich thick plate with considering FG core becomes more flexible; while these results are reverse for thickness ratio.

Keywords: deflection, buckling, and vibration analysis; composite structures; sandwich sandwich reddy plate; FG core; composite piezoelectric layers; electric potential function

1. Introduction

Applying composite material in plate and shell structures is used for new technologies such as aerospace, automotive, and shipbuilding industries. These materials made from two or more constituent materials with various physical or chemical properties. Rapid progress of these materials is because of lightness, strength, when is compared to traditional materials.

Classical plate theory (CPT) is extended by a group of scientists to analyze laminated plates (Reddy 1984, Whitney 1987). In this theory, it is assumed that the shear modulus of isotropic plate in the thickness direction is infinite; as a result shear strains in the thickness direction can be neglected. Because the composite plates have usually greater flexibility than isotropic plate or in other words they have a lower shear modulus of the isotropic plate. Therefore, the assumption of infinite shear modulus for the composite plate is not acceptable. Thus it should be used first-order shear deformation theory (FSDT) or Mindlin plate theory (1951). Brunelle (1971) obtained the

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elastic buckling of transversely isotropic Mindlin plates with two parallel edges simply supported and the remaining two edges subjected to a variety of boundary conditions. He investigated the effect of different boundary conditions on the critical buckling load. Brunelle and Robertson (1974) derived the governing equations of a transversely isotropic subjected to initially stress using Mindlin plate theory, and solved the thick plate equations for simply supported rectangular plates. Nosier and Reddy (1992) studied the free vibration and buckling problems of transversely isotropic symmetrically laminated rectangular plates with various boundary conditions based on FSDT using boundary layer function. Third-order shear deformation theory (TSDT) is introduced firstly by Reddy (1990). Ma and Wang (2004) employed the TSDT to solve the axisymmetric bending and buckling problems of functionally graded (FG) circular plates. They derived the relationships between the solutions of axisymmetric bending and buckling of FG plates based on the TSDT, and the solutions of the homogeneous plates obtained through the CPT. Kang and Leissa (2005) illustrated the exact solutions for the buckling of isotropic rectangular thin plates having two opposite edges simply supported subjected to linearly varying in-plane load using the CPT. Zenkour (2005) analyzed the buckling and free vibration of the simply supported FG sandwich plate. He investigated the effect of the core thickness, relative to the total thickness of the plate on the critical buckling load and natural frequencies. Najafizadeh and Heydari (2004, 2008) studied axisymmetric buckling analysis of thick FG circular plates based on higher-order shear deformation plate theory, under uniform radial compression and different types of thermal loadings. Samsam Shariat and Eslami (2007) presented the mechanical and thermal buckling analysis of thick FG rectangular plate. They used higher-order shear deformation plate theory to obtain the closed form solution for the critical buckling load and simply supported boundary conditions. Kashtalyan and Menshykova (2009) used the three-dimensional elasticity for analysis of sandwich panels with a functionally graded core subjected to transverse loading. The results showed that the use of a FG core instead of a conventional homogeneous one eliminates discontinuity of the in-plane normal and shear stresses across the face sheet–core interfaces, which contribute to the structural failure of the panel. Zenkour and Sobhy (2010) used the sinusoidal shear deformation plate theory to study the thermal buckling of FG sandwich plates. Their results showed the effects of the gradient index, plate aspect ratio, side-to-thickness ratio, loading type and sandwich plate type on the critical buckling for sandwich plates. Bodaghi and Saidi (2012) used the first-order shear deformation plate theory and the neutral surface concept, the equilibrium and stability equations for functionally graded Mindlin plate. Dozio (2013) analyzed the natural frequencies of sandwich plates with FG core via variable-kinematic 2-D Ritz models. His results presented the rectangular sandwich FG plates with various thickness-to-length ratios and various boundary conditions such as clamped, free and simply-supported edges. Yang *et al*. (2014) analyzed the elastic field in a transversely isotropic FG plate with holes by transforming the original three-dimensional (3-*D*) problem into a two dimensional (2-*D*) one. They obtained the general solutions of the governing equations for the 2-*D* problem by four analytic functions $a(\zeta)$, $\beta(\zeta)$, $f(\zeta)$ and $\psi(\zeta)$ when there are no transverse forces acting on the plate surfaces. Oktem and Chaudhuri (2007) presented Levy-type analytical solution for the problem of deformation of finite-dimensional general cross-ply thick rectangular plates. Shen and Li (2008) studied postbuckling of sandwich plates with FG face sheets and temperature-dependent properties. Their results showed that the temperature changes, the volume fraction distribution of FG face sheets, and the substrate-to-face sheet thickness ratio have a significant effect on the critical buckling load and postbuckling behavior of sandwich plates. Palardy and Palazotto (1990) employed the buckling loads and fundamental frequencies of laminated cross-ply plates using the Levy method. Nguyen *et al*. (2015)

used a refined higher-order shear deformation theory for bending, buckling, and vibration analysis of FG sandwich plate. They obtained the equations of motion based on Hamilton's principle and derived the Navier's type and finite element solutions for plate with simply-supported and various boundary conditions, respectively. Using a sinusoidal plate theory, Hamidi *et al*. (2015) presented the thermo-mechanical bending analysis of FG sandwich plates. They assumed the material properties of the sandwich plate faces to vary according to a power law distribution in terms of the volume fractions of the constituents, and the material properties of the core layer is made of an isotropic ceramic material. Moreover, they compared their results with the obtained results by other authors to check the validity of the results. Then, they concluded that the proposed theory is accurate and efficient in predicting the thermo-mechanical behavior of FG sandwich plates. Chen and Liu (1993) investigated the thermal buckling of antisymmetric angle-ply laminated plates with Levy-type boundary conditions by an analytical technique in conjunction with the concept of state-space. Based on TSDT, Hasani Baferani *et al*. (2011) illustrated the free vibration analysis of FG thick rectangular plate resting on two parameter elastic foundation. Using Hamilton's principle, they derived the governing equations of motion, and obtained the natural frequency using Levy type solution. Their results illustrated that the Pasternak (shear) elastic foundation drastically changes the natural frequency. Moreover, they observed that in some boundary conditions, the in-plane displacements have significant effects on natural frequency of thick functionally graded plates. Based on FSDT, Bodaghi and Saidi (2011) presented an exact analytical solution for stability analysis of vertical moderately thick laminated rectangular plates under self weight and top load. Applying an analytical approach, they converted the coupled governing stability equations of the laminated plate into two uncoupled partial differential equations in terms of transverse displacement and an auxiliary function. Using Levy-type solution, the decoupled equations are reduced to two ordinary differential equations. Zenkour and Sobhy (2012) studied the static response of simply supported FG material viscoelastic sandwich plates subjected to transverse uniform loads. Using TSDT and meshless technique, Neves *et al*. (2013) considered bending, free vibration and buckling analysis of isotropic and sandwich FG plates. Jabbari *et al*. (2014) analyzed the thermal buckling analysis of porous circular plate with piezoelectric actuators using FSDT. The obtained results showed that the critical temperature decreases and the plate will be unstable by increasing the porosity. Using Levinson plate theory, Hosseini Hashemi *et al*. (2010) described the free vibration of piezoelectric coupled annular plates. Their results showed that the natural frequencies increase by increasing the thickness of piezoelectric layers. Sobhy (2013) analyzed the buckling and free vibration of exponentially graded sandwich plates using FSDT. Their results showed that the presence of elastic foundations leads to a significant increment in the variation of the frequencies and buckling loads. Thai *et al*. (2014) used a new first-order shear deformation theory for analysis the bending, buckling and free vibration of rectangular plates under various boundary conditions. Sobhy and Zenkour (2015) illustrated thermoelastic deformations of a simply supported FG sandwich plates subjected to a time harmonic sinusoidal temperature field on the top surface and varying through the thickness. Their results showed the influences of the time parameter, power law index, temperature exponent, top-to-bottom surface temperature ratio, side-to-thickness ratio and the foundation parameters on the dynamic bending.

In this article, the electro-elastic solution of a sandwich thick plate with considering FG core and composite face sheets made of piezoelectric layers is considered using third order shear deformation theory. Short circuited condition is applied on the top and bottom of piezoelectric layers. The governing differential equations of the system can be derived using Hamilton's principle and Maxwell's equation. The Navier's type solution for a sandwich rectangular plate with all edges simply supported is used. Moreover, the influences of elastic foundation parameters, thickness ratio, various power law index on the dimensionless deflection, critical buckling load, electric potential function, and the natural frequency of a sandwich rectangular thick plate.

2. The governing equations of a sandwich thick plate

Based on FG core and composite face sheets, a sandwich thick plate with considering piezoelectric layers and the length a , width b and thickness of core $2h$, thickness of face sheets h_p, resting on two-parameter elastic foundation is shown in Fig. 1.

Based on third-order shear deformation plate theory, the displacement fields can be considered as follows

$$u_1(x,y,z)=u(x,y)+z[\psi_x(x,y)-\frac{4}{3}(\frac{z}{H})^2(\psi_x(x,y)+w(x,y)_{,x})] \tag{1}$$

$$u_2(x,y,z)=v(x,y)+z[\psi_y(x,y)-\frac{4}{3}(\frac{z}{H})^2(\psi_y(x,y)+w(x,y)_{,y})] \tag{2}$$

$$u_3(x,y,z)=w(x,y) \tag{3}$$

where u, v, w denote the mid-plane displacements of a sandwich thick rectangular plate along the x, y, z coordinate directions, respectively, and ψ_x and ψ_y are the rotation functions. H is the total

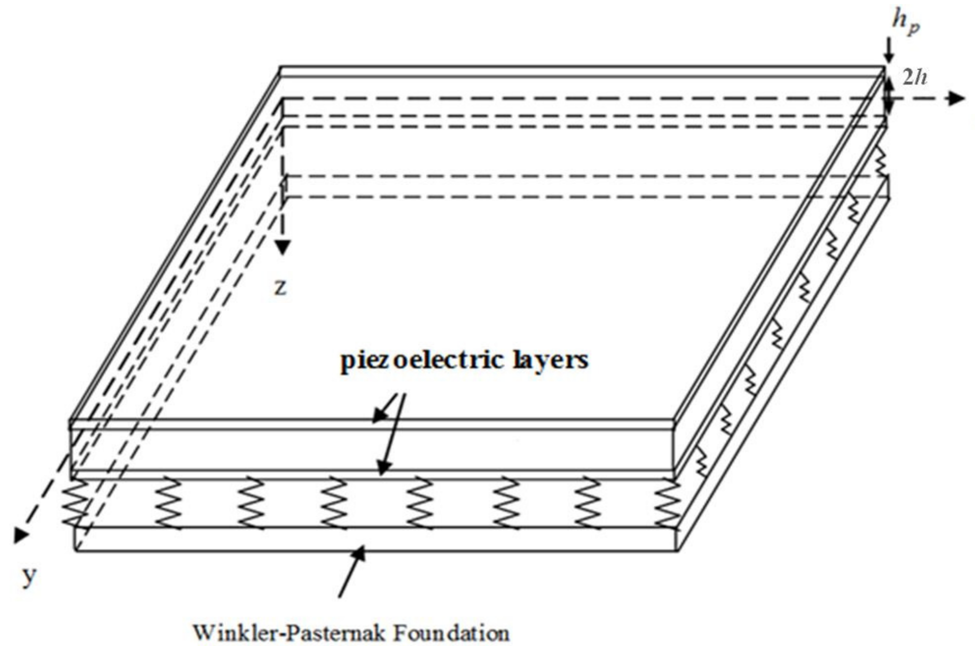

Fig. 1 A schematic of a sandwich rectangular thick plate considering FG core and composite face sheets made of piezoelectric layers resting on Pasternak foundation

thickness of sandwich thick plate.

The linear constitutive equations for FG core of a sandwich thick plate in the plane stress state are expressed as the following form

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{Bmatrix}, \tag{4}$$

where (σ_{11}, σ_{22}) and (ε_{11}, ε_{22}) are the normal stresses and strains, respectively. (τ_{12}, τ_{13}, τ_{23}) and (γ_{12}, γ_{13}, γ_{23}) denote the shear stresses and strains, respectively.

The effective material properties of sandwich plates which change very smoothly and continuously from one surface to another can be expressed by following relations

$$E(z) = E_m + (E_c - E_m)\left(\frac{1}{2} - \frac{z}{2h}\right)^n, \qquad \rho_c(z) = \rho_m + (\rho_c - \rho_m)\left(\frac{1}{2} - \frac{z}{2h}\right)^n \tag{5}$$

where n is the power law index and subscripts m and c denote the metal and ceramic properties, respectively.

The stiffness matrix coefficients (Q_{ij} (i, j = 1, 2, 4, 5, 6)) are defined as follows

$$Q_{11} = Q_{22} = \frac{E(z)}{1-\upsilon^2} \tag{6}$$

$$Q_{12} = Q_{21} = \frac{\upsilon E(z)}{1-\upsilon^2} \tag{7}$$

$$Q_{44} = Q_{55} = Q_{66} = \frac{E(z)}{2(1+\upsilon)} \tag{8}$$

The strain-displacement relationships are considered as follows

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x} \tag{9}$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial y} \tag{10}$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial z} \tag{11}$$

$$\gamma_{12} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \tag{12}$$

$$\gamma_{13} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} \tag{13}$$

$$\gamma_{23} = \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} \tag{14}$$

Substituting Eqs. (1)-(3) into Eqs. (9)-(14), the following kinetic relations are obtained as follows

$$\varepsilon_{11} = \varepsilon_1^o + z(k_1^0 + z^2 k_1^2) \tag{15}$$

$$\varepsilon_{22} = \varepsilon_2^0 + z(k_2^0 + z^2 k_2^2) \tag{16}$$

$$\varepsilon_{33} = 0 \tag{17}$$

$$\gamma_{23} = \varepsilon_4^0 + z^2 k_4^2 \tag{18}$$

$$\gamma_{13} = \varepsilon_5^0 + z^2 k_5^2 \tag{19}$$

$$\gamma_{12} = \varepsilon_6^0 + z(k_6^0 + z^2 k_6^2) \tag{20}$$

where

$$\varepsilon_1^0 = u_{,x} \tag{21}$$

$$k_1^0 = \psi_{x,x} \tag{22}$$

$$k_1^2 = -\frac{4}{3H^2}(\psi_{x,x} + w_{,xx}) \tag{23}$$

$$\varepsilon_2^0 = v_{,y} \tag{24}$$

$$k_2^0 = \psi_{y,y} \tag{25}$$

$$k_2^2 = -\frac{4}{3H^2}(\psi_{y,y} + w_{,yy}) \tag{26}$$

$$\varepsilon_4^0 = \psi_y + w_{,y} \tag{27}$$

$$k_4^2 = -\frac{4}{H^2}(\psi_y + w_{,y}) \tag{28}$$

$$\varepsilon_5^0 = \psi_x + w_{,x} \tag{29}$$

$$k_5^2 = -\frac{4}{H^2}(\psi_x + w_{,x}) \tag{30}$$

$$\varepsilon_6^0 = u_{,y} + v_{,x} \tag{31}$$

$$k_6^0 = \psi_{x,y} + \psi_{y,x} \tag{32}$$

$$k_6^2 = -\frac{4}{3H^2}(\psi_{x,y} + \psi_{y,x} + 2w_{,xy}) \tag{33}$$

The constitutive equations for composite face sheets of a sandwich plate in piezoelectric layers are written as (Arefi and Rahimi 2011, Mohammadimehr *et al*. 2015, Arefi 2015a, b)

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - e_{ijk}E_k \tag{34}$$

where, σ_{ij} and ε_{kl} are the stress and strain components, E_k is electric field, C_{ijkl} and e_{ijk} are the stiffness and piezoelectric coefficients. Extended equations of piezoelectric layers are considered as

$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \tau_{12}\\ \tau_{13}\\ \tau_{23}\end{Bmatrix} = \begin{bmatrix}\bar{C}_{11} & \bar{C}_{12} & 0 & 0 & 0\\ \bar{C}_{12} & \bar{C}_{11} & 0 & 0 & 0\\ 0 & 0 & \bar{C}_{44} & 0 & 0\\ 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & C_{55}\end{bmatrix}\begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\\ \gamma_{13}\\ \gamma_{23}\end{Bmatrix} - \begin{bmatrix}0 & 0 & \bar{e}_{31}\\ 0 & 0 & \bar{e}_{31}\\ 0 & 0 & 0\\ -e_{51} & 0 & 0\\ 0 & -e_{51} & 0\end{bmatrix}\begin{Bmatrix}E_x\\ E_y\\ E_z\end{Bmatrix} \tag{35}$$

where

$$\bar{C}_{11} = C_{11} - \frac{C_{13}^2}{C_{33}} \tag{36}$$

$$\bar{C}_{12} = C_{12} - \frac{C_{13}^2}{C_{33}} \tag{37}$$

$$\bar{e}_{13} = e_{31} - \frac{C_{13}}{C_{33}}e_{33} \tag{38}$$

$$\bar{\eta}_{33} = \eta_{33} + \frac{e_{33}^2}{C_{33}} \tag{39}$$

$$\bar{C}_{44} = \frac{1}{2}(\bar{C}_{11} - \bar{C}_{12}) \tag{40}$$

The electric field E is obtained from an electrostatic potential as the following form

$$E_i = -\phi_{,i} \quad i = x, y, z \tag{41}$$

For piezoelectric layers, we can use a three dimensional electric potential function (Rouzegar and Abad 2015, Arefi and Allam 2015).

$$\phi(x,y,z,t) = f(z)\phi(x,y,t) \tag{42}$$

$$f(z) = \begin{cases} (1-\{\dfrac{z-h-\frac{h_p}{2}}{\frac{h_p}{2}}\}^2), & h \le z \le h+h_p \\ (1-\{\dfrac{-z-h-\frac{h_p}{2}}{\frac{h_p}{2}}\}^2), & -h-h_p \le z \le -h \end{cases} \tag{43}$$

Short-circuited boundary condition is considered for piezoelectric layers at top and bottom. $\phi(x, y)$ must be derived after solution of the problem with considering the planar boundary conditions. Electric displacement for a smart material may be presented as follows (Arefi and Rahimi 2011)

$$D_i = e_{ijk}\varepsilon_{jk} + \eta_{ik}E_k \tag{44}$$

$$\begin{Bmatrix} D_x \\ D_y \\ D_z \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & e_{51} & 0 \\ 0 & 0 & 0 & 0 & e_{51} \\ \bar{e}_{31} & \bar{e}_{31} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{Bmatrix} + \begin{bmatrix} \eta_{11} & 0 & 0 \\ 0 & \eta_{11} & 0 \\ 0 & 0 & \bar{\eta}_{33} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ E_z \end{Bmatrix} \tag{45}$$

And in the extended form, we have

$$D_x = e_{51}\gamma_{13} \tag{46}$$

$$D_y = e_{51}\gamma_{23} \tag{47}$$

$$D_z = \bar{e}_{31}\varepsilon_{11} + \bar{e}_{31}\varepsilon_{22} + \bar{\eta}_{33}E_z \tag{48}$$

The governing differential equations of motion for the sandwich thick plate are derived using the Hamilton's principle which is given by

$$\int_0^t (\delta T - \delta U - \delta W)dt = 0 \tag{49}$$

where δT, δU, and δW are the variations of kinetic energy, strain energy, the work done by external applied forces, respectively.

Variations of the kinetic energy for sandwich plate can be described as follows

$$\begin{aligned} \delta T &= \int_V \rho_i \frac{\partial u_i}{\partial t}\delta(\frac{\partial u_i}{\partial t})dV = \int_V \rho_i (\dot{u}_1\delta\dot{u}_1 + \dot{u}_2\delta\dot{u}_2 + \dot{u}_3\delta\dot{u}_3)dV \\ &= \int_A \int_{\frac{-H}{2}}^{\frac{H}{2}} \rho_i (\dot{u}_1\delta\dot{u}_1 + \dot{u}_2\delta\dot{u}_2 + \dot{u}_3\delta\dot{u}_3)dzdA \end{aligned} \tag{50}$$

$$\delta T = -\int_A (\delta u\,(I_0\ddot{u} + I_1\ddot{\psi}_x - C_1 I_3(\ddot{\psi}_x + \frac{\partial \ddot{w}}{\partial x})) + \delta v\,(I_0\ddot{v} + I_1\ddot{\psi}_y - C_1 I_3\ddot{\psi}_y + \frac{\partial \ddot{w}}{\partial y}))$$
$$+\delta\psi_x\,(I_1\ddot{u} + I_2\ddot{\psi}_x - C_1(I_3\ddot{u} + 2I_4\ddot{\psi}_x + I_4\frac{\partial \ddot{w}}{\partial x}) + C_1^2 I_6(\ddot{\psi}_x + \frac{\partial \ddot{w}}{\partial x}))$$
$$+\delta\psi_y\,(I_1\ddot{v} + I_2\ddot{\psi}_y - C_1(I_3\ddot{v} + 2I_4\ddot{\psi}_y + I_4\frac{\partial \ddot{w}}{\partial y}) + C_1^2 I_6(\ddot{\psi}_x + \frac{\partial \ddot{w}}{\partial y})) \tag{51}$$
$$+\delta w\,(C_1 I_3(\frac{\partial \ddot{u}}{\partial x} + \frac{\partial \ddot{v}}{\partial y}) + C_1 I_4(\frac{\partial \ddot{\psi}_x}{\partial x} + \frac{\partial \ddot{\psi}_y}{\partial y}) - C_1^2 I_6(\frac{\partial \ddot{\psi}_x}{\partial x} + \frac{\partial \ddot{\psi}_y}{\partial y}$$
$$-\frac{\partial^2 \ddot{w}}{\partial x^2} - \frac{\partial^2 \ddot{w}}{\partial y^2}) + I_0\ddot{w}))dA$$

where

$$I_0 = \int_{-\frac{H}{2}}^{\frac{H}{2}} \rho_i dz = \int_{-h-h_p}^{-h} \rho_p dz + \int_{-h}^{h} \rho_c(z)dz + \int_{h}^{h+h_p} \rho_p dz \tag{52}$$

$$I_1 = \int_{-\frac{H}{2}}^{\frac{H}{2}} \rho_i z dz = \int_{-h-h_p}^{-h} \rho_p z dz + \int_{-h}^{h} \rho_c(z) z dz + \int_{h}^{h+h_p} \rho_p z dz \tag{53}$$

$$I_2 = \int_{-\frac{H}{2}}^{\frac{H}{2}} \rho_i z^2 dz = \int_{-h-h_p}^{-h} \rho_p z^2 dz + \int_{-h}^{h} \rho_c(z) z^2 dz + \int_{h}^{h+h_p} \rho_p z^2 dz \tag{54}$$

$$I_3 = \int_{-\frac{H}{2}}^{\frac{H}{2}} \rho_i z^3 dz = \int_{-h-h_p}^{-h} \rho_p z^3 dz + \int_{-h}^{h} \rho_c(z) z^3 dz + \int_{h}^{h+h_p} \rho_p z^3 dz \tag{55}$$

$$I_4 = \int_{-\frac{H}{2}}^{\frac{H}{2}} \rho_i z^4 dz = \int_{-h-h_p}^{-h} \rho_p z^4 dz + \int_{-h}^{h} \rho_c(z) z^4 dz + \int_{h}^{h+h_p} \rho_p z^4 dz \tag{56}$$

$$I_6 = \int_{-\frac{H}{2}}^{\frac{H}{2}} \rho_i z^6 dz = \int_{-h-h_p}^{-h} \rho_p z^6 dz + \int_{-h}^{h} \rho_c(z) z^6 dz + \int_{h}^{h+h_p} \rho_p z^6 dz \tag{57}$$

$$C_1 = \frac{4}{3H^2} \tag{58}$$

Variations of the strain energy for sandwich thick plate can be expressed as

$$\delta U = \int_V \sigma_{ij}\delta\varepsilon_{ij}\, dV = \int_V (\sigma_{11}\delta\varepsilon_{11} + \sigma_{22}\delta\varepsilon_{22} + \sigma_{33}\delta\varepsilon_{33} + \tau_{12}\delta\gamma_{12} + \tau_{13}\delta\gamma_{13} + \tau_{23}\delta\gamma_{23})dV \tag{59}$$

Variations of work done due to the external load and elastic foundation can be considered as follows

$$\delta W = -\int P(x,y)\delta w\, dx + \int F_{elastic}\,\delta w\, dx \tag{60}$$

According to Fig. 1, the force due to the elastic medium is obtained as (Mohammadimehr *et al*. 2010, Ghorbanpour Arani *et al*. 2011)

$$F_{elastic} = Kw\,(w) - KG\,\nabla^2 w \tag{61a}$$

where Kw and KG are the spring constant of Winkler type and the shear constant of Pasternak type for elastic foundation, respectively. The Eq. (61a) is known to Pasternak foundation that in addition to the transverse force (Kw (w)), the shear force ($KG\ \nabla_w^2$) can be also withstands, while the Winkler foundation can be endure only the transverse force in which is defined as the following form

$$F_{elastic} = Kw\,(w) \tag{61b}$$

The equilibrium equations of sandwich resting on Pasternak foundation using the Hamilton's principle can be derived as the following form

$$\delta u : N_{1,x} + N_{6,y} = I_0\ddot{u} + I_1\ddot{\psi}_x - C_1 I_3(\ddot{\psi}_x + \frac{\partial \ddot{w}}{\partial x}) \tag{62}$$

$$\delta v : N_{2,y} + N_{6,x} = I_0\ddot{v} + I_1\ddot{\psi}_y - C_1 I_3(\ddot{\psi}_y + \frac{\partial \ddot{w}}{\partial y}) \tag{63}$$

$$\begin{aligned}&\delta\psi_x : M_{1,x} + M_{6,y} - Q_1 - \frac{4\lambda}{3H^2}(P_{1,x} + P_{6,y}) + \frac{4\lambda}{H^2}R_1 = I_1\ddot{u} + I_2\ddot{\psi}_x \\ &-C_1(I_3\ddot{u} + 2I_4\ddot{\psi}_x + I_4\frac{\partial \ddot{w}}{\partial x}) + C_1^2 I_6(\ddot{\psi}_x + \frac{\partial \ddot{w}}{\partial x})\end{aligned} \tag{64}$$

$$\begin{aligned}&\delta\psi_y : M_{2,y} + M_{6,x} - Q_2 - \frac{4\lambda}{3H^2}(P_{2,y} + P_{6,x}) + \frac{4\lambda}{H^2}R_2 = I_1\ddot{v} + I_2\ddot{\psi}_y \\ &-C_1(I_3\ddot{v} + 2I_4\ddot{\psi}_y + I_4\frac{\partial \ddot{w}}{\partial y}) + C_1^2 I_6(\ddot{\psi}_x + \frac{\partial \ddot{w}}{\partial y})\end{aligned} \tag{65}$$

$$\begin{aligned}&\delta w : Q_{1,x} + Q_{2,y} + \frac{4\lambda}{3H^2}(P_{1,xx} + P_{2,yy} + 2P_{6,xy}) - \frac{4\lambda}{H^2}(R_{2,y} + R_{1,x}) + (-Kw\,(w) + KG\nabla^2 w) + P(x,y) \\ &= C_1 I_3(\frac{\partial \ddot{u}}{\partial x} + \frac{\partial \ddot{v}}{\partial y}) + C_1 I_4(\frac{\partial \ddot{\psi}_x}{\partial x} + \frac{\partial \ddot{\psi}_y}{\partial y}) + C_1^2 I_6(\frac{\partial \ddot{\psi}_x}{\partial x} + \frac{\partial \ddot{\psi}_y}{\partial y} - \frac{\partial^2 \ddot{w}}{\partial x^2} - \frac{\partial^2 \ddot{w}}{\partial y^2}) + I_0\ddot{w})\end{aligned} \tag{66}$$

where the λ parameter is equal to one and zero for TSDT and FSDT, respectively. N_i, M_i (i = 1, 2, 6) denote the resultant forces and moments, respectively, also R_i, P_i are the higher order resultant shear forces and moments, respectively, and Q_i is the transverse shear forces which are defined by the following expressions

$$(\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix}, \begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix}, \begin{Bmatrix} P_1 \\ P_2 \\ P_6 \end{Bmatrix}_i) = \int_{-\frac{H}{2}}^{\frac{H}{2}} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} (1, z, z^3)\,dz \tag{67}$$

$$(\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}, \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}) = \int_{-\frac{H}{2}}^{\frac{H}{2}} \begin{Bmatrix} \tau_{13} \\ \tau_{23} \end{Bmatrix} (1, z^2) dz \tag{68}$$

Substituting Eqs. (9)-(14) and (35) into Eqs. (67) and (68) yields

$$N_1 = A_{11} u_{,x} + A_{12} v_{,y} + \mu_1 \phi \tag{69}$$

$$N_2 = A_{11} v_{,y} + A_{12} u_{,x} + \mu_1 \phi \tag{70}$$

$$N_6 = A_{66} (v_{,x} + u_{,y}) \tag{71}$$

$$M_1 = B_{11} \psi_{x,x} + B_{12} \psi_{y,y} + D_{11} w_{,xx} + D_{12} w_{,yy} + \mu_2 \phi \tag{72}$$

$$M_2 = B_{22} \psi_{y,y} + B_{12} \psi_{x,x} + D_{12} w_{,xx} + D_{22} w_{,yy} + \mu_2 \phi \tag{73}$$

$$M_6 = F_{11} (\psi_{x,y} + \psi_{y,x}) + F_{12} w_{,xy} \tag{74}$$

$$P_1 = H_{11} \psi_{x,x} + H_{12} \psi_{y,y} + K_{11} w_{,xx} + K_{12} w_{,yy} + \mu_3 \phi \tag{75}$$

$$P_2 = H_{12} \psi_{x,x} + H_{22} \psi_{y,y} + K_{12} w_{,xx} + K_{11} w_{,yy} + \mu_3 \phi \tag{76}$$

$$P_6 = L_{11} (\psi_{x,y} + \psi_{y,x}) + L_{12} w_{,xy} \tag{77}$$

$$Q_1 = S_{11} (\psi_x + w_{,x}) \tag{78}$$

$$Q_2 = S_{11} (\psi_y + w_{,y}) \tag{79}$$

$$R_1 = S_{22} (\psi_x + w_{,x}) \tag{80}$$

$$R_2 = S_{22} (\psi_y + w_{,y}) \tag{81}$$

where the above coefficients are defined in Appendix A.

The electric potential in piezoelectric layers satisfies Maxwell's equation as the following integral form (Arefi 2015a, b)

$$\int_h^{h+h_p} (\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}) dz + \int_{-h-h_p}^{-h} (\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}) dz = 0 \tag{82}$$

Substituting Eqs. (46)-(48) into Eq. (82) yields

$$\lambda_1 (\psi_{x,x} + \psi_{y,y}) + \lambda_2 (w_{,xx} + w_{,yy}) + \lambda_3 \phi = 0 \tag{83}$$

$$\phi = \frac{-1}{\lambda_3}(\lambda_1(\psi_{x,x} + \psi_{y,y}) + \lambda_2(w_{,xx} + w_{,yy})) \tag{84}$$

$$\phi = -\lambda_4(\psi_{x,x} + \psi_{y,y}) - \lambda_5(w_{,xx} + w_{,yy})) \tag{85}$$

By substituting Eqs. (69)-(81) into Eqs. (62)-(66), the governing equations of motion for sandwich thick plate based on TSDT embedded in an elastic medium are obtained as follows

$$\begin{aligned} &\delta u : A_{11}u_{,xx} + (A_{12}+A_{66})v_{,xy} + A_{66}u_{,yy} + \mu_1(-\lambda_4(\psi_{x,xx} + \psi_{y,xy}) - \lambda_5(w_{,xxx} + w_{,xyy})) \\ &= I_0\ddot{u} + I_1\ddot{\psi}_x - C_1 I_3(\ddot{\psi}_x + \frac{\partial \ddot{w}}{\partial x}) \end{aligned} \tag{86}$$

$$\begin{aligned} &\delta v : A_{11}v_{,yy} + (A_{12}+A_{66})u_{,xy} + A_{66}v_{,xx} + \mu_1(-\lambda_4(\psi_{x,xy} + \psi_{y,yy}) - \lambda_5(w_{,xxy} + w_{,yyy})) \\ &= I_0\ddot{v} + I_1\ddot{\psi}_y - C_1 I_3(\ddot{\psi}_y + \frac{\partial \ddot{w}}{\partial y}) \end{aligned} \tag{87}$$

$$\begin{aligned} &\delta\psi_x : B_{11}\psi_{x,xx} + B_{12}\psi_{y,xy} + D_{11}w_{,xxx} + D_{12}w_{,xyy} + \mu_2(-\lambda_4(\psi_{x,xx} + \psi_{y,xy}) - \lambda_5(w_{,xxx} + w_{,xyy})) \\ &+F_{11}\psi_{x,yy} + F_{11}\psi_{y,xy} + F_{12}w_{,xyy} - S_{11}\psi_x - S_{11}w_{,x} - \frac{4}{3H^2}(H_{11}\psi_{x,xx} + H_{12}\psi_{y,xy} \\ &+K_{11}w_{,xxx} + K_{12}w_{,xyy} + \mu_3(-\lambda_4(\psi_{x,xx} + \psi_{y,xy}) - \lambda_5(w_{,xxx} + w_{,xyy})) + L_{11}\psi_{x,yy} + L_{11}\psi_{y,xy} \\ &+L_{12}w_{,xyy}) + \frac{4}{H^2}(S_{22}\psi_x + S_{22}w_{,x}) = I_1\ddot{u} + I_2\ddot{\psi}_x - C_1(I_3\ddot{u} + 2I_4\ddot{\psi}_x + I_4\frac{\partial \ddot{w}}{\partial x}) + C_1^2 I_6(\ddot{\psi}_x + \frac{\partial \ddot{w}}{\partial x}) \end{aligned} \tag{88}$$

$$\begin{aligned} &\delta\psi_y : B_{22}\psi_{y,yy} + B_{12}\psi_{x,xy} + D_{22}w_{,yyy} + D_{12}w_{,xxy} + \mu_2(-\lambda_4(\psi_{x,xy} + \psi_{y,yy}) - \lambda_5(w_{,xxy} + w_{,yyy})) \\ &+F_{11}\psi_{x,xy} + F_{11}\psi_{y,xx} + F_{12}w_{,xxy} - S_{11}\psi_y - S_{11}w_{,y} - \frac{4}{3H^2}(H_{22}\psi_{y,yy} + H_{12}\psi_{x,xy} + K_{11}w_{,yyy} \\ &+K_{12}w_{,xxy} + \mu_3(-\lambda_4(\psi_{x,xy} + \psi_{y,yy}) - \lambda_5(w_{,xxy} + w_{,yyy})) + L_{11}\psi_{x,xy} + L_{11}\psi_{y,xx} + L_{12}w_{,xxy}) \\ &+\frac{4}{H^2}(S_{22}\psi_y + S_{22}w_{,y}) = I_1\ddot{v} + I_2\ddot{\psi}_y - C_1(I_3\ddot{v} + 2I_4\ddot{\psi}_y + I_4\frac{\partial \ddot{w}}{\partial y}) + C_1^2 I_6(\ddot{\psi}_x + \frac{\partial \ddot{w}}{\partial y}) \end{aligned} \tag{89}$$

$$\begin{aligned} &\delta w : S_{11}\psi_{x,x} + S_{11}w_{,xx} + S_{11}\psi_{y,y} + S_{11}w_{,yy} + \frac{4}{3H^2}(H_{11}\psi_{x,xxx} + H_{12}\psi_{y,xxy} + K_{11}w_{,xxxx} + K_{12}w_{,xxyy} \\ &+\mu_3(-\lambda_4(\psi_{x,xxx} + \psi_{y,xxy}) - \lambda_5(w_{,xxxx} + w_{,xxyy})) + H_{22}\psi_{y,yyy} + H_{12}\psi_{x,xyy} + K_{11}w_{,yyyy} + K_{12}w_{,xxyy} \\ &+\mu_3(-\lambda_4(\psi_{x,xyy} + \psi_{y,yyy}) - \lambda_5(w_{,xxyy} + w_{,yyyy})) + 2L_{11}\psi_{x,xyy} + 2L_{11}\psi_{y,xxy} + 2L_{12}w_{,xxyy}) \\ &-\frac{4}{H^2}(S_{22}\psi_{y,y} + S_{22}w_{,yy} + S_{22}\psi_{x,x} + S_{22}w_{,xx}) + (-Kw\, w + KG\nabla^2 w) + P(x,y) \\ &= C_1 I_3(\frac{\partial \ddot{u}}{\partial x} + \frac{\partial \ddot{v}}{\partial y}) + C_1 I_4(\frac{\partial \ddot{\psi}_x}{\partial x} + \frac{\partial \ddot{\psi}_y}{\partial y}) + C_1^2 I_6(\frac{\partial \ddot{\psi}_x}{\partial x} + \frac{\partial \ddot{\psi}_y}{\partial y} - \frac{\partial^2 \ddot{w}}{\partial x^2} - \frac{\partial^2 \ddot{w}}{\partial y^2}) + I_0\ddot{w}) \end{aligned} \tag{90}$$

3. The Navier's type solution for sandwich thick plate

Analytical solutions for a simply supported rectangular sandwich plate are obtained using Navier's solution technique. Simply supported boundary conditions of all edges for sandwich plate are given by

$$u(x,0)=u(x,b)=0 \tag{91}$$

$$\psi_X(x,0)=\psi_X(x,b)=0 \tag{92}$$

$$w(x,0)=w(x,b)=0 \tag{93}$$

$$v(0,y)=v(a,y)=0 \tag{94}$$

$$\psi_y(0,y)=\psi_y(a,y)=0 \tag{95}$$

$$w(0,y)=w(a,y)=0 \tag{96}$$

Using Navier's type solution, one can be written as follows

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}U_{mn}\cos(\alpha x)\sin(\beta y)e^{i\omega t} \tag{97}$$

$$v(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}V_{mn}\sin(\alpha x)\cos(\beta y)e^{i\omega t} \tag{98}$$

$$\psi_x(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Psi_{x\,mn}\cos(\alpha x)\sin(\beta y)e^{i\omega t} \tag{99}$$

$$\psi_y(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Psi_{y\,mn}\sin(\alpha x)\cos(\beta y)e^{i\omega t} \tag{100}$$

$$w(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{mn}\sin(\alpha x)\sin(\beta y)e^{i\omega t} \tag{101}$$

$$p(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Q_{mn}\sin(\alpha x)\sin(\beta y)e^{i\omega t} \tag{102}$$

where for rectangular uniformly distributed load ($p(x, y, t) = P_0$), thus we have

$$Q_{mn}=\frac{4}{ab}\int_0^b\int_0^a P_0\sin\alpha x\,\sin\beta y dxdy \tag{103}$$

Substituting Eqs. (97)-(103) into Eqs. (86)-(90), the matrix form of bending equations for a sandwich thick plate is written as

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} \end{bmatrix} \begin{Bmatrix} U_{mn} \\ V_{mn} \\ \Psi_x mn \\ \Psi_y mn \\ W_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ Q_{mn} \end{Bmatrix} \tag{104}$$

The dimensionless deflection of sandwich plate is defined as follows (Kim and Reddy 2013)

$$\overline{w} = \frac{E_m h^3}{P_0 a^4} w \tag{105}$$

The matrix form of free vibration equations for a sandwich plate is considered as

$$\left\{ [S] - \omega^2 [M] \right\} \{U\} = \{0\} \tag{106a}$$

where the elements of the mass and stiffness matrices are given in Appendix B.

To obtain the natural frequency for Eq. (106a), the determinant of coefficient matrix should be equal to zero, thus we have

$$\left| \left([S] - \omega^2 [M] \right) \right| = 0 \text{ or } \det\left([S] - \omega^2 [M] \right) = 0 \tag{106b}$$

The dimensionless natural frequency of sandwich plate is defined as follows (Kim and Reddy 2013)

$$\overline{\omega} = \sqrt{\frac{\rho_m a^4}{E_m h^2}} \omega \tag{107}$$

The matrix form of buckling equations for a sandwich plate is written as

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} - N_0(\alpha^2 + k\beta^2) \end{bmatrix} \begin{Bmatrix} U_{mn} \\ V_{mn} \\ \Psi_x mn \\ \Psi_y mn \\ W_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad k = \frac{N_{xx}}{N_{yy}} \tag{108}$$

$$\begin{Bmatrix} U_{mn} \\ V_{mn} \\ \Psi_x mn \\ \Psi_y mn \end{Bmatrix} = \left[\overline{C} \right] \begin{Bmatrix} S_{15} \\ S_{25} \\ S_{35} \\ S_{45} \end{Bmatrix} \{W_{mn}\} \tag{109}$$

where

$$\left[\overline{C}\right]=\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix}^{-1} \tag{110}$$

Using Eq. (108), we obtain the expression for the critical buckling load N_0 of sandwich plate as follows

$$N_0=\frac{1}{\alpha^2+k\,\beta^2}\left(S_{55}-\begin{Bmatrix} S_{51} \\ S_{52} \\ S_{53} \\ S_{54} \end{Bmatrix}^T\left[\overline{C}\right]\begin{Bmatrix} S_{15} \\ S_{25} \\ S_{35} \\ S_{45} \end{Bmatrix}\right) \tag{111}$$

The dimensionless buckling load of sandwich plate is written as follows (Kim and Reddy, 2013)

$$\overline{N}_0=\frac{a^2}{E_m h^3}N_0 \tag{112}$$

4. Numerical results and discussions

Numerical results for bending, buckling, and free vibration are presented for symmetric rectangular FG core and composite piezoelectric layer plate resting on two-parameter elastic foundations with all edges simply supported.

Table 1 The mechanical and electrical properties of sandwich plate

Property	Core plate		Piezoelectric layer
	Al	*Alumina*	*PZT- 4*
E (GPa)	70	380	--
υ	0.3	0.3	--
C_{11} (GPa)	--	--	132
C_{12} (GPa)	--	--	71
C_{33} (GPa)	--	--	115
C_{13} (GPa)	--	--	73
C_{55} (GPa)	--	--	26
e_{31} (Cm^{-2})	--	--	-4.1
e_{33} (Cm^{-2})	--	--	14.1
e_{15} (Cm^{-2})	--	--	10.5
η_{11} (n.F.m^{-1})	--	--	7.124
η_{33} (n.F.m^{-1})	--	--	5.841
ρ (Kg.m^{-3})	2707	3800	7500

Table 2 Natural frequencies of the simply supported sandwich plate

Power law index	$\frac{2h}{a}$	$\frac{h_p}{2a}$	Various theory	First natural frequency
0	0.05	0.1	Present	426.645
			VRPT (Rouzegar and Abad 2015)	426.818
			FSDT (Askari Farsangi and Saidi 2013)	426.662
		0.2	Present	417.938
			VRPT(Rouzegar and Abad 2015)	408.836
			FSDT(Askari Farsangi and Saidi 2013)	408.475
	0.1	0.1	Present	827.364
			FRPT(Rouzegar and Abad 2015)	827.520
			FSDT(Askari Farsangi and Saidi 2013)	826.463
		0.2	Present	788.495
			VRPT(Rouzegar and Abad 2015)	788.433
			FSDT(Askari Farsangi and Saidi 2013)	786.011
0.5	0.05	0.1	Present	364.562
			VRPT(Rouzegar and Abad 2015)	369.195
			FSDT(Askari Farsangi and Saidi 2013)	369.015
		0.2	Present	365.849
			VRPT(Rouzegar and Abad 2015)	362.655
			FSDT(Askari Farsangi and Saidi 2013)	362.269
	0.1	0.1	Present	718.344
			FRPT(Rouzegar and Abad 2015)	716.563
			FSDT(Askari Farsangi and Saidi 2013)	715.319
		0.2	Present	718.344
			VRPT(Rouzegar and Abad 2015)	716.563
			FSDT(Askari Farsangi and Saidi 2013)	715.319
1	0.05	0.1	Present	344.644
			VRPT(Rouzegar and Abad 2015)	340.006
			FSDT(Askari Farsangi and Saidi 2013)	339.859
		0.2	Present	346.344
			VRPT(Rouzegar and Abad 2015)	340.673
			FSDT(Askari Farsangi and Saidi 2013)	340.311
	0.1	0.1	Present	660.345
			FRPT(Rouzegar and Abad 2015)	659.565
			FSDT(Askari Farsangi and Saidi 2013)	658.555
		0.2	Present	657.243
			VRPT(Rouzegar and Abad 2015)	656.105
			FSDT(Askari Farsangi and Saidi 2013)	653.652

A simply supported FG plates coupled with piezoelectric layers is considered. The FG core plate is made of aluminum and alumina, and two *PZT-4* piezoelectric layers are attached to its top and bottom surfaces with the side of 40 cm. The mechanical and electrical of aluminum, Alumina and *PZT-4* are listed in Table 1 (Rouzegar and Abad 2015).

4.1 Natural vibration of sandwich plate

The natural frequencies of the simply supported sandwich plate are obtained using Eq. (106). The fundamental natural frequency is obtained when $m = 1$ and $n = 1$.

First natural frequency for different theories of plate and power law indices is presented in Table 2. It is shown that the present results without considering electric fields in x and y directions has a good agreement with the obtained results by variable refined plate theory (VRPT) (Rouzegar

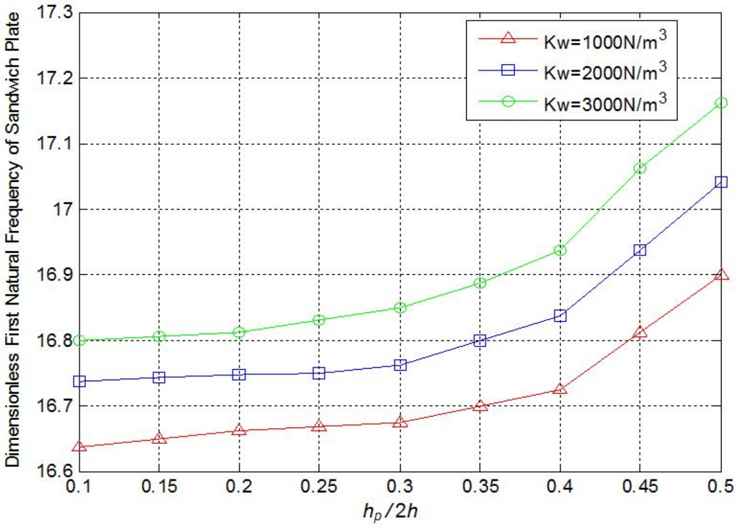

Fig. 2 The effect of spring constant of Winkler type on dimensionless first natural frequency ($a = b = 400$ mm, $2h = 5$ mm, $n = 2$, $KG = 0$)

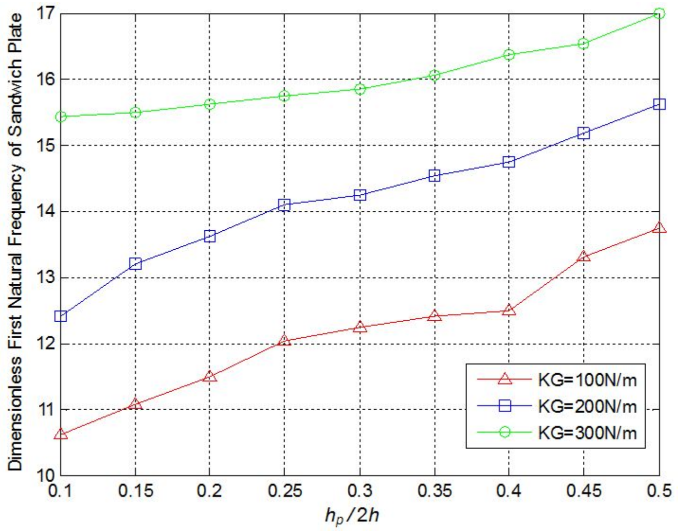

Fig. 3 The effect of shear constant of Pasternak type on dimensionless first natural frequency ($a = b = 400$ mm, $2h = 5$ mm, $n = 2$, $Kw = 0$)

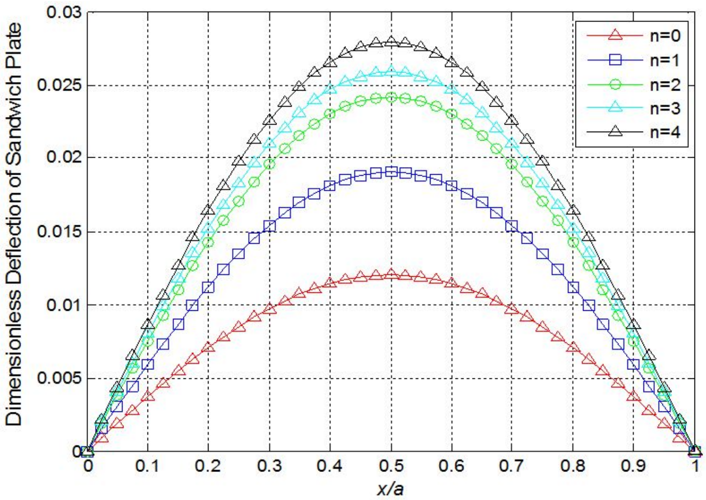

Fig. 4 The effect of power law index on the dimensionless deflection ($a = b = 400$ mm, $h = 2.5$ mm, $h_p = 1$, $Kw = 0$, $KG = 0$, $P0 = 100$ N/m^2)

and Abad 2015) and FSDT (Askari Farsangi and Saidi 2013). It is observed that with increasing power law indices, the natural frequency decreases and vice versa for $h_p/2a$ in which the power law index is constant.

The natural frequencies of the simply supported sandwich plate are obtained using Eq. (106).

In Figs. 2 and 3, we can see the effects of spring and shear constants on dimensionless fundamental natural frequency of sandwich plate. With increasing of the spring (Kw) and shear KG constants and thickness ratio, the dimensionless natural frequency of sandwich plate increases. It is due to that with increasing these parameters, the sandwich plate becomes stiffer.

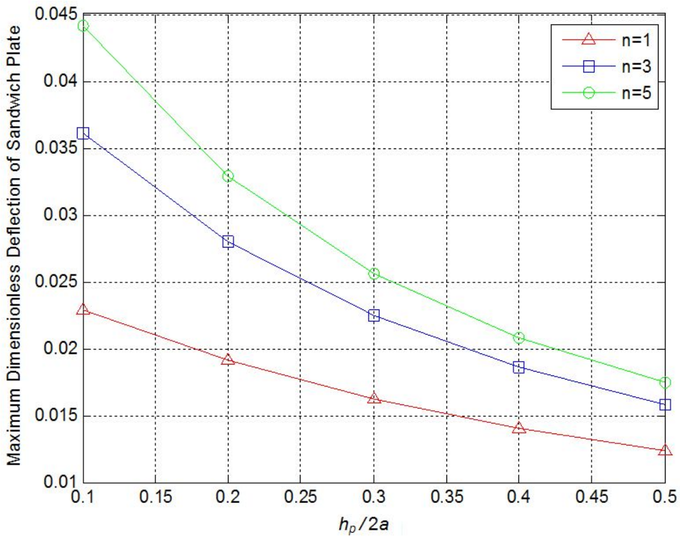

Fig. 5 The effect of thickness ratio on the dimensionless deflection ($a = b = 400$ mm, $h = 2.5$ mm, $Kw = 0$, $KG = 0$, $P0 = 100$ N/m^2)

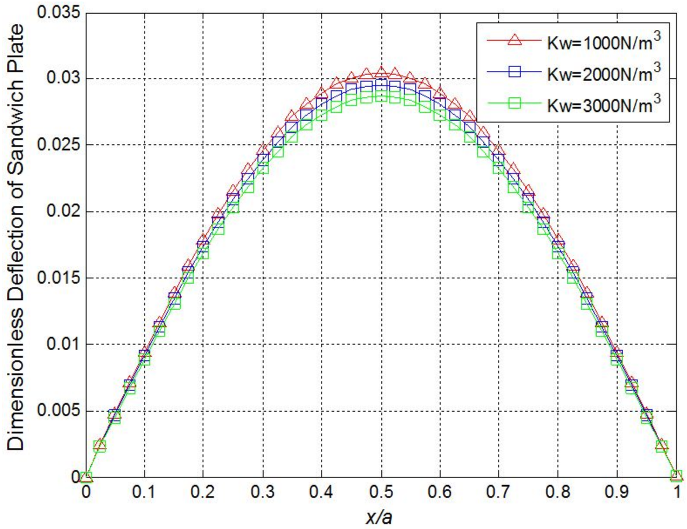

Fig. 6 The effect of spring constant of Winkler type on dimensionless deflection
($a = b = 400$ mm, $h = 2.5$ mm, $h_p = 1$, $n = 3$, $KG = 0$, $P0 = 100$ N/m^2)

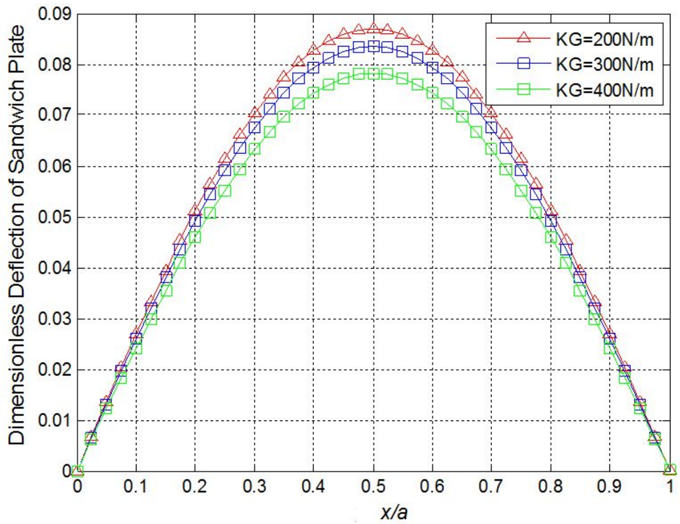

Fig. 7 The effect of shear constant of Pasternak type on dimensionless deflection
($a = b = 400$ mm, $h = 2.5$ mm, $h_p = 1$, $n = 3$, $Kw = 0$, $P0 = 100$ N/m^2)

4.2 Bending of sandwich plate

For bending analysis of sandwich plate, the uniformly distributed load is applied on top surface $z = H/2$ in Fig. 4. This figure shows that with increasing power law index, dimensionless deflection of the sandwich plate increases. Fig. 5 shows that with increasing thickness ratio, dimensionless deflection of sandwich plate decreases. Figs. 6 and 7 show that the effect of the foundation elastic on the dimensionless deflection of sandwich plate. It is concluded that the dimensionless deflection of sandwich plate with considering FG core and composite piezoelectric layers reduces with a change the elastic foundation constants. It is observed that the boundary conditions for edges simply supported is satisfied as well as.

4.3 The Buckling analysis of sandwich plate

The critical buckling loads of sandwich plate are obtained using Eq. (111). The compressive loads N_{xx} and N_{yy} are assumed that is equal to each other for sandwich plate. The minimum critical

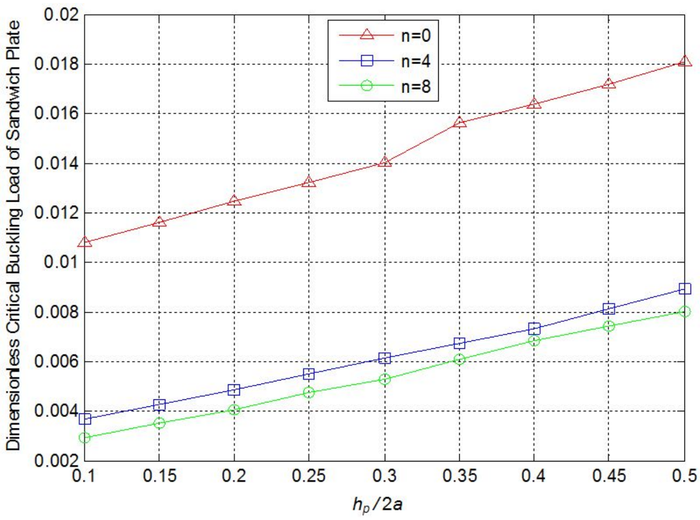

Fig. 8 The effect of power law index on the dimensionless critical buckling load ($a = b = 400$ mm, $h = 2.5$ mm, $h_p = 1$, $Kw = 0$, $KG = 0$)

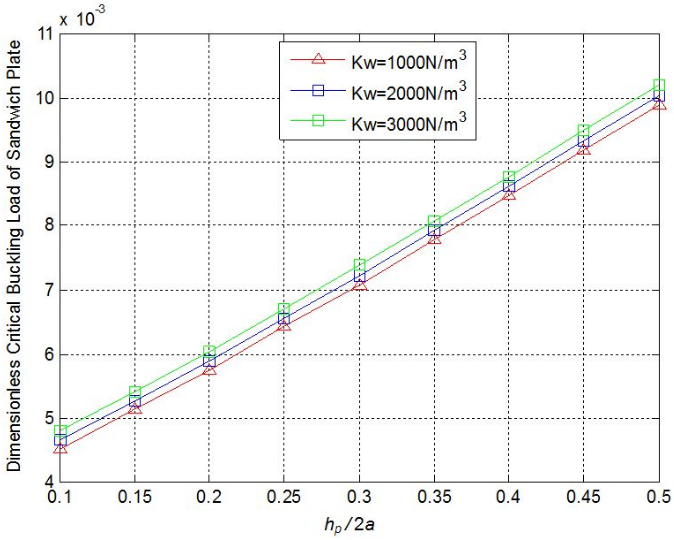

Fig. 9 The effect of Kw on dimensionless critical buckling load ($a = b = 400$ mm, $h = 2.5$ mm, $h_p = 1$, $n = 2$, $KG = 0$)

load is obtained when $m = 1$ and $n = 1$. Fig. 7 shows that with increasing the power law index, the dimensionless critical buckling load decreases and vice versa for elastic foundation parameters in Figs. 8 and 9 because of the sandwich plate becomes stiffer.

4.4 Electric potential function

The electrical potential function for sandwich plate is obtained using Eq. (85). Figs. 11 and 12

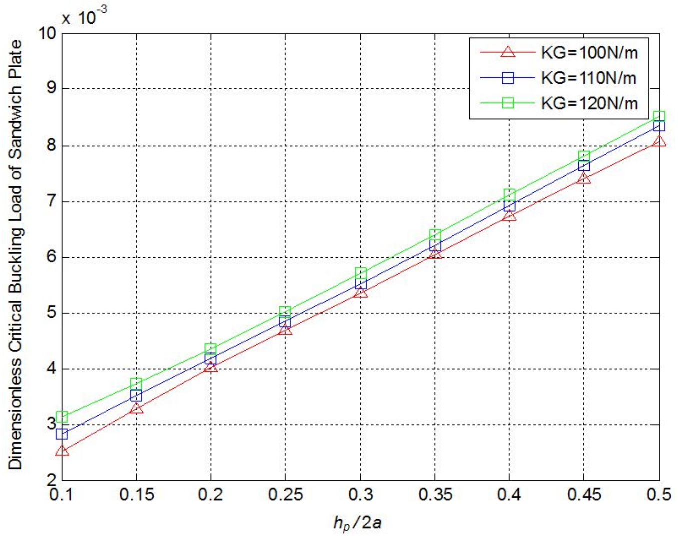

Fig. 10 The effect of KG on dimensionless critical buckling load ($a = b = 400$ mm, $h = 2.5$ mm, $h_p = 1$, $n = 2$, $Kw = 0$)

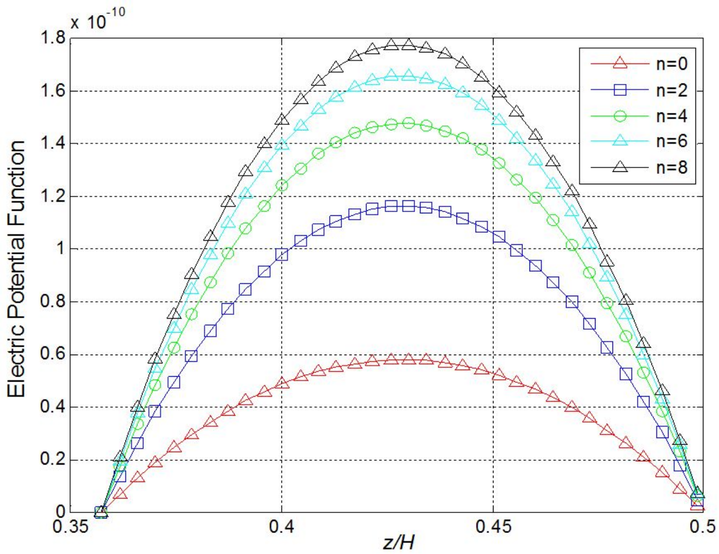

Fig. 11 Effect of power law indices on electrical potential function ($a = b = 400$ mm, $h = 2.5$ mm, $h_p = 1$, $Kw = 0$, $KG = 0$, $P0 = 1000$ N/m^2)

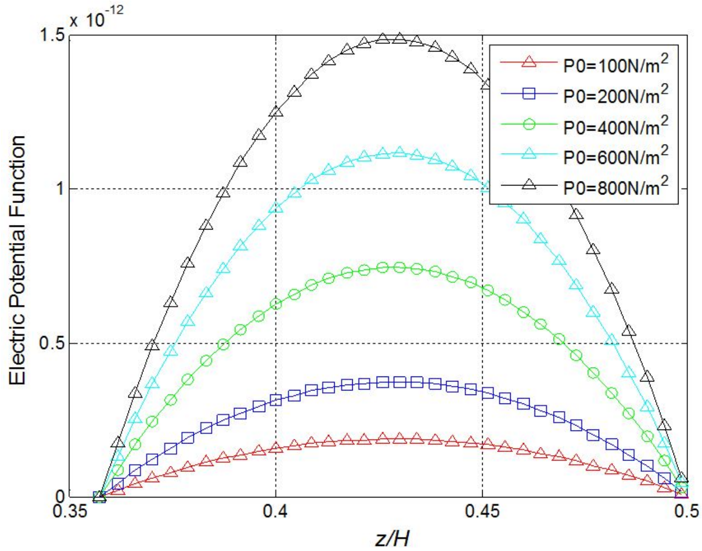

Fig. 12 Effect of normal load on electrical potential function
($a = b = 400$ mm, $h = 2.5$ mm, $h_p = 1$, $Kw = 0$, $KG = 0$, $n = 2$)

depict that with increasing the power law index and the normal pressure, the electrical potential function of sandwich plate increases, respectively. Fig. 13 shows the effect of thickness ratio on electric potential function. It can be seen from this figure that the maximum electric potential of sandwich thick plate reduces with an increase in the thickness ratio. Figs. 14 and 15 illustrate the influence of the Winkler and Pasternak coefficients on electrical potential function. It is shown that

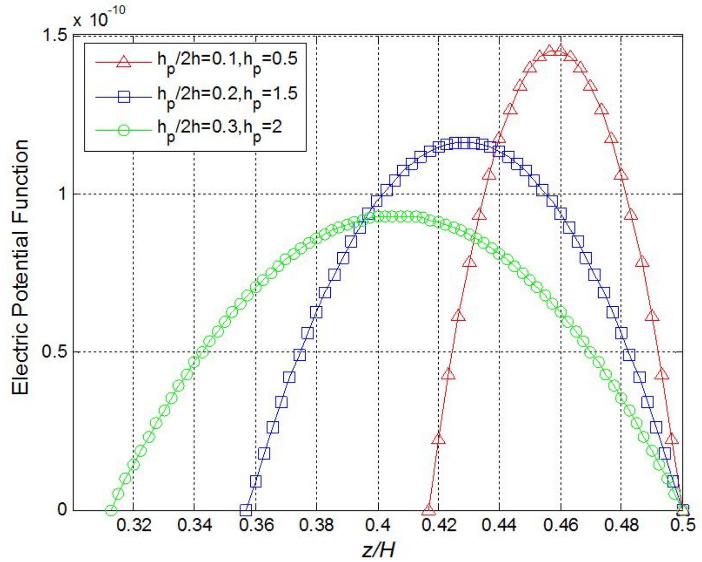

Fig. 13 Effect of thickness ratio on electrical potential function
($a = b = 400$ mm, $h = 2.5$ mm, $Kw = 0$, $KG = 0$, $P0 = 1000$ N/m^2)

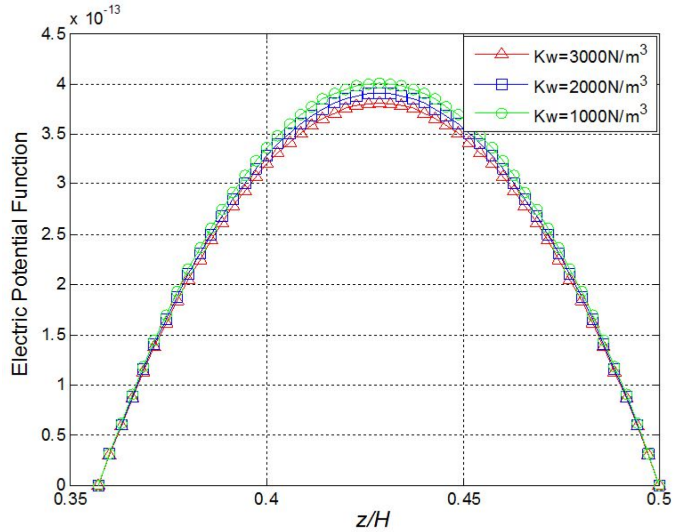

Fig. 14 Effect of Winkler parameter on electrical potential function
($a = b = 400$ mm, $h = 2.5$ mm, $h_p = 1$, $n = 2$, $KG = 0$, $P0 = 10$ N/m^2)

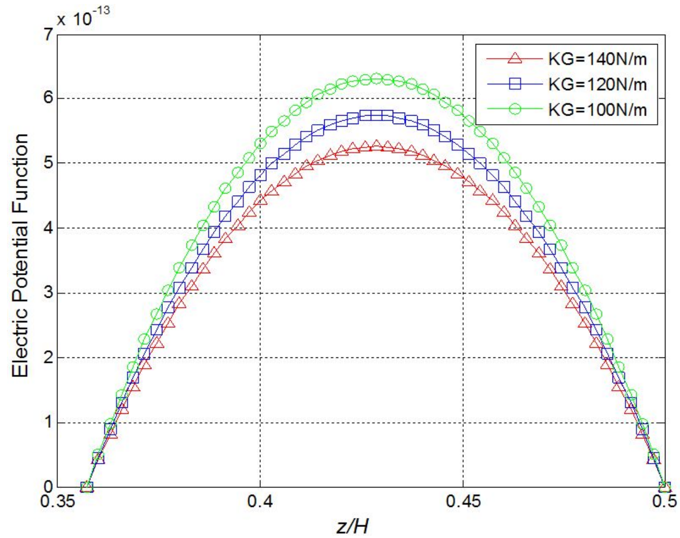

Fig. 15 Effect of Pasternak parameter on electrical potential function
($a = b = 400$ mm, $h = 2.5$ mm, $h_p = 1$, $n = 2$, $Kw = 0$, $P0 = 10$ N/m^2)

with increasing Winkler and Pasternak coefficients, the electrical potential function of sandwich plate reduces. As presented in these figures, electric potential is reduced with increasing the both parameters of elastic foundation. This decreasing is due to increasing the stiffness of the structure by increasing the parameters of elastic foundation. Increasing two parameters of elastic foundation tend to decrease the deflection and consequently decrease electric potential.

5. Conclusions

In this paper, based on TSDT, the bending, free vibration, buckling analysis and electric potential function of the sandwich thick plate considering FG core and composite piezoelectric layers on Pasternak foundation was investigated. Using variational method, the governing equations of motion for the sandwich thick plate considering FG core and composite piezoelectric layers were derived. Then to satisfy all edges simply supported boundary conditions, the Navier's type solution is used to obtain the dimensionless deflection, critical buckling load and natural frequency of the sandwich thick plate were obtained.

The results of this research showed that with increasing the elastic foundation parameters, the dimensionless natural frequency and critical buckling load increase and vice versa for dimensionless deflection and electrical potential function, because of the sandwich thick plate considering FG core and composite piezoelectric layers become stiffer. Moreover, the dimensionless natural frequency and critical buckling load diminish with an increase in the power law index, and vice versa for dimensionless deflection and electrical potential function, because of the sandwich thick plate becomes more flexible. The dimensionless deflection of sandwich plate decreases with an increase in the thickness ratio ($hp/2h$) and this result is similar to maximum electric potential function and vice versa for dimensionless natural frequency and critical buckling load.

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CC

Appendix A

$$A_{11} = \int_{-h}^{h} Q_{11} dz + 2\int_{h}^{h+h_p} \overline{C}_{11}\, dz \tag{A1}$$

$$A_{12} = \int_{-h}^{h} Q_{12} dz + 2\int_{h}^{h+h_p} \overline{C}_{12}\, dz \tag{A2}$$

$$A_{66} = 2\int_{h}^{h+h_p} \overline{C}_{44} dz + 2\int_{-h}^{h} Q_{44} dz \tag{A3}$$

$$B_{11} = \int_{-h}^{h} Q_{11} (z^2 - \frac{4z^4}{3H^2}) dz + 2\int_{h}^{h+h_p} \overline{C}_{11} (z^2 - \frac{4z^4}{3H^2}) dz \tag{A4}$$

$$B_{12} = \int_{-h}^{h} Q_{12} (z^2 - \frac{4z^4}{3H^2}) dz + 2\int_{h}^{h+h_p} \overline{C}_{12} (z^2 - \frac{4z^4}{3H^2}) dz \tag{A5}$$

$$B_{22} = \int_{-h}^{h} Q_{22} (z^2 - \frac{4z^4}{3H^2}) dz + 2\int_{h}^{h+h_p} \overline{C}_{11} (z^2 - \frac{4z^4}{3H^2}) dz \tag{A6}$$

$$D_{11} = \int_{-h}^{h} Q_{11} (-\frac{4z^4}{3H^2}) dz + 2\int_{h}^{h+h_p} \overline{C}_{11} (-\frac{4z^4}{3H^2}) dz \tag{A7}$$

$$D_{12} = \int_{-h}^{h} Q_{12} (-\frac{4z^4}{3H^2}) dz + 2\int_{h}^{h+h_p} \overline{C}_{12} (-\frac{4z^4}{3H^2}) dz \tag{A8}$$

$$D_{22} = \int_{-h}^{h} Q_{22} (-\frac{4z^4}{3H^2}) dz + 2\int_{h}^{h+h_p} \overline{C}_{11} (-\frac{4z^4}{3H^2}) dz \tag{A9}$$

$$F_{11} = \int_{-h}^{h} Q_{44} (z^2 - \frac{4z^4}{3H^2}) dz + 2\int_{h}^{h+h_p} \overline{C}_{44} (z^2 - \frac{4z^4}{3H^2}) dz \tag{A10}$$

$$F_{12} = \int_{-h}^{h} Q_{44} (-\frac{8z^4}{3H^2}) dz + 2\int_{h}^{h+h_p} \overline{C}_{44} (-\frac{8z^4}{3H^2}) dz \tag{A11}$$

$$H_{11} = \int_{-h}^{h} Q_{11} (z^4 - \frac{4z^6}{3H^2}) dz + 2\int_{h}^{h+h_p} \overline{C}_{11} (z^4 - \frac{4z^6}{3H^2}) dz \tag{A12}$$

$$H_{12} = \int_{-h}^{h} Q_{12} (z^4 - \frac{4z^6}{3H^2}) dz + 2\int_{h}^{h+h_p} \overline{C}_{12} (z^4 - \frac{4z^6}{3H^2}) dz \tag{A13}$$

$$H_{22} = \int_{-h}^{h} Q_{22} (z^4 - \frac{4z^6}{3H^2}) dz + 2\int_{h}^{h+h_p} \overline{C}_{11} (z^4 - \frac{4z^6}{3H^2}) dz \tag{A14}$$

$$K_{11} = \int_{-h}^{h} Q_{11} (-\frac{4z^6}{3H^2}) dz + 2\int_{h}^{h+h_p} \overline{C}_{11} (-\frac{4z^6}{3H^2}) dz \tag{A15}$$

$$K_{12}=\int_{-h}^{h}Q_{12}(-\frac{4z^6}{3H^2})dz+2\int_{h}^{h+h_p}\overline{C}_{12}(-\frac{4z^6}{3H^2})dz \tag{A16}$$

$$L_{11}=\int_{-h}^{h}Q_{44}(z^4-\frac{4z^6}{3H^2})dz+2\int_{h}^{h+h_p}\overline{C}_{44}(z^4-\frac{4z^6}{3H^2})dz \tag{A17}$$

$$L_{12}=\int_{-h}^{h}Q_{44}(-\frac{8z^6}{3H^2})dz+2\int_{h}^{h+h_p}\overline{C}_{44}(-\frac{8z^6}{3H^2})dz \tag{A18}$$

$$S_{11}=\int_{-h}^{h}Q_{55}(1-\frac{4z^2}{H^2})dz+2\int_{h}^{h+h_p}C_{55}(1-\frac{4z^2}{H^2})dz \tag{A19}$$

$$S_{22}=\int_{-h}^{h}Q_{55}(z^2-\frac{4z^4}{H^2})dz+2\int_{h}^{h+h_p}C_{55}(z^2-\frac{4z^4}{H^2})dz \tag{A20}$$

$$\mu_1=\int_{h}^{h+h_p}\overline{e}_{31}f(z)_{,z}\,dz+\int_{-h-h_p}^{-h}\overline{e}_{31}f(z)_{,z}\,dz \tag{A21}$$

$$\mu_2=\int_{h}^{h+h_p}\overline{e}_{31}f(z)_{,z}\,zdz+\int_{-h-h_p}^{-h}\overline{e}_{31}f(z)_{,z}\,zdz \tag{A22}$$

$$\mu_3=\int_{h}^{h+h_p}\overline{e}_{31}f(z)_{,z}\,z^3dz+\int_{-h-h_p}^{-h}\overline{e}_{31}f(z)_{,z}\,z^3dz \tag{A23}$$

$$\mu_4=\int_{h}^{h+h_p}-e_{51}f(z)dz+\int_{-h-h_p}^{-h}-e_{51}f(z)dz \tag{A24}$$

$$\mu_5=\int_{h}^{h+h_p}-e_{51}f(z)z^2dz+\int_{-h-h_p}^{-h}-e_{51}f(z)z^2dz \tag{A25}$$

$$\lambda_1=\int_{-h-h_p}^{-h}e_{51}(1-\frac{4z^2}{H^2})+\overline{e}_{31}(1-\frac{4z^2}{H^2})dz+\int_{h}^{h+h_p}e_{51}(1-\frac{4z^2}{H^2})+\overline{e}_{31}(1-\frac{4z^2}{H^2})dz \tag{A26}$$

$$\lambda_2=\int_{-h-h_p}^{-h}e_{51}(1-\frac{4z^2}{H^2})+\overline{e}_{31}(-\frac{4z^2}{H^2})dz+\int_{h}^{h+h_p}e_{51}(1-\frac{4z^2}{H^2})+\overline{e}_{31}(-\frac{4z^2}{H^2})dz \tag{A27}$$

$$\lambda_3=\int_{-h-h_p}^{-h}-\overline{\eta}_{33}f_{,zz}\,dz+\int_{h}^{h+h_p}-\overline{\eta}_{33}f_{,zz}\,dz \tag{A28}$$

$$\lambda_4=\frac{\lambda_1}{\lambda_3} \tag{A29}$$

$$\lambda_5=\frac{\lambda_2}{\lambda_3} \tag{A30}$$

Appendix B

$$S_{11} = -A_{11}\alpha^2 - A_{66}\beta^2 \tag{B1}$$

$$S_{12} = -(A_{12} + A_{66})\alpha\beta \tag{B2}$$

$$S_{13} = \mu_1\lambda_4\alpha^2 \tag{B3}$$

$$S_{14} = \mu_1\lambda_4\alpha\beta \tag{B4}$$

$$S_{15} = \mu_1\lambda_5(\alpha^3 + \alpha\beta^2) \tag{B5}$$

$$S_{21} = -(A_{12} + A_{66})\alpha\beta \tag{B6}$$

$$S_{22} = -A_{11}\beta^2 - A_{66}\alpha^2 \tag{B7}$$

$$S_{23} = \mu_1\lambda_4\alpha\beta \tag{B8}$$

$$S_{24} = \mu_1\lambda_4\beta^2 \tag{B9}$$

$$S_{25} = \mu_1\lambda_5(\alpha^2\beta + \beta^3) \tag{B10}$$

$$S_{31} = 0 \tag{B11}$$

$$S_{32} = 0 \tag{B12}$$

$$S_{33} = -B_{11}\alpha^2 + \mu_2\lambda_4\alpha^2 - F_{11}\beta^2 - S_{11} + \frac{4H_{11}}{3H^2}\alpha^2 - \frac{4\mu_3\lambda_4}{3H^2}\alpha^2 + \frac{4L_{11}}{3H^2}\beta^2 + \frac{4S_{22}}{H^2} \tag{B13}$$

$$S_{34} = -B_{12}\alpha\beta + \mu_2\lambda_4\alpha\beta - F_{11}\alpha\beta + \frac{4H_{12}}{3H^2}\alpha\beta + \frac{4\mu_3\lambda_4}{3H^2}\alpha\beta - \frac{4L_{11}}{3H^2}\alpha\beta \tag{B14}$$

$$\begin{aligned} S_{35} = &-D_{11}\alpha^3 - D_{12}\alpha\beta^2 + \mu_2\lambda_5(\alpha^3 + \alpha\beta^2) - F_{12}\alpha\beta - S_{11}\alpha + \frac{4K_{11}}{3H^2}\alpha^3 + \frac{4K_{12}}{3H^2}\alpha\beta^2 \\ &- \frac{4\mu_3\lambda_5}{3H^2}\alpha^3 - \frac{4\mu_3\lambda_5}{3H^2}\alpha\beta^2 + \frac{4L_{12}}{3H^2}\alpha\beta^2 + \frac{4S_{22}}{H^2}\alpha \end{aligned} \tag{B15}$$

$$S_{41} = 0 \tag{B16}$$

$$S_{42} = 0 \tag{B17}$$

$$S_{43} = -B_{12}\alpha\beta + \mu_2\lambda_4\alpha\beta - F_{11}\alpha\beta + \frac{4H_{12}}{3H^2}\alpha\beta + \frac{4\mu_3\lambda_4}{3H^2}\alpha\beta - \frac{4L_{11}}{3H^2}\alpha\beta \quad \text{(B18)}$$

$$S_{44} = -B_{22}\beta^2 + \mu_2\lambda_4\beta^2 - F_{11}\alpha^2 - S_{11} + \frac{4H_{22}}{3H^2}\beta^2 - \frac{4\mu_3\lambda_4}{3H^2}\beta^2 - \frac{4L_{11}}{3H^2}\alpha^2 + \frac{4S_{22}}{H^2} \quad \text{(B19)}$$

$$\begin{aligned} S_{45} &= -D_{22}\beta^3 - D_{12}\alpha^2\beta + \mu_2\lambda_5(\alpha^2\beta + \beta^3) - F_{12}\alpha^2\beta - S_{11}\beta + \frac{4K_{11}}{3H^2}\beta^3 + \frac{4K_{12}}{3H^2}\alpha^2\beta \\ &- \frac{4\mu_3\lambda_5}{3H^2}\alpha^2\beta - \frac{4\mu_3\lambda_5}{3H^2}\beta^3 - \frac{4L_{12}}{3H^2}\alpha^2\beta + \frac{4S_{22}}{H^2}\beta \end{aligned} \quad \text{(B20)}$$

$$S_{51} = 0 \quad \text{(B21)}$$

$$S_{52} = 0 \quad \text{(B22)}$$

$$S_{53} = -S_{11}\alpha - \frac{4H_{11}}{3H^2}\alpha^3 - \frac{4\mu_3\lambda_4}{3H^2}\alpha^3 + \frac{4H_{12}}{3H^2}\alpha\beta^2 + \frac{4\mu_3\lambda_4}{3H^2}\alpha\beta - \frac{8L_{11}}{3H^2}\alpha\beta^2 + \frac{4S_{22}}{H^2}\alpha \quad \text{(B23)}$$

$$\begin{aligned} S_{54} &= -S_{11}\beta + \frac{4H_{12}}{3H^2}\alpha^2\beta - \frac{4\mu_3\lambda_4}{3H^2}\alpha^2\beta + \frac{8K}{3H^2}\alpha^2\beta + \frac{4H_{22}}{3H^2}\beta^3 - \frac{4\mu_3\lambda_4}{3H^2}\beta^3 \\ &+ \frac{8L_{11}}{3H^2}\alpha^2\beta + \frac{8S_{22}}{3H^2}\beta \end{aligned} \quad \text{(B24)}$$

$$\begin{aligned} S_{55} &= -S_{11}(\alpha^2 + \beta^2) + \frac{4K_{11}}{3H^2}\alpha^4 + \frac{4K_{12}}{3H^2}\alpha^2\beta^2 - \frac{4\mu_3\lambda_4}{3H^2}\alpha^4 - \frac{4\mu_3\lambda_5}{3H^2}\alpha^2\beta^2 \\ &+ \frac{4K_{11}}{3H^2}\beta^4 + \frac{4K_{12}}{3H^2}\alpha^2\beta^2 - \frac{4\mu_3\lambda_5}{3H^2}\alpha^2\beta^2 - \frac{4\mu_3\lambda_5}{3H^2}\beta^4 + \frac{8L_{12}}{3H^2}\alpha^2\beta^2 \\ &+ \frac{4S_{22}}{H^2}(\alpha^2 + \beta^2) - K_w + K_G(-\alpha^2 - \beta^2) \end{aligned} \quad \text{(B25)}$$

$$m_{11} = I_0 \quad \text{(B26)}$$

$$m_{12} = 0 \quad \text{(B27)}$$

$$m_{13} = I_1 - C_1 I_3 \quad \text{(B28)}$$

$$m_{14} = 0 \quad \text{(B29)}$$

$$m_{15} = -C_1 I_3 \alpha \quad \text{(B30)}$$

$$m_{21} = 0 \tag{B31}$$

$$m_{22} = I_0 \tag{B32}$$

$$m_{23} = 0 \tag{B33}$$

$$m_{24} = I_1 - C_1 I_3 \tag{B34}$$

$$m_{25} = -C_1 I_3 \beta \tag{B35}$$

$$m_{31} = I_1 - C_1 I_3 \tag{B36}$$

$$m_{32} = 0 \tag{B37}$$

$$m_{33} = I_2 - 2C_1 I_4 + C_1^2 I_6 \tag{B38}$$

$$m_{34} = 0 \tag{B39}$$

$$m_{35} = -C_1 I_4 \alpha + C_1^2 I_6 \alpha \tag{B40}$$

$$m_{41} = 0 \tag{B41}$$

$$m_{42} = I_1 - C_1 I_3 \tag{B42}$$

$$m_{43} = 0 \tag{B43}$$

$$m_{44} = I_2 - 2C_1 I_4 + C_1^2 I_6 \tag{B44}$$

$$m_{45} = -C_1 I_4 \beta + C_1^2 I_6 \beta \tag{B45}$$

$$m_{51} = -C_1 I_3 \alpha \tag{B46}$$

$$m_{52} = -C_1 I_3 \beta \tag{B47}$$

$$m_{53} = -\alpha (C_1 I_4 - C_1^2 I_6) \tag{B48}$$

$$m_{54} = -\beta (C_1 I_4 - C_1^2 I_6) \tag{B49}$$

$$m_{55} = -C_1^2 I_6 \alpha^2 - C_1^2 I_6 \beta^2 + I_0 \tag{B50}$$