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Analytical solution for bending analysis of functionally graded beam

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Abstract. In this paper, a refined exponential shear deformation beam theory is developed for bending analysis of functionally graded beams. The theory account for parabolic variation of transverse shear strain through the depth of the beam and satisfies the zero traction boundary conditions on the surfaces of the beam without using shear correction factors. Contrary to the others refined theories elaborated, where the stretching effect is neglected, in the current investigation this so-called "stretching effect" is taken into consideration. The material properties of the functionally graded beam are assumed to vary according to power law distribution of the volume fraction of the constituents. Based on the present shear deformation beam theory, the equilibrium equations are derived from the principle of virtual displacements. Analytical solutions for static are obtained. Numerical examples are presented to verify the accuracy of the present theory.

Keywords: beam; static; shear deformation theory; strain; stretching effect

1. Introduction

Composite materials have been successfully used in aircraft and other engineering applications for many years because of their excellent strength to weight and stiffness to weight ratios. Recently, advanced composite materials known as functionally graded material have attracted much attention in many engineering applications due to their advantages of being able to resist high temperature gradient while maintaining structural integrity (Koizumi 1997). The functionally graded materials (FGMs) are microscopically inhomogeneous, in which the mechanical properties vary smoothly and continuously from one surface to the other. They are usually made from a mixture of ceramics and metals to attain the significant requirement of material properties (Benachour *et al.* 2011, El Meiche *et al.* 2011, Bouderba *et al.* 2013, Ait Amar Meziane *et al.* 2014, Khalfi *et al.* 2014, Mahi *et al.* 2015).

Due to the increased relevance of the FGMs structural components in the design of engineering structures, many studies have been reported on the static, and vibration analyses of functionally graded (FG) plates. Aydogdu (2008) presented the vibration of multi-walled carbon nanotubes by

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generalized shear deformation theory. Sina *et al*. (2009) provided an analytical method for free vibration analysis of functionally graded beams.

Aydogdu (2007) studied the thermal buckling analysis of cross-ply laminated composite beams with general boundary conditions. Li (2008) investigated static bending and transverse vibration of FGM Timoshenko beams, in which by introducing a new function, the governing equations for bending and vibration of FGM beams were decoupled and the deflection, rotational angle and the resultant force and moment were expressed only in the terms of this new function. Benatta *et al*. (2009) proposed an analytical solution to the bending problem of a symmetric FG beam by including warping of the cross-section and shear deformation effect. Thai and Vo (2012) presented a Bending and free vibration of functionally graded beams using various higher-order shear deformation beam theories. Ould larbi latifa *et al*. (2013) developed an efficient shear deformation beam theory based on neutral surface position for bending and free vibration of functionally graded beams. Allahverdizadeh *et al*. (2014) proposed a nonlinear vibration analysis of FGER sandwich beams. Kadoli *et al*. (2008) studied the static behavior of an FG beam by using higher-order shear deformation theory and finite element method.

Recently, Hadji *et al*. (2014) studied the static and free vibration of FGM beam using a higher order shear deformation theory. Wattanasakulpong and Ungbhakorn (2014) studied linear and nonlinear vibration problems of elastically end restrained FG beams having porosities. Bourada *et al*. (2015) proposed a new simple shear and normal deformations theory for functionally graded Beams. Houari *et al*. (2013) presented the Thermoelastic bending analysis of functionally graded sandwich plates using a new higher order shear and normal deformation theory with stretching effects. The stretching effect was included also in the analysis of the mechanical responses of thick FG plates (Mantari and Guedes Soares 2014, Thai and Kim 2013, Saidi *et al*. 2013, Bessaim *et al*. 2013, Bousahla *et al*. 2014, Fekrar *et al*. 2014, Belabed *et al*. 2014, Hebali *et al*. 2014, Hamidi *et al*. 2015, Mantari and Granados 2015). Bourada *et al*. (2015) investigated also effects of thickness stretching in FG beams.

In the present study, the bending of simply supported FG beams was investigated by using a refined exponential shear deformation beam theory with $(\varepsilon_z \neq 0)$. Contrary to the others refined theories elaborated, where the stretching effect is neglected, in the current investigation this o-called "stretching effect" is taken into consideration. The most interesting feature of this theory is that it accounts for a parabolic variation of the transverse shear strains across the thickness and satisfies the zero traction boundary conditions on the top and bottom surfaces of the beam without using shear correction factors. Then, the present theory together with Hamilton's principle, are employed to extract the motion equations of the functionally graded beams. Analytical solutions for static and free vibration are obtained. Numerical examples are presented to verify the accuracy of the present theory.

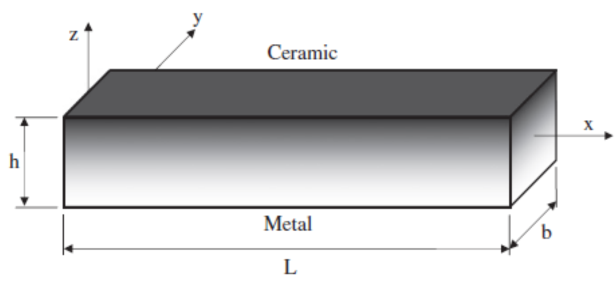

Fig. 1 Geometry and coordinate of a FG beam

2. Problem formulation

Consider a functionally graded beam with length L and rectangular cross section $b \times h$, with b being the width and h being the height as shown in Fig. 1. The beam is made of isotropic material with material properties varying smoothly in the thickness direction.

2.1 Material properties

The properties of FGM vary continuously due to the gradually changing volume fraction of the constituent materials (ceramic and metal), usually in the thickness direction only. The power-law function is commonly used to describe these variations of materials properties. The expression given below represents the profile for the volume fraction.

$$V_C = \left(\frac{z}{h} + \frac{1}{2}\right)^k \tag{1}$$

k is a parameter that dictates material variation profile through the thickness. The value of k equal to zero represents a fully ceramic beam, whereas infinite k indicates a fully metallic beam, and for different values of k one can obtain different volume fractions of metal.

The material properties of FG beams are assumed to vary continuously through the depth of the beam by the rule of mixture (Marur 1999, Tounsi *et al*. 2013a, Bachir Bouiadjra *et al*. 2013, Zidi *et al*. 2014) as

$$P(z) = \left(P_t - P_b\right) V_C + P_b \tag{1b}$$

where P denotes a generic material property like modulus, P_t and P_b denotes the property of the top and bottom faces of the beam respectively, Here, it is assumed that modules E, G and v vary according to the Eq. (1).

2.2 Kinematics and constitutive equations

The displacement field of the proposed theory takes the simpler form as follows

$$\begin{aligned} u(x,z) &= u_0(x,t) - z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x}, \\ u(x,z) &= w_b(x) + w_s + g(z)\varphi_z(x), \end{aligned} \tag{2}$$

Clearly, the displacement field in Eq. (2) contains only four unknowns (u, w_b, w_s, φ_z). The strains associated with the displacements in Eq. (2) are

$$\varepsilon_x = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_b}{\partial x^2} - f(z)\frac{\partial^2 w_s}{\partial x^2} \tag{3a}$$

$$\varepsilon_z = g'(z)\,\varphi_z \tag{3b}$$

$$\gamma_{xz} = g(z)\left(\frac{\partial w_s}{\partial x} + \frac{\partial \varphi_z}{\partial x}\right) \tag{3c}$$

Where $f(z) = z - ze^{-2(z/h)^2}$ and $g(z) = 1 - f'(z)$. It can be seen from Eq. (3c) that the transverse shears strain γ_{xz} is equal to zero at the top $(z = h/2)$ and bottom $(z = -h/2)$ surfaces of the beam, thus satisfying the zero transverse shear stress conditions.

The state of stress in the beam is given by the generalized Hooke's law as follows

$$\sigma_x = Q_{11}(z)\varepsilon_x + Q_{13}(z)\varepsilon_z \tag{4a}$$

$$\tau_{xz} = Q_{55}(z)\gamma_{xz} \tag{4b}$$

$$\sigma_z = Q_{13}(z)\varepsilon_x + Q_{33}(z)\varepsilon_z \tag{4c}$$

The Q_{ij} expressions in terms of engineering constants are

$$Q_{11}(z) = Q_{33}(z) = \frac{E(z)}{1-\upsilon^2}, \quad Q_{13}(z) = \nu\, Q_{11}(z), \quad Q_{55}(z) = \frac{E(z)}{2(1+\upsilon)} \tag{4d}$$

2.3 Governing equations

The governing equations of equilibrium can be derived by using the principle of virtual displacements. The principle of virtual work in the present case yields

$$\int_0^L \int_{-h/2}^{h/2} [\sigma_x \delta\varepsilon_x + \sigma_z \delta\varepsilon_z + \tau_{xz}\delta\gamma_{xz}] dz dx - \int_0^L q\delta(w_b + w_s + g\varphi_z) dx \tag{5}$$

Substituting Eqs. (3) and (4) into Eq. (5) and integrating through the thickness of the beam, Eq. (5) can be rewritten as

$$\begin{aligned} &\int_0^L N_x \delta\frac{\partial u_0}{\partial x} - M_x \delta\frac{\partial^2 w_b}{\partial x^2} - P_x \delta\frac{\partial^2 w_s}{\partial x^2} + R_z \delta\varphi_z + Q_{xz}\delta\left(\frac{\partial w_s}{\partial x} + \frac{\partial \varphi_z}{\partial x}\right) dx \\ &- \int_0^L q\delta(w_b + w_s + g\varphi_z) dx = 0 \end{aligned} \tag{6}$$

where N, M, P and Q are the stress resultants defined by

$$(N, M_x, P_x) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1, z, f)\, \sigma_x dz, \quad Q = \int_{-\frac{h}{2}}^{\frac{h}{2}} g\tau_{xz} dz \quad \text{and} \quad R_z = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_z g'(z) dz \tag{7}$$

The governing equations of equilibrium can be derived from Eq. (6) by integrating the displacement gradients by parts and setting the coefficients zero δu_0, δw_b, δw_s and $\delta\varphi_z$ separately. Thus one can obtain the equilibrium equations associated with the present exponential shear deformation theory

$$\delta u_0 : \frac{\partial N_x}{\partial x} = 0 \tag{8a}$$

$$\delta w_b : \frac{\partial^2 M_x^b}{\partial x^2} + q = 0 \tag{8b}$$

$$\delta w_s : \frac{\partial^2 P_x}{\partial x^2} + \frac{\partial Q_{xz}}{\partial x} + q = 0 \tag{8c}$$

$$\delta \varphi_z : -R_z + \frac{\partial Q_{xz}}{\partial x} + gq = 0 \tag{8d}$$

Eqs. (8a)-(8d) can be expressed in terms of displacements (u_0, w_b, w_s, φ_z) by using Eqs. (2), (3), (4) and (7) as follows

$$A_{11}\frac{\partial^2 u_0}{\partial x^2} - B_{11}\frac{\partial^3 w_b}{\partial x^3} - B_{11}^s\frac{\partial^3 w_s}{\partial x^3} + X_{13}\frac{\partial \varphi_z}{\partial x} = 0 \tag{9a}$$

$$B_{11}\frac{\partial^3 u_0}{\partial x^3} - D_{11}\frac{\partial^4 w_b}{\partial x^4} - D_{11}^s\frac{\partial^4 w_s}{\partial x^4} + Y_{13}\frac{\partial^2 \varphi_z}{\partial x^2} + q = 0 \tag{9b}$$

$$B_{11}^s\frac{\partial^3 u_0}{\partial x^3} - D_{11}^s\frac{\partial^4 w_b}{\partial x^4} - H_{11}\frac{\partial^4 w_s}{\partial x^4} + A_{55}^s\frac{\partial^2 w_s}{\partial x^2} + \left(Y_{13}^s + A_{55}^s\right)\frac{\partial^2 \varphi_z}{\partial x^2} + q = 0 \tag{9c}$$

$$-X_{13}\frac{\partial u_0}{\partial x} + Y_{13}\frac{\partial^2 w_b}{\partial x^2} + \left(Y_{13}^s + A_{55}^s\right)\frac{\partial^2 w_s}{\partial x^2} + A_{55}^s\frac{\partial^2 \varphi_z}{\partial x^2} - Z_{33}\varphi_z + gq = 0 \tag{9d}$$

where A_{11}, D_{11}, etc., are the beam stiffness, defined by

$$A_{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11}dz, \quad B_{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11}zdz, \quad B_{11}^s = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11}fdz, \quad X_{13} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{13}g'dz, \tag{10a}$$

$$D_{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11}z^2dz, \quad D_{11}^s = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11}z.fdz, \quad Y_{13} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{13}z.g'dz, \quad H_{11}^s = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11}f^2dz, \tag{10b}$$

$$Y_{13}^s = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{13}.f.g'.dz, \quad Z_{33} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{13}\left[g'\right]^2 dz, \quad A_{55}^s = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{55}g^2dz, \tag{10c}$$

3. Analytical solution

The equilibrium equations admit the Navier solutions for simply supported beams. The variables u_0, w_b, w_s, φ_z, can be written by assuming the following variations.

$$\begin{Bmatrix} u_0 \\ w_b \\ w_s \\ \varphi_z \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} U_m \cos(\lambda x) e^{i\omega t} \\ w_{bm} \sin(\lambda x) e^{i\omega t} \\ w_{sm} \sin(\lambda x) e^{i\omega t} \\ \phi_{zm} \sin(\lambda x) e^{i\omega t} \end{Bmatrix} \tag{11}$$

where U_m, W_{bm}, W_{sm} and ϕ_{zm} are arbitrary parameters to be determined, ω is the eigenfrequency associated with mth eigenmode, and $\lambda = m\pi/L$. The transverse load q is also expanded in Fourier series as

$$q(x) = \sum_{m=1}^{\infty} Q_m \sin(\lambda\, x) \tag{12}$$

where Q_m is the load amplitude calculated from

$$Q_m = \frac{2}{L} \int_0^L q(x) \sin(\lambda\, x) dx \tag{13}$$

The coefficients Q_m are given below for some typical loads. For the case of uniform distributed load, we have

$$Q_m = \frac{4Q_0}{m\pi}, \quad (m = 1, 3, 5...) \tag{14}$$

Substituting the expressions of u_0, w_b, w_s, φ_z from Eqs. (11) and (12) into the equilibrium equations of Eq. (9), the analytical solutions can be obtained from the following equations

$$\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{12} & m_{22} & m_{23} & m_{24} \\ m_{13} & m_{23} & m_{33} & m_{34} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix} \right) \begin{Bmatrix} U_m \\ W_{bm} \\ W_{sm} \\ \phi_{zm} \end{Bmatrix} = \begin{Bmatrix} 0 \\ Q_m \\ Q_m \\ gQ_m \end{Bmatrix} \tag{15}$$

where

$$\begin{aligned} &a_{11} = A_{11}\lambda^2,\ a_{12} = -B_{11}\lambda^3,\ a_{13} = -B_{11}^s\lambda^3,\ a_{14} = -X_{13}\lambda,\ a_{22} = D_{11}\lambda^4,\ a_{23} = D_{11}^s\lambda^4, \\ &a_{24} = Y_{13}\lambda^2,\ a_{33} = H_{11}\lambda^4 + A_{55}^s\lambda^2,\ a_{34} = Y_{13}^s\lambda^2 + A_{55}^s\lambda^2,\ a_{44} = A_{55}^s\lambda^2 + Z_{33} \end{aligned} \tag{16a}$$

$$\begin{aligned} &m_{11} = I_0,\ m_{12} = -I_1\lambda,\ m_{13} = J_1\lambda,\ m_{14} = 0,\ m_{22} = I_0 + I_2\lambda^2,\ m_{23} = I_0 + J_2\lambda, \\ &m_{24} = L_1,\ m_{33} = I_0 + K_2\lambda^2,\ m_{34} = L_1,\ m_{44} = L_2 \end{aligned} \tag{16b}$$

4. Results and discussion

In this section, various numerical examples are presented and discussed to verify the accuracy of present theories in predicting the bending and free vibration responses of simply supported FG beams. The FG beam is taken to be made of aluminum and alumina with the following material properties:

Ceramic (P_C: Alumina, Al_2O_3): E_c = 380 GPa; ν = 0.3.
Metal (P_M: Aluminium, Al): E_m = 70 GPa; ν = 0.3.

And their properties change through the thickness of the beam according to power-law. The bottom surfaces of the FG beams are aluminum rich, whereas the top surfaces of the FG beams are alumina rich.

For convenience, the following dimensionless form is used

$$\bar{w} = 100\frac{E_m h^3}{q_0 L^4} w\left(\frac{L}{2}\right), \quad \bar{u} = 100\frac{E_m h^3}{q_0 L^4} u\left(0, -\frac{h}{2}\right), \quad \bar{\sigma}_x = \frac{h}{q_0 L}\sigma_x\left(\frac{L}{2}, \frac{h}{2}\right), \quad \bar{\tau}_{xz} = \frac{h}{q_0 L}\tau_{xz}(0,0),$$

4.1 Results for bending analysis

Table 1 contains nondimensional deflection and stresses of FG beams under uniform load q_0 for different values of power law index k and span-to-depth ratio L / h. The obtained results are compared with the analytical solutions given by Li *et al*. (2010), the results of Ould Larbi Latifa *et al*. (2013) and the results of Hadji *et al*. (2014). It can be observed that our results with $(\varepsilon_z \neq 0)$ are in an excellent agreement to those predicted using the higher order shear deformation theory of Hadji *et al*. (2014), Ould Larbi *et al*. (2013) and Li *et al*. (2010) with $(\varepsilon_z = 0)$ for all values of power law index p and span-to-depth ratio L/h. However, the small difference found between the results is due to that the theories presented by Hadji *et al*. (2014), Ould Larbi *et al*. (2013) and Li *et al*. (2010) ignore the thickness stretching effect.

Table 1 Nondimensional deflections and stresses of FG beams under uniform load

k	Method	L/h = 5				L/h = 20			
		$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$	$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$
0	Li *et al*. (2010)	3.1657	0.9402	3.8020	0.7500	2.8962	0.2306	15.0130	0.7500
	Ould Larbi *et al*. (2013)	3.1651	0.9406	3.8043	0.7489	2.8962	0.2305	15.0136	0.7625
	Hadji *et al*. (2014)	3.1654	0.9398	3.8019	0.7330	2.8962	0.2306	15.0129	0.7437
	Present ($\varepsilon_z \neq 0$)	3.1673	0.9233	3.9129	0.7883	2.8807	0.2290	15.4891	0.7890
0.5	Li *et al*. (2010)	4.8292	1.6603	4.9925	0.7676	4.4645	0.4087	19.7005	0.7676
	Ould Larbi *et al*. (2013)	4.8282	1.6608	4.9956	0.7660	4.4644	0.4087	19.7013	0.7795
	Hadji *et al*. (2014)	4.8285	1.6596	4.9923	0.7501	4.4644	0.4087	19.7002	0.7614
	Present ($\varepsilon_z \neq 0$)	4.8045	1.6091	5.1538	0.8053	4.4160	0.3998	20.3969	0.8057

Table 1 Continued

k	Method	$L/h = 5$				$L/h = 20$			
		$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$	$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$
1	Li *et al*. (2010)	6.2599	2.3045	5.8837	0.7500	5.8049	0.5686	23.2054	0.7500
	Ould Larbi *et al*. (2013)	6.2590	2.3052	5.8875	0.7489	5.8049	0.5685	23.2063	0.7625
	Hadji *et al*. (2014)	6.2594	2.3038	5.8835	0.7330	5.8049	0.5685	23.2051	0.7437
	Present ($\varepsilon_z \neq 0$)	6.1805	2.2115	6.0709	0.7883	5.6965	0.5498	24.0095	0.7890
2	Li *et al*. (2010)	8.0602	3.1134	6.8812	0.6787	7.4415	0.7691	27.0989	0.6787
	Ould Larbi *et al*. (2013)	8.0683	3.1146	6.8878	0.6870	7.4421	0.7691	27.1005	0.7005
	Hadji *et al*. (2014)	8.0677	3.1129	6.8824	0.6704	7.4421	0.7691	27.0989	0.6812
	Present ($\varepsilon_z \neq 0$)	7.9106	2.9629	7.0925	0.7274	7.2458	0.7366	27.9844	0.7287
5	Li *et al*. (2010)	9.7802	3.7089	8.1030	0.5790	8.8151	0.9133	31.8112	0.5790
	Ould Larbi *et al*. (2013)	9.8345	3.7128	8.1187	0.6084	8.8186	0.9134	31.8151	0.6218
	Hadji *et al*. (2014)	9.8281	3.7100	8.1104	0.5904	8.8182	0.9134	31.8127	0.6013
	Present ($\varepsilon_z \neq 0$)	9.6933	3.5429	8.3581	0.6513	8.6182	0.8775	32.8183	0.6540
10	Li *et al*. (2010)	10.8979	3.8860	9.7063	0.6436	9.6879	0.9536	38.1372	0.6436
	Ould Larbi *et al*. (2013)	10.9413	3.8898	9.7203	0.6640	9.6907	0.9537	38.1408	0.6788
	Hadji *et al*. (2014)	10.9381	3.8863	9.7119	0.6465	9.6905	0.9536	38.1382	0.6586
	Present ($\varepsilon_z \neq 0$)	10.8680	3.7462	9.9878	0.7064	9.5513	0.9262	39.2717	0.7091

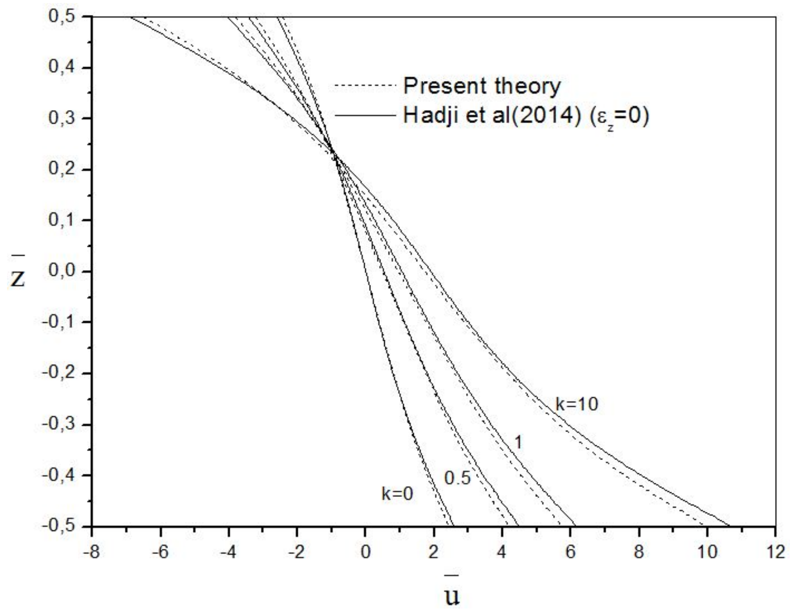

Fig. 2 The variation of the axial displacement $\bar{u}$ through-the-thickness of a FG beam ($L = 2h$)

Figs. 2-4 show the variations of axial displacement $\bar{u}$, axial stress $\bar{\sigma}_x$, and transverse shear stress $\bar{\tau}_{xz}$, respectively, through the depth of a very deep beam ($L = 2h$) under uniform load. In general, the present theory and the shear deformation beam models of Hadji *et al*. (2014) give almost identical results.

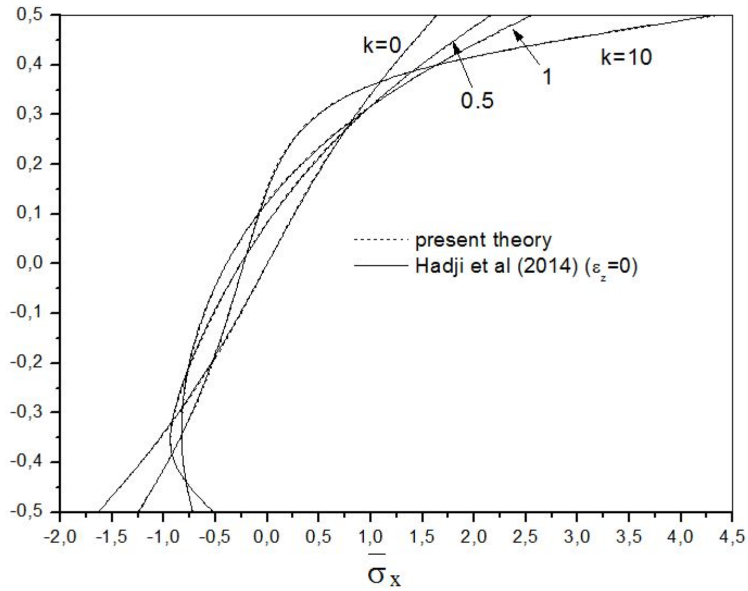

Fig. 3 The variation of the axial stress $\bar{\sigma}_x$ through-the-thickness of a FG beam ($L = 2h$)

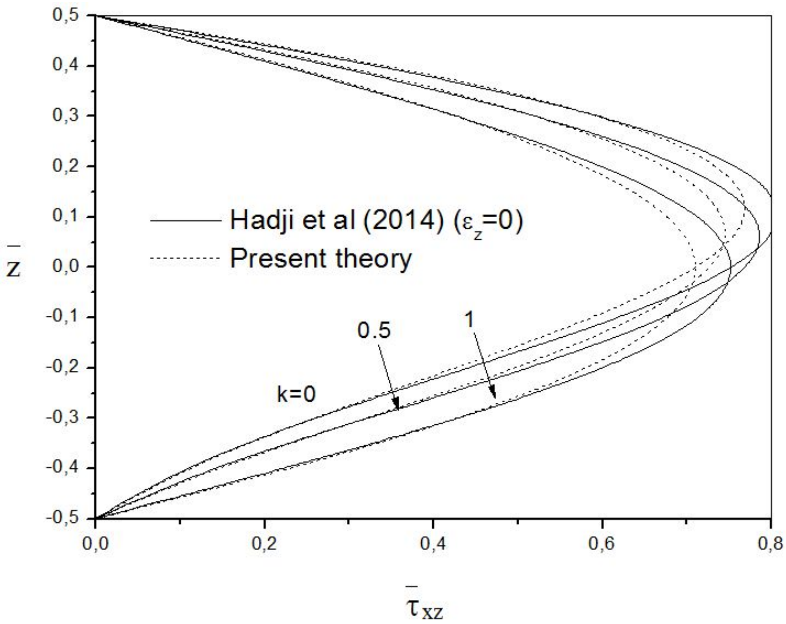

Fig. 4 The variation of the transverse shear stress $\bar{\tau}_{xz}$ through-the-thickness of a FG beam ($L = 2h$)

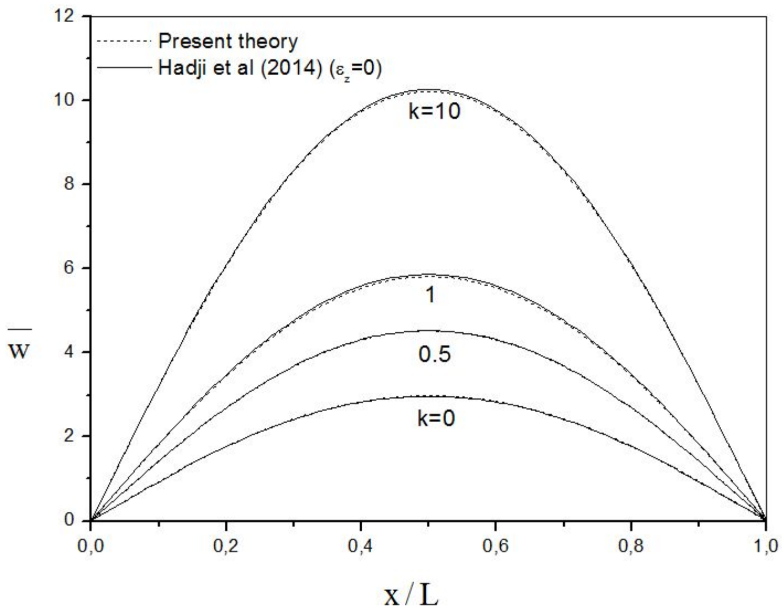

Fig. 5 Variation of the transverse displacement $\overline{w}$ versus non-dimensional length of a FG beam (L = 5h)

Fig. 5 illustrates the variation of the non-dimensional transversal displacement $\overline{w}$ versus non-dimensional length for different power law index k. It can be seen also that the present beam theory gives almost identical results to Hadji *et al*. (2014). In addition, the results show that the increase of the power law index k leads to an increase of transversal displacement $\overline{w}$.

5. Conclusions

A refined exponential shear deformation theory is proposed for bending analysis of functionally graded beams. The theory accounts for the stretching and shear deformation effects without requiring a shear correction factor. It is based on the assumption that the transverse displacements consist of bending, shear and thickness stretching parts. Based on the present refined exponential beam theory, the equilibrium equations are derived from the principle of virtual displacements. Numerical examples show that the proposed theory gives solutions which are almost identical with those obtained using other shear deformation theories. In future, this theoretical formulation can be extended to FRP plates (Draiche *et al*. 2014, Nedri *et al*. 2014) and nanostructures (Benzair *et al*. 2008, Heireche *et al*. 2008, Amara *et al*. 2010, Berrabah *et al*. 2013, Tounsi *et al*. 2013b, c, d, 2015, Semmah *et al*. 2014, Benguediab *et al*. 2014, Belkorissat *et al*. 2015, Larbi Chaht *et al*. 2015).

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