Steel and Composite Structures, *Vol. 19, No. 5 (2015) 1259-1277*
DOI: http://dx.doi.org/10.12989/scs.2015.19.5.1259

On bending, buckling and vibration responses of functionally graded carbon nanotube-reinforced composite beams

S.H. Tagrara [1,2], Abdelkader Benachour [1,2],
Mohamed Bachir Bouiadjra [2,3] and Abdelouahed Tounsi *[1,2,3,4]

[1] *Material and Hydrology Laboratory, University of Sidi Bel Abbes, Faculty of Technology, Civil Engineering Department, Algeria*
[2] *Algerian National Thematic Agency of Research in Science and Technology (ATRST), Algeria*
[3] *Laboratoire des Structures et Matériaux Avancés dans le Génie Civil et Travaux Publics, Université de Sidi Bel Abbes, Faculté de Technologie, Département de génie civil, Algeria*
[4] *Laboratoire de Modélisation et Simulation Multi-échelle, Département de Physique, Faculté des Sciences Exactes, Département de Physique, Université de Sidi Bel Abbés, Algeria*

(*Received January 14, 2015, Revised May 05, 2015, Accepted May 07, 2015*)

Abstract. In this work, a trigonometric refined beam theory for the bending, buckling and free vibration analysis of carbon nanotube-reinforced composite (CNTRC) beams resting on elastic foundation is developed. The significant feature of this model is that, in addition to including the shear deformation effect, it deals with only 3 unknowns as the Timoshenko beam (TBM) without including a shear correction factor. The single-walled carbon nanotubes (SWCNTs) are aligned and distributed in polymeric matrix with different patterns of reinforcement. The material properties of the CNTRC beams are assessed by employing the rule of mixture. To examine accuracy of the present theory, several comparison studies are investigated. Furthermore, the effects of different parameters of the beam on the bending, buckling and free vibration responses of CNTRC beam are discussed.

Keywords: CNTRC beams; bending; buckling; free vibration; elastic foundation

1. Introduction

Recently, Carbon nanotubes (CNTs) become a new class of fiber reinforcement in polymer matrix composites due to their superior mechanical, electrical, and thermal properties (Thostenson *et al*. 2001, Esawi and Farag 2007, Besseghier *et al*. 2015, Adda Bedia *et al*. 2015) and have taken a considerable research interests in the materials engineering community (Tounsi *et al*. 2013a, Benguediab *et al*. 2014). Compared with the classical carbon fiber-reinforced polymer composites, carbon nanotube-reinforced composites (CNTRCs) have the potential of improving increased strength and stiffness. The polymer composites reinforced by aligned CNT arrays were investigated in the first time by Ajayan *et al*. (1994). From then, many researchers (Odegard *et al*. 2003, Thostenson and Chou 2003, Griebel and Hamaekers 2004, Fidelus *et al*. 2005, Hu *et al*. 2005, Zhu *et al*. 2007) studied the material characteristics of CNTRCs. Using a finite element,

*Corresponding author, Professor, E-mail: tou_abdel@yahoo.com

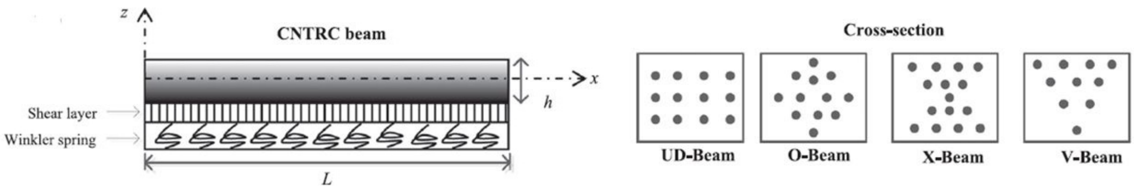
http://www.techno-press.org/?journal=scs&subpage=8 ISSN: 1229-9367 (Print), 1598-6233 (Online)

Ashrafi and Hubert (2006) analyzed the elastic properties of carbon nanotube array/polymer composites. Xu *et al*. (2006) studied the thermal response of single-walled carbon nanotubes (SWCNT) polymer–matrix composites. Han and Elliott (2007) investigated the elastic characteristics of polymer/carbon nanotube composites by employing molecular dynamics (MD). Wuite and Adali (2005) presented a multi-scale analysis of bending deflection and stresses of CNT-reinforced polymer composite beams. Vodenitcharova and Zhang (2006) presented a continuum mechanics model for the uniform bending of a nanocomposite beam with circular cross section, comprising of a SWCNT and a matrix. Ray and Batra (2007) presented a new carbon CNT-reinforced 1–3 piezoelectric composite for the active control of smart structures. A micromechanical analysis has been developed to determine the effective piezoelectric and elastic modulus of nanocomposite. Based on Timoshenko beam theory, Ke *et al*. (2013) studied the dynamic stability response of functionally graded (FG) nanocomposite beams reinforced by SWCNTs. Bakhti *et al*. (2013) studied the nonlinear cylindrical bending behavior of FG nanocomposite plates reinforced by SWCNTs using an efficient and simple refined theory. Kaci *et al*. (2012) investigated the nonlinear cylindrical bending of simply supported FG nanocomposite plates reinforced by SWCNTs. Zhu *et al*. (2012) employed the finite element method to examine the free vibration of FG-CNT reinforced composite plates. Wattanasakulpong and Ungbhakorn (2013) studied the bending, buckling and vibration behaviors of carbon nanotube-reinforced composite (CNTRC) beams where several higher-order shear deformation theories are presented and discussed in details. Lei *et al*. (2013a) utilized the element-free kp-Ritz method to investigate the free vibration of FG-CNT reinforced composite rectangular plates in a thermal environment. Lei *et al*. (2013b) presented a large deflection analysis of FG-CNT reinforced composite plates by considering different boundary conditions. Alibeigloo (2014) studied the bending behavior of a CNT reinforced composite rectangular host plate attached to thin piezoelectric layers subjected to thermal load and or electric field. Aydogdu (2014) investigated the vibration response of aligned carbon nanotube reinforced composite beams by using the Ritz method. Wu and Li (2014) presented a unified formulation of finite prism techniques based on Reissner's mixed variational theorem for analysis of 3D free vibration of FG-CNT reinforced composite plates and laminated fiber-reinforced composite plates. Recently, the stability of FG sandwich plate was studied by Swaminathan and Naveenkumar (2014) using a higher order refined computational models. In literature survey, we can found also some studies dealing about beams resting on elastic foundations such as (Yesilce 2010, Yesilce and Catal 2009).

In the present work, the bending, buckling and vibration response of the CNTRC beams is investigated using a trigonometric refined beam theory. This theory is based on assumption that the in-plane and transverse displacements consist of bending and shear components, in which the bending components do not contribute toward shear forces and, likewise, the shear components do not contribute toward bending moments. The simply supported CNTRC beams are supported by the Pasternak elastic foundation, including a shear layer and Winkler spring. Novel analytical solutions of deflections, stresses, buckling loads, natural frequencies are developed and discussed in details. Several aspects of spring parameters, thickness ratios, CNT volume fractions, types of CNT distribution, etc., which have considerable impact on the analytical solutions are also studied.

2. Functionally graded carbon nanotube-reinforced composites beams

The CNTRC beam under the present study is made from a mixture of the SWCNTs and isotropic polymer matrix. Fig. 1(a) shows a CNTRC beam, having length (L) and thickness (h),



(a) (b)

Fig. 1 Geometry of a CNTRC beam on elastic foundation (a); and cross sections of different patterns of reinforcement (b)

supported by the Pasternak elastic foundation. Four different patterns of reinforcement over the cross sections are considered in this study as is indicated in Fig. 1(b).

The material properties of CNTRC beams can be computed utilizing the rule of mixture which gives the effective Young's modulus and shear modulus of CNTRC beams as (Bakhti *et al*. 2013, Kaci *et al*. 2012, Wattanasakulpong and Ungbhakorn 2013).

$$E_{11} = \eta_1 V_{cnt} E_{11}^{cnt} + V_p E_p \tag{1a}$$

$$\frac{\eta_2}{E_{22}} = \frac{V_{cnt}}{E_{22}^{cnt}} + \frac{V_p}{E_p} \tag{1b}$$

$$\frac{\eta_3}{G_{12}} = \frac{V_{cnt}}{G_{12}^{cnt}} + \frac{V_p}{G_p} \tag{1c}$$

where E_{11}^{cnt}; E_{22}^{cnt} and G_{12}^{cnt} are the Young's modulus and shear modulus of SWCNT, respectively and E^p and G^p are the corresponding material properties of the polymer matrix. Also, V_{cnt} and V_p are the volume fractions for carbon nanotube and the polymer matrix, respectively, with the relation of $V_{cnt} + V_p = 1$. To introduce the size-dependent material properties of SWCNT, the CNT efficiency parameters, η_i (i = 1, 2, 3), are considered. They can be obtained from matching the elastic moduli of CNTRCs estimated by the MD simulation with the numerical results determined by the rule of mixture (Han and Elliott 2007). By employing the same rule, Poisson's ratio (ν) and mass density (ρ) of the CNTRC beams are expressed as

$$\nu = V_{cnt} \nu^{cnt} + V_p \nu^p, \quad \rho = V_{cnt} \rho^{cnt} + V_p \rho^p \tag{2}$$

where ν^{cnt}, ν^p and ρ^{cnt}, ρ^p are the Poisson's ratios and densities of the CNT and polymer matrix respectively. For different patterns of carbon nanotube reinforcement distributed within the cross sections of the beams as shown in Fig. 1(b), the continuous mathematical functions employing for introducing the distributions of material constituents are expressed below

UD-Beam

$$V_{cnt} = V_{cnt}^* \tag{3a}$$

O-Beam

$$V_{cnt} = 2\left(1 - 2\frac{|z|}{h}\right)V_{cnt}^{*} \tag{3b}$$

X-Beam

$$V_{cnt} = 4\frac{|z|}{h}V_{cnt}^{*} \tag{3c}$$

V-Beam

$$V_{cnt} = \left(1 + \frac{2z}{h}\right)V_{cnt}^{*} \tag{3d}$$

where V_{cnt}^{*} is the considered volume fraction of CNTs, which can be determined from the following equation

$$V_{cnt}^{*} = \frac{W_{cnt}}{W_{cnt} + (\rho^{cnt}/\rho^{m})(1 - W_{cnt})} \tag{4}$$

where W_{cnt} is the mass fraction of CNTs. From Eq. (3), it can be seen that the O-, X-and V-Beams are some types of functionally graded beams in which their material constituents are varied continuously within their thicknesses; while, the UD-Beam has uniformly distributed CNT reinforcement. In this work, the CNT efficiency parameters (η_i) associated with the considered volume fraction (V_{cnt}^{*}) are: $\eta_1 =$ 1.2833 and $\eta_2 = \eta_3 =$ 1.0566 for the case of $V_{cnt}^{*} = 0.12$; $\eta_1 = 1.3414$ and $\eta_2 = \eta_3 = 1.7101$ for the case of $V_{cnt}^{*} = 0.17$; $\eta_1 = 1.3238$ and $\eta_2 = \eta_3 =$ 1.7380 for the case of $V_{cnt}^{*} = 0.28$ (Yas and Samadi 2012).

3. Theory and formulations

3.1 Kinematics and constitutive equations

The displacement field of the present theory, based on Ould Larbi *et al*. (2013), Al-Basyouni *et al*. (2015), Bourada *et al*. (2015) and Bennai *et al*. (2015) beam theory, can be obtained as

$$u(x,z,t) = u_0(x,t) - z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x} \tag{5a}$$

$$w(x,z,t) = w_b(x,t) + w_s(x,t) \tag{5b}$$

where u_0 is the axial displacement, w_b and w_s are the bending and shear components of transverse displacement along the mid-plane of the beam. In this work, the shape function $f(z)$ is chosen based on a trigonometric function as (Bouderba *et al*. 2013, Tounsi *et al*. 2013b)

$$f(z) = z - \frac{h}{\pi}\sin\left(\frac{\pi}{h}z\right) \tag{6}$$

The non-zero strains are given by

$$\varepsilon_x = \varepsilon_x^0 + z\, k_x^b + f(z)\, k_x^s \tag{7a}$$

$$\gamma_{xz} = g(z)\, \gamma_{xz}^s \tag{7b}$$

where

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x}, \quad k_x^b = -\frac{\partial^2 w_b}{\partial x^2}, \quad k_x^s = -\frac{\partial^2 w_s}{\partial x^2}, \quad \gamma_{xz}^s = \frac{\partial w_s}{\partial x}, \tag{7c}$$

$$g(z) = 1 - f'(z) \quad \text{and} \quad f'(z) = \frac{df(z)}{dz} \tag{7d}$$

By assuming that the material of CNTRC beam obeys Hooke’s law, the stresses in the beam become

$$\sigma_x = Q_{11}(z)\, \varepsilon_x \quad \text{and} \quad \tau_{xz} = Q_{55}(z)\, \gamma_{xz} \tag{8a}$$

$$Q_{11}(z) = \frac{E_{11}(z)}{1-\nu^2} \quad \text{and} \quad Q_{55}(z) = G_{12}(z) \tag{8b}$$

3.2 Equations of motion

Hamilton’s principle is employed herein to determine the equations of motion as follows (Talha and Singh 2010, Li *et al*. 2010, El Meiche *et al*. 2011, Benachour *et al*. 2011, Shahrjerdi *et al*. 2011, Jha *et al*. 2012, 2013, Berrabah *et al*. 2013, Nedri *et al*. 2014, Belabed *et al*. 2014, Hebali *et al*. 2014, Mahi *et al*. 2015, Ziane *et al*. 2015, Zemri *et al*. 2015)

$$\int_{t_1}^{t_2} (\delta U + \delta V - \delta K)\, dt = 0 \tag{9}$$

where δU is the virtual variation of the strain energy; δV is the virtual variation of the potential energy; and δK is the virtual variation of the kinetic energy.

The variation of the strain energy of the beam can be stated as

$$\begin{aligned} \delta U &= \int_0^L \int_{-h/2}^{h/2} (\sigma_x \delta\, \varepsilon_x + \tau_{xz} \delta\, \gamma_{xz})\, dz dx \\ &= \int_0^L \left(N \frac{d\delta\, u_0}{dx} - M_b \frac{d^2 \delta\, w_b}{dx^2} - M_s \frac{d^2 \delta\, w_s}{dx^2} + Q \frac{d\delta\, w_s}{dx} \right) dx \end{aligned} \tag{10}$$

where N, M_b, M_s and Q are the stress resultants defined as

$$(N, M_b, M_s) = \int_{-h/2}^{h/2} (1, z, f)\, \sigma_x dz \quad \text{and} \quad Q = \int_{-h/2}^{h/2} g\, \tau_{xz} dz \tag{11}$$

The variation of the potential energy by the transverse load q, the axial compressive force N_{x0} and the density of reaction force of foundation f_e can be written as (Akavci 2014, Zidi *et al*. 2014, Yaghoobi and Fereidoon 2014, Yaghoobi *et al*. 2014)

$$\delta V = -\int_0^L \left[(q + f_e)(\delta w_b + \delta w_s) + N_{x0} \frac{d(w_b + w_s)}{dx} \frac{d(\delta w_b + \delta w_s)}{dx} \right] dx \tag{12}$$

where the reaction force of foundation is given by (Khalfi *et al*. 2014. Ait Amar Meziane *et al*. 2014)

$$f_e = K_w w - K_s \frac{\partial^2 w}{\partial x^2} \tag{13}$$

where K_w and K_s are the Winkler and shearing layer spring constants which can be determined from $K_w = \beta_w A_{110} / L^2$ and $K_s = \beta_s A_{110}$ in which β_w and β_s are the corresponding spring constant factors. It is also defined that A_{110} is the extension stiffness or the value of A_{11} of a homogeneous beam made of pure matrix material.

The variation of the kinetic energy can be expressed as

$$\begin{aligned}\delta K &= \int_0^L \int_{-h/2}^{h/2} \rho(z)\left[\dot{u}\delta\dot{u} + \dot{w}\delta\dot{w}\right] dz dx \\ &= \int_0^L \left\{ I_0\left[\dot{u}_0\delta\dot{u}_0 + (\dot{w}_b + \dot{w}_s)(\delta\dot{w}_b + \delta\dot{w}_s)\right] - I_1\left(\dot{u}_0\frac{d\delta\dot{w}_b}{dx} + \frac{d\dot{w}_b}{dx}\delta\dot{u}_0\right)\right. \\ &\quad + I_2\left(\frac{d\dot{w}_b}{dx}\frac{d\delta\dot{w}_b}{dx}\right) - J_1\left(\dot{u}_0\frac{d\delta\dot{w}_s}{dx} + \frac{d\dot{w}_s}{dx}\delta\dot{u}_0\right) + K_2\left(\frac{d\dot{w}_s}{dx}\frac{d\delta\dot{w}_s}{dx}\right) \\ &\quad \left. + J_2\left(\frac{d\dot{w}_b}{dx}\frac{d\delta\dot{w}_s}{dx} + \frac{d\dot{w}_s}{dx}\frac{d\delta\dot{w}_b}{dx}\right)\right\} dx\end{aligned} \tag{14}$$

where dot-superscript convention indicates the differentiation with respect to the time variable t; $\rho(z)$ is the mass density; and (I_0, I_1, J_1, I_2, J_2, K_2) are the mass inertias defined as

$$(I_0, I_1, J_1, I_2, J_2, K_2) = \int_{-h/2}^{h/2} (1, z, f, z^2, z\,f, f^2)\rho(z) dz \tag{15}$$

Substituting the expressions for δU, δV, and δK from Eqs. (10), (12), and (14) into Eq. (9) and integrating by parts versus both space and time variables, and collecting the coefficients of δu_0, δw_b, and δw_s, the following equations of motion of the CNTRC beam are obtained

$$\delta u_0 : \; \frac{dN}{dx} = I_0\ddot{u}_0 - I_1\frac{d\ddot{w}_b}{dx} - J_1\frac{d\ddot{w}_s}{dx} \tag{16a}$$

$$\delta w_b : \; \frac{d^2 M_b}{dx^2} + q - f_e + N_{x0}\frac{d^2(w_b + w_s)}{dx^2} = I_0(\ddot{w}_b + \ddot{w}_s) + I_1\frac{d\ddot{u}_0}{dx} - I_2\frac{d^2\ddot{w}_b}{dx^2} - J_2\frac{d^2\ddot{w}_s}{dx^2} \tag{16b}$$

$$\delta w_s: \ \frac{d^2 M_s}{dx^2} + \frac{dQ}{dx} + q - f_e + N_{x0} \frac{d^2(w_b + w_s)}{dx^2} = I_0(\ddot{w}_b + \ddot{w}_s) + J_1 \frac{d\ddot{u}_0}{dx} - J_2 \frac{d^2 \ddot{w}_b}{dx^2} - K_2 \frac{d^2 \ddot{w}_s}{dx^2} \quad (16c)$$

By substituting Eq. (7) into Eq. (8) and the subsequent results into Eq. (11), the constitutive equations for the stress resultants are obtained as

$$N = A_{11} \frac{du_0}{dx} - B_{11} \frac{d^2 w_b}{dx^2} - B_{11}^s \frac{d^2 w_s}{dx^2} \quad (17a)$$

$$M_b = B_{11} \frac{du_0}{dx} - D_{11} \frac{d^2 w_b}{dx^2} - D_{11}^s \frac{d^2 w_s}{dx^2} \quad (17b)$$

$$M_s = B_{11}^s \frac{du_0}{dx} - D_{11}^s \frac{d^2 w_b}{dx^2} - H_{11}^s \frac{d^2 w_s}{dx^2} \quad (17c)$$

$$Q = A_{55}^s \frac{dw_s}{dx} \quad (17d)$$

where A_{11}, B_{11}, etc., are the beam stiffness, defined by

$$\left(A_{11}, B_{11}, D_{11}, B_{11}^s, D_{11}^s, H_{11}^s\right) = \int_{-h/2}^{h/2} Q_{11}\left(1, z, z^2, f(z), z\, f(z), f^2(z)\right) dz \quad (18a)$$

where

$$A_{55}^s = \int_{-h/2}^{h/2} Q_{55}\left[g(z)\right]^2 dz, \quad (18b)$$

Eq. (16) can be expressed in terms of displacements (u_0, w_b, w_s) by using Eqs. (17) and (16) as follows

$$A_{11} \frac{\partial^2 u_0}{\partial x^2} - B_{11} \frac{\partial^3 w_b}{\partial x^3} - B_{11}^s \frac{\partial^3 w_s}{\partial x^3} = I_0 \ddot{u}_0 - I_1 \frac{d\ddot{w}_b}{dx} - J_1 \frac{d\ddot{w}_s}{dx} \quad (19a)$$

$$\begin{aligned} &B_{11} \frac{\partial^3 u_0}{\partial x^3} - D_{11} \frac{\partial^4 w_b}{\partial x^4} - D_{11}^s \frac{\partial^4 w_s}{\partial x^4} + q - f_e + N_{x0} \frac{d^2(w_b + w_s)}{dx^2} \\ &= I_0(\ddot{w}_b + \ddot{w}_s) + I_1 \frac{d\ddot{u}_0}{dx} - I_2 \frac{d^2 \ddot{w}_b}{dx^2} - J_2 \frac{d^2 \ddot{w}_s}{dx^2} \end{aligned} \quad (19b)$$

$$\begin{aligned} &B_{11}^s \frac{\partial^3 u_0}{\partial x^3} - D_{11}^s \frac{\partial^4 w_b}{\partial x^4} - H_{11}^s \frac{\partial^4 w_s}{\partial x^4} + A_{55}^s \frac{\partial^2 w_s}{\partial x^2} + q - f_e + N_{x0} \frac{d^2(w_b + w_s)}{dx^2} \\ &= I_0(\ddot{w}_b + \ddot{w}_s) + J_1 \frac{d\ddot{u}_0}{dx} - J_2 \frac{d^2 \ddot{w}_b}{dx^2} - K_2 \frac{d^2 \ddot{w}_s}{dx^2} \end{aligned} \quad (19c)$$

3.3 Analytical solution

The Navier solution method is employed to obtain the analytical solutions for a simply supported CNTRC beam. The solution is assumed to be of the form

$$\begin{Bmatrix} u_0 \\ w_b \\ w_s \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} U_m \cos(\lambda\, x)\, e^{i\,\omega\, t} \\ W_{bm} \sin(\lambda\, x)\, e^{i\,\omega\, t} \\ W_{sm} \sin(\lambda\, x)\, e^{i\,\omega\, t} \end{Bmatrix} \tag{20}$$

where U_m, W_{bm}, and W_{sm} are arbitrary parameters to be determined, ω is the eigenfrequency associated with mth eigenmode, and $\lambda = m\pi\,/\,L$. The transverse load q is also expanded in Fourier series as

$$q(x) = \sum_{m=1}^{\infty} Q_m \sin(\lambda\, x) \tag{21}$$

where Q_m is the load amplitude calculated from

$$Q_m = \frac{2}{L}\int_0^L q(x)\sin(\lambda\, x)dx \tag{22}$$

The coefficients Q_m are given below for some typical loads. For the case of a sinusoidally distributed load, we have

$$m = 1 \quad \text{and} \quad Q_1 = q_0 \tag{23a}$$

and for the case of uniform distributed load, we have

$$Q_m = \frac{4q_0}{m\pi}, \quad (m = 1,3,5....) \tag{23b}$$

Substituting the expansions of u_0, w_b, w_s, and q from Eqs. (20) and (21) into the equations of motion Eq. (19), the analytical solutions can be obtained from the following equations

$$\left(\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} \right) \begin{Bmatrix} U_m \\ W_{bm} \\ W_{sm} \end{Bmatrix} = \begin{Bmatrix} 0 \\ Q_m \\ Q_m \end{Bmatrix} \tag{24}$$

where

$$\begin{aligned} &S_{11} = A_{11}\lambda^2, \quad S_{12} = -B_{11}\lambda^3, \quad S_{13} = -B_{11}^s\lambda^3, \quad S_{22} = D_{11}\lambda^4 + K_W + K_S\lambda^2 + N_{X0}\lambda^2, \\ &S_{23} = D_{11}^s\lambda^4 + K_W + K_S\lambda^2 + N_{X0}\lambda^2, \quad S_{33} = H_{11}^s\lambda^4 + A_{55}^s\lambda^2 + K_W + K_S\lambda^2 + N_{X0}\lambda^2 \end{aligned} \tag{25a}$$

$$\begin{aligned} &m_{11} = I_0, \quad m_{12} = -I_1\lambda, \quad m_{13} = -J_1\lambda, \\ &m_{22} = I_0 + I_2\lambda^2, \quad m_{23} = I_0 + J_2\lambda^2, \quad m_{33} = I_0 + K_2\lambda^2 \end{aligned} \tag{25b}$$

4. Numerical results and discussion

In this section, numerical results of bending, buckling and vibrations behaviors of CNTRC beams are presented and discussed. The effective material characteristics of CNTRC beams at ambient temperature employed throughout this work are given as follows. Poly methyl methacrylate (PMMA) is utilized as the matrix and its material properties are: $v^p = 0.3$; $\rho^p = 1190$ kg/m^3 and $E^p = 2.5$ GPa. For reinforcement material, the armchair (10, 10) SWCNTs is chosen with the following properties (Yas and Samadi 2012): $v^{cnt} = 0.19$; ρcn = 1400 kg/m^3; $E_{11}^{cnt} = 600$ GPa; $E_{22}^{cnt} = 10$ GPa and $G_{12}^{cnt} = 17.2$ GPa.

For convenience, the following nondimensionalizations are employed

- For bending analysis: $\bar{w} = 100\dfrac{E_p h^3}{q_0 L^4} w, \quad \bar{\sigma}_x = \dfrac{h}{q_0 L}\sigma_x\left(\dfrac{L}{2},\dfrac{h}{2}\right), \quad \bar{\tau}_{xz} = \dfrac{h}{q_0 L}\tau_{xz}(0,0),$
- For buckling: $\bar{N} = \dfrac{N_{x0}}{A_{110}}$
- For vibration analysis: $\bar{\omega} = \omega L\sqrt{\dfrac{I_{00}}{A_{110}}}$

where A_{110} and I_{00} are $A_{11}I_{00}$ and I_0 I0 of beam made of pure matrix material, respectively.

4.1 Bending analysis of CNTRC beams

For bending analysis of UD beams with and without elastic foundations, the present method agree well with the bending results of Wattanasakulpong and Ungbhakorn (2013) using third shear deformation theory as shown in Tables 1 and 2. It can be observed that the beams supported by elastic foundation have lower displacements and stresses compared to those of the beams without elastic foundation. Moreover, increasing amount of CNTs makes the CNTRC beams stiffer.

Figs. 2 and 3 present respectively the effect of both Winkler modulus parameter and the Pasternak shear modulus on the deflection of different types of CNTRC beams under uniform load. It is observed that as the Winkler and the Pasternak shear parameters increase the transverse

Table 1 Dimensionless displacements and stress of UD-Beam with and without elastic foundation under uniform loads

V_{cnt}^*	L/h	Theory	$\beta_w = 0, \beta_s = 0$			$\beta_w = 0.1, \beta_s = 0.02$		
			$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$
0.12	10	Ref [(a)]	0.704	8.399	0.701	0.594	7.053	0.602
		Present	0.703	8.458	0.718	0.593	7.103	0.617
	15	Ref [(a)]	0.524	11.849	0.716	0.400	9.556	0.568
		Present	0.524	11.888	0.736	0.400	9.019	0.584
	20	Ref [(a)]	0.461	15.448	0.725	0.311	10.316	0.520
		Present	0.460	15.479	0.746	0.311	10.336	0.536

Table 1 Continued

V_{cnt}^{*}	L/h	Theory	$\beta_w = 0, \beta_s = 0$			$\beta_w = 0.1, \beta_s = 0.02$		
			$\overline{w}$	$\overline{\sigma}_x$	$\overline{\tau}_{xz}$	$\overline{w}$	$\overline{\sigma}_x$	$\overline{\tau}_{xz}$
0.17	10	Ref [(a)]	0.449	8.268	0.704	0.403	7.374	0.638
		Present	0.448	8.319	0.722	0.401	7.419	0.654
	15	Ref [(a)]	0.344	11.762	0.719	0.286	9.737	0.614
		Present	0.344	11.796	0.739	0.286	9.764	0.631
	20	Ref [(a)]	0.307	15.384	0.726	0.232	11.568	0.575
		Present	0.307	15.410	0.748	0.232	11.587	0.592
0.28	10	Ref [(a)]	0.325	8.562	0.697	0.299	7.869	0.647
		Present	0.324	8.631	0.713	0.299	7.933	0.662
	15	Ref [(a)]	0.235	11.959	0.714	0.206	10.469	0.638
		Present	0.234	12.004	0.733	0.206	10.511	0.655
	20	Ref [(a)]	0.203	15.530	0.723	0.167	12.751	0.613
		Present	0.203	15.566	0.743	0.167	12.781	0.631

(a) Taken from Wattanasakulpong and Ungbhakorn (2013)

Table 2 Dimensionless displacements and stress of UD-Beam with and without elastic foundation under sinusoidal loads

V_{cnt}^{*}	L/h	Theory	$\beta_w = 0, \beta_s = 0$			$\beta_w = 0.1, \beta_s = 0.02$		
			$\overline{w}$	$\overline{\sigma}_x$	$\overline{\tau}_{xz}$	$\overline{w}$	$\overline{\sigma}_x$	$\overline{\tau}_{xz}$
0.12	10	Ref [(a)]	0.704	8.399	0.701	0.594	7.053	0.602
		Present	0.703	8.458	0.718	0.593	7.103	0.617
	15	Ref [(a)]	0.524	11.849	0.716	0.400	9.556	0.568
		Present	0.524	11.888	0.736	0.400	9.019	0.584
	20	Ref [(a)]	0.461	15.448	0.725	0.311	10.316	0.520
		Present	0.460	15.479	0.746	0.311	10.336	0.536
0.17	10	Ref [(a)]	0.449	8.268	0.704	0.403	7.374	0.638
		Present	0.448	8.319	0.722	0.401	7.419	0.654
	15	Ref [(a)]	0.344	11.762	0.719	0.286	9.737	0.614
		Present	0.344	11.796	0.739	0.286	9.764	0.631
	20	Ref [(a)]	0.307	15.384	0.726	0.232	11.568	0.575
		Present	0.307	15.410	0.748	0.232	11.587	0.592
0.28	10	Ref [(a)]	0.325	8.562	0.697	0.299	7.869	0.647
		Present	0.324	8.631	0.713	0.299	7.933	0.662
	15	Ref [(a)]	0.235	11.959	0.714	0.206	10.469	0.638
		Present	0.234	12.004	0.733	0.206	10.511	0.655
	20	Ref [(a)]	0.203	15.530	0.723	0.167	12.751	0.613
		Present	0.203	15.566	0.743	0.167	12.781	0.631

[(a)] Taken from Wattanasakulpong and Ungbhakorn (2013)

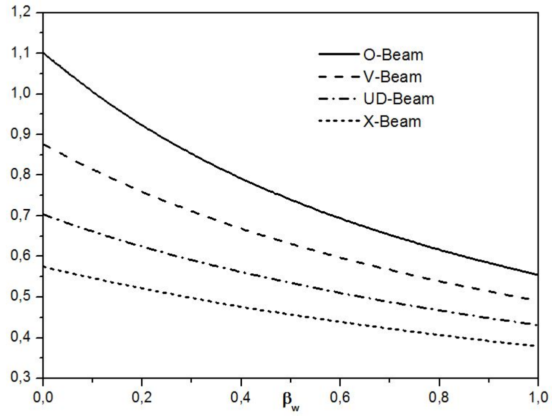


Fig. 2 Effect of Winkler modulus parameter on the dimensionless transverse displacements of CNTRC beams under uniform load (L/h = 10; β_s = 0; V_{cnt}^* = 0.12)

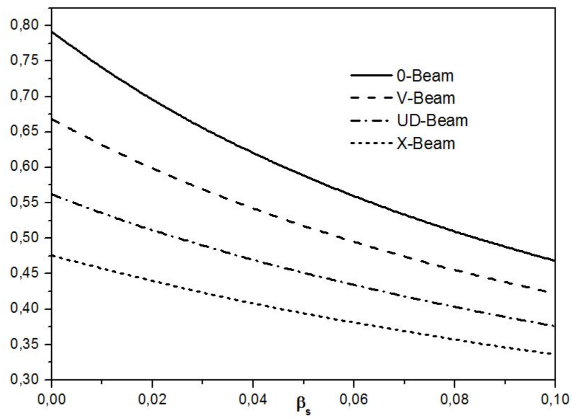


Fig. 3 Effect of Pasternak shear modulus parameter on the dimensionless transverse displacements of CNTRC beams under uniform load (L/h = 10; β_s = 0.4; V_{cnt}^* = 0.12)

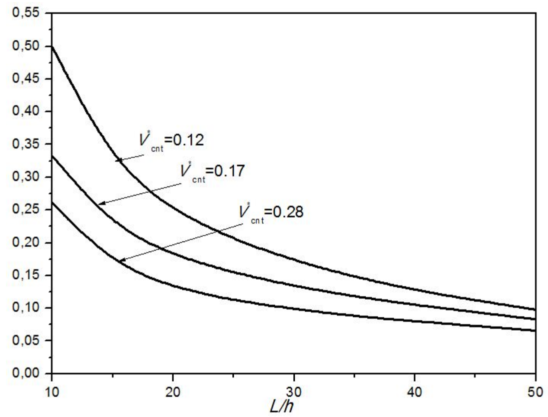


Fig. 4 Dimensionless transverse displacements of X Beam on elastic foundation with various thickness ratios (β_w = 0.1, β_s = 0.02)

displacement decreases. This decreasing trend is attributed to the stiffness of the elastic medium. Indeed, it is found from Eq. (24) that the foundation parameters appear in the stiffness matrix [*S*] and at last increase the total stiffness of the CNTRC beam. It can be also observed that the strongest beam is the X-Beam with the smallest deflection, and followed by the UD-, V-and O-Beams, respectively.

The influence of volume fractions of CNTs on the deflection of the strongest beam (X-beam) is demonstrated in Fig. 4 by employing the trigonometric refined beam theory. Increasing the volume fractions of CNTs leads to reduction in the deflections. The dramatic reduction of the deflections is observed in the range of L / h = 10 to 30.

4.2 Buckling analysis of CNTRC beams

In this section, numerical results of buckling analysis of CNTRC beams are discussed. The present results based on the refined trigonometric beam theory are in a good agreement with the buckling results of third shear deformation CNTRC beam theory and Timoshenko CNTRC beams documented by Wattanasakulpong and Ungbhakorn (2013) and Yas and Samadi (2012), respectively, as shown in Table 3. Because of the presence of stretching-bending coupling

Table 3 Comparison of critical loads for CNTRC beam with and without elastic foundation ($L / h = 15$, $V^*_{cnt} = 0.12$)

Source	$\beta_w = 0, \beta_s = 0$			$\beta_w = 0.1, \beta_s = 0.02$		
	UD	O	X	UD	O	X
FSDT [(a)]	0.1032	0.0604	0.1367	0.1333	0.0905	0.1668
TSDT [(a)]	0.0985	0.0575	0.1291	0.1287	0.0876	0.1590
Ref [(b)]	0.0986	0.0588	0.1288	0.1287	0.0889	0.1590
Present	0.0985	0.0575	0.1291	0.1286	0.0876	0.1592

[(a)] Taken from Wattanasakulpong and Ungbhakorn (2013)
[(b)] Taken from Yas and Samadi (2012)

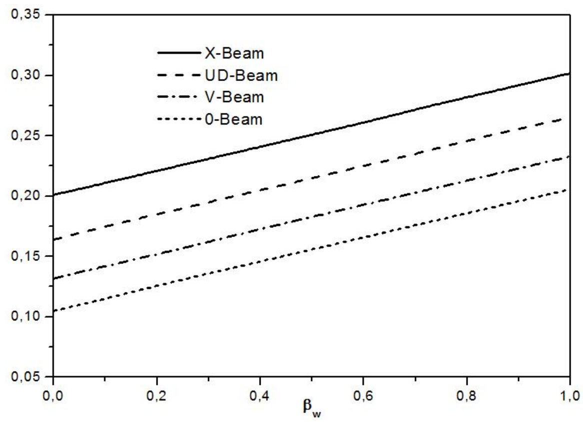


Fig. 5 Effect of Winkler modulus parameter on the critical buckling loads of CNTRC beams ($L/h = 10$; $\beta_s = 0$; $V^*_{cnt} = 0.12$)

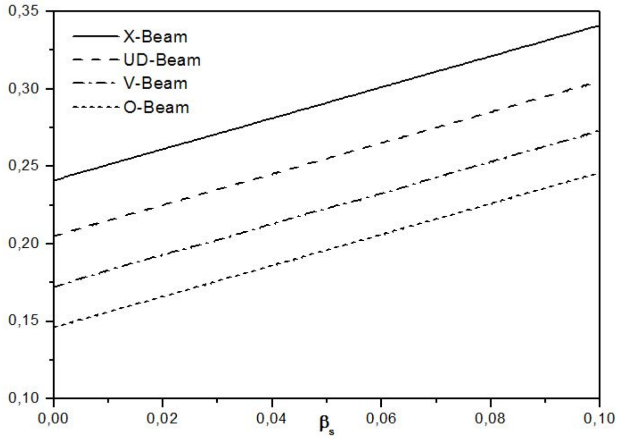


Fig. 6 Effect of Pasternak shear modulus parameter the critical buckling loads of CNTRC beams (L/h = 10; β_s = 0.4; V_{cnt}^* = 0.12)

characteristic in V-Beam due to its asymmetry, this coupling engenders deflections and bending moments when the beam is under compressive loading. Therefore, V-Beam has no bifurcation-type buckling (Liew *et al*. 2003). According to bending study, the X-Beam is the strongest beam that supports the largest buckling load and followed by the UD-Beam and O-Beam.

Figs. 5 and 6 show respectively the influence of both Winkler modulus parameter and the Pasternak shear modulus on the buckling load of different types of CNTRC beams. It is observed that the buckling loads increase linearly as the increase of the spring constant factors.

The effect of volume fractions of CNTs on the critical buckling load of the strongest beam (X-beam) is shown in Fig. 7 using the trigonometric refined beam theory. Decreasing the volume fractions of CNTs leads to reduction in the buckling loads. The dramatic reduction of the buckling loads is observed in the range of L / h = 10 to 30.

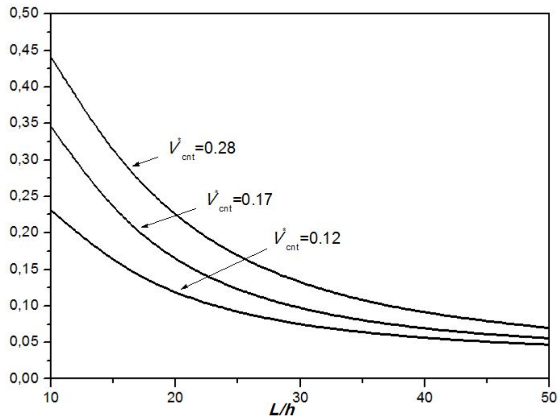


Fig. 7 Dimensionless critical buckling loads of X Beam on elastic foundation with various thickness ratios (β_w = 0.1, β_s = 0.02)

Table 4 Comparison of fundamental frequencies for CNTRC beam with and without elastic foundation (L/h = 15, V_{cnt}^{*} = 0.12)

Source	$\beta_w = 0, \beta_s = 0$				$\beta_w = 0.1, \beta_s = 0.02$			
	UD	O	X	V	UD	O	X	V
FSDT [a]	0.9976	0.7628	1.1485	0.8592	1.1339	0.9339	1.2688	1.0142
TSDT [a]	0.9749	0.7446	1.1163	0.8443	1.1140	0.9192	1.2397	1.0016
Ref [b]	0.9753	0.7527	1.1150	0.9453	1.1144	0.9258	1.2386	1.0883
Present	0.9749	0.7446	1.1163	0.8442	1.1140	0.9192	1.2397	1.0015

[a] Taken from Wattanasakulpong and Ungbhakorn (2013)
[b] Taken from Yas and Samadi (2012)

4.3 Vibration analysis of CNTRC beams

In order to prove the validity of the present formulation in the case of vibration analysis, the computed frequencies of CNTRC beams are numerically compared with those of Wattanasakulpong and Ungbhakorn (2013) and Yas and Samadi (2012) in Table 4. It can be observed that the results obtained from the proposed formulation are in excellent agreement with those obtained from previous results, especially with the third shear deformation theory used by Wattanasakulpong and Ungbhakorn (2013). It is also found that the O-Beam has the lowest natural frequency, while the X-Beam, the highest.

Figs. 8 and 9 show the variation of the fundamental frequency parameter ($\bar{\omega}$) of different types of CNTRC beams with Winkler modulus parameter and the Pasternak shear modulus, respectively. It can be deduced from Figs. 8 and 9 that frequency of the X-Beam are higher than those of beams with other CNTs distributions. It is also seen that the frequencies increase almost linearly as the increase of the spring constant factors.

The effects of CNT volume fractions and the thickness ratios on frequency parameter of the

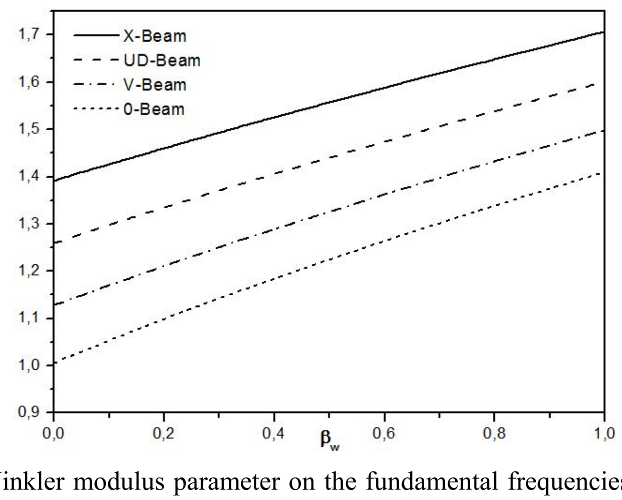


Fig. 8 Effect of Winkler modulus parameter on the fundamental frequencies of CNTRC beams (L/h = 10; β_s = 0; V_{cnt}^{*} = 0.12)

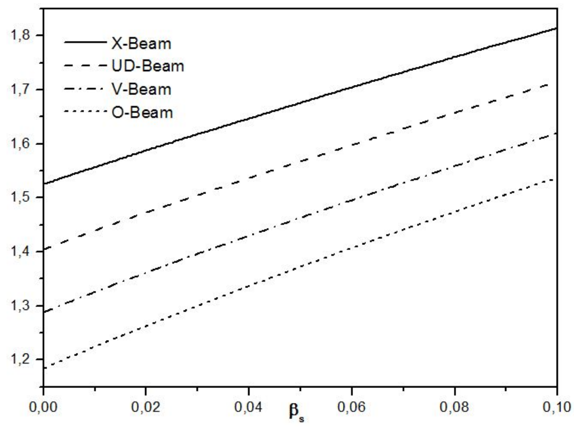


Fig. 9 Effect of Pasternak shear modulus parameter the fundamental frequencies of CNTRC beams (L/h = 10; β_s = 0.4; V_{cnt}^{*} = 0.12)

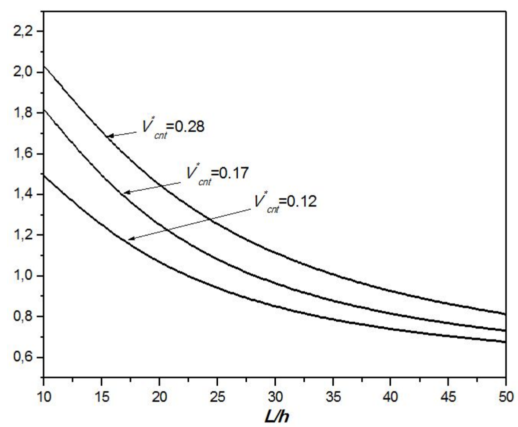


Fig. 10 Dimensionless fundamental frequencies of X Beam on elastic foundation with various thickness ratios (β_w = 0.1, β_s = 0.02)

X-Beam are shown in Fig. 10. The increase of CNT volume fractions conducts to an increase of frequencies. As is seen in the case of buckling analysis, the increase of thickness ratios leads to a decrease of frequencies, especially in the range of L/h = 10 to 30.

5. Conclusions

In this work, a trigonometric refined beam theory is used to investigate the bending, free vibrations and buckling of nanocomposite beams reinforced by single-walled carbon nanotubes resting on Pasternak elastic foundation. The equations of motion have been obtained using the Hamilton's principle. The accuracy of the present theoretical method is numerically checked by comparison with some available results. From the numerical results, it is found that the X-Beam is the strongest among different types of CNTRC beams in supporting the flexure and buckling loads, while the O-Beam is the weakest. An improvement of the present formulation will be considered in the future work to consider the thickness stretching effect by using quasi-3D shear deformation theories (Bessaim *et al*. 2013, Saidi *et al*. 2013, Bousahla *et al*. 2014, Hamidi et *al*. 2015, Fekrar *et al*. 2014, Hebali *et al*. 2014, Houari *et al*. 2013, Larbi Chaht *et al*. 2015, Meradjah *et al*. 2015, Sayyad and Ghugal 2014).

Acknowledgments

This research was supported by the Algerian National Thematic Agency of Research in Science and Technology (ATRST) and university of Sidi Bel Abbes (UDL SBA) in Algeria.

References

Adda Bedia, W., Benzair, A., Semmah, A., Tounsi, A. and Mahmoud, S.R. (2015), "On the thermal buckling characteristics of armchair single-walled carbon nanotube embedded in an elastic medium based on nonlocal continuum elasticity", *Braz. J. Phys*., **45**(2), 225-233.

Ait Amar Meziane, M., Abdelaziz, H.H. and Tounsi, A. (2014), "An efficient and simple refined theory for buckling and free vibration of exponentially graded sandwich plates under various boundary conditions", *J. Sandw. Struct. Mater*., **16**(3), 293-318.

Ait Yahia, S., Ait Atmane, H., Houari, M.S.A. and Tounsi, A. (2015), "Wave propagation in functionally graded plates with porosities using various higher-order shear deformation plate theories", *Struct. Eng. Mech*., *Int. J*., **53**(6), 1143-1165.

Ajayan, P.M., Stephan, O., Colliex, C. and Trauth, D. (1994), "Aligned carbon nanotube arrays formed by cutting a polymer resin–nanotube composite", *Science*, **256**(5176), 1212-1214.

Akavci, S.S. (2014), "An efficient shear deformation theory for free vibration of functionally graded thick rectangular plates on elastic foundation", *Compos. Struct*., **108**, 667-676.

Al-Basyouni, K.S., Tounsi, A. and Mahmoud, S.R. (2015), "Size dependent bending and vibration analysis of functionally graded micro beams based on modified couple stress theory and neutral surface position", *Compos. Struct*., **125**, 621-630.

Alibeigloo, A. (2014), "Three-dimensional thermoelasticity solution of functionally graded carbon nanotube reinforced composite plate embedded in piezoelectric sensor and actuator layers", *Compos. Struct*., **118**, 482-495.

Ashrafi, B. and Hubert, P. (2006), "Modeling the elastic properties of carbon nanotube array/polymer composites", *Compos. Sci. Technol*., **66**(3-4), 387-396.

Aydogdu, M. (2014), "On the vibration of aligned carbon nanotube reinforced composite beams", *Adv. Nano Res*., *Int. J*., **2**(4), 199-210.

Bakhti, K., Kaci, A., Bousahla, A.A., Houari, M.S.A., Tounsi, A. and Adda Bedia, E.A. (2013), "Large deformation analysis for functionally graded carbon nanotube-reinforced composite plates using an efficient and simple refined theory", *Steel Compos. Struct*., *Int. J*., **14**(4), 335-347.

Belabed, Z., Houari, M.S.A., Tounsi, A., Mahmoud, S.R. and Anwar Bég, O. (2014), "An efficient and simple higher order shear and normal deformation theory for functionally graded material (FGM) plates", *Compos*.: *Part B*, **60**, 274-283.

Benachour, A., Daouadji, H.T., Ait Atmane, H., Tounsi, A. and Meftah, S.A. (2011), "A four variable refined plate theory for free vibrations of functionally graded plates with arbitrary gradient", *Compos. Part B*, **42**(6), 1386-1394.

Benguediab, S., Tounsi, A., Zidour, M. and Semmah, A. (2014), "Chirality and scale rffects on mechanical buckling properties of zigzag double-walled carbon nanotubes", *Compos. Part B*, **57**, 21-24.

Bennai, R., Ait Atmane, H. and Tounsi, A. (2015), "A new higher-order shear and normal deformation theory for functionally graded sandwich beams", *Steel Compos. Struct*., *Int. J*., **19**(3), 521-546.

Berrabah, H.M., Tounsi, A., Semmah, A. and Adda Bedia, E.A. (2013), "Comparison of various refined nonlocal beam theories for bending, vibration and buckling analysis of nanobeams", *Struct. Eng. Mech., Int. J.*, **48**(3), 351-365.
Bessaim, A., Houari, M.S.A., Tounsi, A., Mahmoud, S.R. and Adda Bedia, E.A. (2013), "A new higher-order shear and normal deformation theory for the static and free vibration analysis of sandwich plates with functionally graded isotropic face sheets", *J. Sandw. Struct. Mater.*, **15**(6), 671-703.
Besseghier, A., Heireche, H., Bousahla, A.A., Tounsi, A. and Benzair, A. (2015), "Nonlinear vibration properties of a zigzag single-walled carbon nanotube embedded in a polymer matrix", *Adv. Nano Res., Int. J.*, **3**(1), 29-37.
Bouderba, B., Houari, M.S.A. and Tounsi, A. (2013), "Thermomechanical bending response of FGM thick plates resting on Winkler–Pasternak elastic foundations", *Steel Compos. Struct., Int. J.*, **14**(1), 85-104.
Bourada, M., Kaci, A., Houari, M.S.A. and Tounsi, A. (2015), "A new simple shear and normal deformations theory for functionally graded beams", *Steel Compos. Struct., Int. J.*, **18**(2), 409-423.
Bousahla, A.A., Houari, M.S.A., Tounsi, A. and Adda Bedia, E.A. (2014), "A novel higher order shear and normal deformation theory based on neutral surface position for bending analysis of advanced composite plates", *Int. J. Computat. Method.*, **11**(6), 1350082.
Draiche, K., Tounsi, A. and Khalfi, Y. (2014), "A trigonometric four variable plate theory for free vibration of rectangular composite plates with patch mass", *Steel Compos. Struct., Int. J.*, **17**(1), 69-81.
El Meiche, N., Tounsi, A., Ziane, N., Mechab, I. and Adda Bedia, E.A. (2011), "A new hyperbolic shear deformation theory for buckling and vibration of functionally graded sandwich plate", *Int. J. Mech. Sci.*, **53**(4), 237-247.
Esawi, A.M.K. and Farag, M.M. (2007), "Carbon nanotube reinforced composites: Potential and current challenges", *Mater. Des.*, **28**(9), 2394-2401.
Fekrar, A., Houari, M.S.A., Tounsi, A. and Mahmoud, S.R. (2014), "A new five-unknown refined theory based on neutral surface position for bending analysis of exponential graded plates", *Meccanica*, **49**(4), 795-810.
Fidelus, J.D., Wiesel, E., Gojny, F.H., Schulte, K. and Wagner, H.D. (2005), "Thermo-mechanical properties of randomly oriented carbon/epoxy nanocomposites", *Compos. Part A*, **36**(11), 1555-1561.
Griebel, M. and Hamaekers, J. (2004), "Molecular dynamics simulations of the elastic moduli of polymer-carbon nanotube composites", *Comput. Method. Appl. Mech. Eng.*, **193**(17-20), 1773-1788.
Hamidi, A., Houari, M.S.A., Mahmoud, S.R. and Tounsi, A. (2015), "A sinusoidal plate theory with 5-unknowns and stretching effect for thermomechanical bending of functionally graded sandwich plates", *Steel Compos. Struct., Int. J.*, **18**(1), 235-253.
Han, Y. and Elliott, J. (2007), "Molecular dynamics simulations of the elastic properties of polymer/carbon nanotube composites", *Comp. Mater. Sci.*, **39**(2), 315-323.
Hebali, H., Tounsi, A., Houari, M.S.A., Bessaim, A. and Adda Bedia, E.A. (2014), "A new quasi-3D hyperbolic shear deformation theory for the static and free vibration analysis of functionally graded plates", *ASCE J. Eng. Mech.*, **140**(2), 374-383.
Houari, M.S.A., Tounsi, A. and Anwar Bég, O. (2013), "Thermoelastic bending analysis of functionally graded sandwich plates using a new higher order shear and normal deformation theory", *Int. J. Mech. Sci.*, **76**, 467-479.
Hu, N., Fukunaga, H., Lu, C., Kameyama, M. and Yan, B. (2005), "Prediction of elastic properties of carbon nanotube reinforced composites", *P. Roy. Soc. A*,, **461**(2058), 1685-1710.
Jha, D.K., Kant, T. and Singh, R.K. (2012), "Higher order shear and normal deformation theory for natural frequency of functionally graded rectangular plates", *Nucl. Eng. Des.*, **250**, 8-13.
Jha, D.K., Kant, T. and Singh, R.K. (2013), "A critical review of recent research on functionally graded plates", *Compos. Struct.*, **96**, 833-849.
Kaci, A., Tounsi, A., Bakhti, K. and Adda Bedia, E.A. (2012), "Nonlinear cylindrical bending of functionally graded carbon nanotube-reinforced composite plates", *Steel Compos. Struct., Int. J.*, **12**(6), 491-504.
Ke, L.L., Yang, J. and Kitipornchai, S. (2013), "Dynamic stability of functionally graded carbon

nanotube-reinforced composite beams", *Mech. Adv. Mater. Struct.*, **20**(1), 28-37.
Khalfi, Y., Houari, M.S.A. and Tounsi, A. (2014), "A refined and simple shear deformation theory for thermal buckling of solar functionally graded plates on elastic foundation", *Int. J. Computat. Method.*, **11**(5), 135007.
Larbi Chaht, F., Kaci, A., Houari, M.S.A., Tounsi, A., Anwar Bég, O. and Mahmoud, S.R. (2015), "Bending and buckling analyses of functionally graded material (FGM) size-dependent nanoscale beams including the thickness stretching effect", *Steel Compos. Struct., Int. J.*, **18**(2), 425-442.
Lei, Z.X., Liew, K.M. and Yu, J.L. (2013a), "Free vibration analysis of functionally graded carbon nanotube-reinforced composite plates using the element-free kp-Ritz method in thermal environment", *Compos. Struct.*, **106**, 128-138.
Lei, Z.X., Liew, K.M. and Yu, J.L. (2013b), "Large deflection analysis of functionally graded carbon nanotube-reinforced composite plates by the element-free kp-Ritz method", *Comput. Method. Appl. Mech. Eng.*, **256**, 189-199.
Li, X.F., Wang, B.L. and Han, J.C. (2010), "A higher-order theory for static and dynamic analyses of functionally graded beams", *Arch. Appl. Mech.*, **80**(10), 1197-1212.
Liew, K.M., Yang, J. and Kitipornchai, S. (2003), "Postbuckling of piezoelectric FGM plates subject to thermo-electro-mechanical loading", *Int. J. Solid Strut.*, **40**(15), 3869-3892.
Mahi, A., Adda Bedia, E.A. and Tounsi, A. (2015), "A new hyperbolic shear deformation theory for bending and free vibration analysis of isotropic, functionally graded, sandwich and laminated composite plates", *Appl. Math. Model.*, **39**(9), 2489-2508.
Meradjah, M., Kaci, A., Houari, M.S.A., Tounsi, A. and Mahmoud, S.R. (2015), "A new higher order shear and normal deformation theory for functionally graded beams", *Steel Compos. Struct., Int. J.*, **18**(3), 793-809.
Nedri, K., El Meiche, N. and Tounsi, A. (2014), "Free vibration analysis of laminated composite plates resting on elastic foundations by using a refined hyperbolic shear deformation theory", *Mech. Compos. Mater.*, **49**(6), 641-650.
Odegard, G.M., Gates, T.S., Wise, K.E., Park, C. and Siochi, E.J. (2003), "Constitutive modelling of nanotube-reinforced polymer composites", *Compos. Sci. Technol.*, **63**(11), 1671-1687.
Ould Larbi, L, Kaci, A., Houari, M.S.A. and Tounsi, A. (2013), "An efficient shear deformation beam theory based on neutral surface position for bending and free vibration of functionally graded beams", *Mech. Based Des. Struct. Mach.*, **41**(4), 421-433.
Ray, M.C. and Batra, R.C. (2007), "A single-walled carbon nanotube reinforced 1-3 piezoelectric composite for active control of smart structures", *Smart Mater. Struct.*, **16**(5), 1936-1947.
Saidi, H., Houari, M.S.A., Tounsi, A. and Adda Bedia, E.A. (2013), "Thermo-mechanical bending response with stretching effect of functionally graded sandwich plates using a novel shear deformation theory", *Steel Compos. Struct., Int. J.*, **15**(2), 221-245.
Sayyad, A.S. and Ghugal, Y.M. (2014), "Flexure of cross-ply laminated plates using equivalent single layer trigonometric shear deformation theory", *Struct. Eng. Mech., Int. J.*, **51**(5), 867-891.
Shahrjerdi, A., Mustapha, F., Bayat, M. and Majid, D.L.A. (2011), "Free vibration analysis of solar functionally graded plates with temperature-dependent material properties using second order shear deformation theory", *J. Mech. Sci. Technol.*, **25**(9), 2195-2209.
Swaminathan, K. and Naveenkumar, D.T. (2014), "Higher order refined computational models for the stability analysis of FGM plates: Analytical solutions", *Eur. J. Mech. A/Solids*, **47**, 349-361.
Talha, M. and Singh, B.N. (2010), "Static response and free vibration analysis of FGM plates using higher order shear deformation theory", *Appl. Math. Model.*, **34**(12), 3991-4011.
Thostenson, E.T. and Chou, T.W. (2003), "On the elastic properties of carbon nanotube-based composites: Modelling and characterization", *J. Phys. A–Appl. Phys.*, **36**(5), 573-582.
Thostenson, E.T., Ren, Z.F. and Chou, T.W. (2001), "Advances in the science and technology of carbon nanotubes and their composites: A review", *Compos. Sci. Technol.*, **61**(13), 1899-1912.
Tounsi, A., Benguediab, S., Adda Bedia, E.A., Semmah, A. and Zidour, M. (2013a), "Nonlocal effects on thermal buckling properties of double-walled carbon nanotubes", *Adv. Nano Res., Int. J.*, **1**(1), 1-11.

Tounsi, A., Houari, M.S.A., Benyoucef, S. and Adda Bedia, E.A. (2013b), "A refined trigonometric shear deformation theory for thermoelastic bending of functionally graded sandwich plates", *Aerosp. Sci. Technol.*, **24**(1), 209-220.

Vodenitcharova, T. and Zhang, L.C. (2006), "Bending and local buckling of a nanocomposite beam reinforced by a single-walled carbon nanotube", *Int. J. Solids Struct.*, **43**(10), 3006-3024.

Wattanasakulpong, N. and Ungbhakorn, V. (2013), "Analytical solutions for bending, buckling and vibration responses of carbon nanotube-reinforced composite beams resting on elastic foundation", *Computat. Mater. Sci.*, **71**, 201-208.

Wu, C.-P. and Li, H.-Y. (2014), "Three-dimensional free vibration analysis of functionally graded carbon nanotube-reinforced composite plates with various boundary conditions", *J. Vib. Control*, 1077546314528367.

Wuite, J. and Adali, S. (2005), "Deflection and stress behaviour of nanocomposite reinforced beams using a multiscale analysis", *Compos. Struct.*, **71**(3-4), 388-396.

Xu, Y.S., Ray, G. and Abdel-Magid, B. (2006), "Thermal behavior of single walled carbon nanotube polymer–matrix composites", *Compos. Part A*, **37**(1), 114-121.

Yas, M.H. and Samadi, N. (2012), "Free vibrations and buckling analysis of carbon nanotube-reinforced composite Timoshenko beams on elastic foundation", *Int. J. Pres. Vess. Pip.*, **98**, 119-128.

Yaghoobi, H. and Fereidoon, A. (2014), "Mechanical and thermal buckling analysis of functionally graded plates resting on elastic foundations: An assessment of a simple refined *n*th-order shear deformation theory", *Compos.: Part B*, **62**, 54-64.

Yaghoobi, H., Valipour, M.S., Fereidoon, A. and Khoshnevisrad, P. (2014), "Analytical study on post-buckling and nonlinear free vibration analysis of FG beams resting on nonlinear elastic foundation under thermo-mechanical loading using VIM", *Steel Compos. Struct., Int. J.*, **17**(5), 753-776.

Yesilce, Y. (2010), "Effect of axial force on the free vibration of Reddy–Bickford multi- span beam carrying multiple spring-mass systems", *J. Vib. Control*, **16**(1), 11-32.

Yesilce, Y. and Catal, S. (2009), "Free vibration of axially loaded Reddy–Bickford beam on elastic soil using the differential transform method", *Struct. Eng. Mech., Int. J.*, **31**(4), 453-476.

Zemri, A., Houari, M.S.A., Bousahla, A.A. and Tounsi, A. (2015), "A mechanical response of functionally graded nanoscale beam: An assessment of a refined nonlocal shear deformation theory beam theory", *Struct. Eng. Mech., Int. J.*, **54**(4), 693-710.

Zhu, R., Pan, E. and Roy, A.K. (2007), "Molecular dynamics study of the stress-strain behavior of carbon-nanotube reinforced Epon 862 composites", *Mater. Sci. Eng. A.*, **447**(1-2), 51-57.

Zhu, P., Lei, Z.X. and Liew, K.M. (2012), "Static and free vibration analyses of carbon nanotube reinforced composite plates using finite element method with first order shear deformation plate theory", *Compos. Struct.*, **94**(4), 1450-1460.

Ziane, N., Meftah, S.A., Ruta, G., Tounsi, A. and Adda Bedia, E.A. (2015), "Investigation of the Instability of FGM box beams", *Struct. Eng. Mech., Int. J.*, **54**(3), 579-595.

Zidi, M., Tounsi, A., Houari, M.S.A., Adda Bedia, E.A. and Anwar Bég, O. (2014), "Bending analysis of FGM plates under hygro-thermo-mechanical loading using a four variable refined plate theory", *Aerosp. Sci. Technol.*, **34**, 24-34.

CC