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A new higher-order shear and normal deformation theory for functionally graded sandwich beams

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Abstract. A new refined hyperbolic shear and normal deformation beam theory is developed to study the free vibration and buckling of functionally graded (FG) sandwich beams under various boundary conditions. The effects of transverse shear strains as well as the transverse normal strain are taken into account. Material properties of the sandwich beam faces are assumed to be graded in the thickness direction according to a simple power-law distribution in terms of the volume fractions of the constituents. The core layer is still homogeneous and made of an isotropic material. Equations of motion are derived from Hamilton's principle. Analytical solutions for the bending, free vibration and buckling analyses are obtained for simply supported sandwich beams. Illustrative examples are given to show the effects of varying gradients, thickness stretching, boundary conditions, and thickness to length ratios on the bending, free vibration and buckling of functionally graded sandwich beams.

Keywords: functionally graded sandwich beam; refined shear deformation theory; stretching effect

1. Introduction

Sandwich structures are applied in mechanical and civil engineering since the middle of 20th century. It offers great potential for large civil infrastructure projects, such as industrial buildings and vehicular bridges. In recent years, the functionally graded materials (FGMs) are taken into account in the sandwich structure industries. The FG sandwich structures commonly exist in two types: FG facesheet homogeneous core and homogeneous facesheet FG core. For the case of homogeneous core, the softcore is commonly employed because of the light weight and high bending stiffness in the structural design. The homogeneous hardcore is also used in other fields such as control or in the thermal environments. With the wide application of FG sandwich structures, understanding vibration and buckling of FG sandwich structures becomes an important task. For functionally graded materials, great progress has been made in elasticity theory as well as

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plates and beams. However, for FG sandwich plates/beams, related studies are very limited. Etemadi *et al*. (2009) investigated the low velocity impact behavior of sandwich panels with a FG core using a three-dimensional finite element simulation. Bhangale and Ganesan (2006) studied vibration and buckling analysis of a FG sandwich beam having constrained viscoelastic layer in thermal environment by using finite element formulation. Amirani *et al*. (2009) used the element free Galerkin method for free vibration analysis of sandwich beam with FG core. Bui *et al*. (2013) investigated transient responses and natural frequencies of sandwich beams with inhomogeneous FG core using a truly meshfree radial point interpolation method. Natarajan and Manickam (2012) examined the bending and free vibration response of two types of FG sandwich plates. Bourada *et al*. (2012) investigated the thermal buckling response of FG sandwich plates. Based on the first-order shear deformation plate theory (FSDT), Yaghoobi and Yaghoobi (2013) examined the buckling behavior of sandwich plates with FG face sheets resting on elastic foundation. Kettaf *et al*. (2013) proposed a new hyperbolic shear displacement model for thermal buckling behavior of FG sandwich plates. Tounsi *et al*. (2013) analytically investigated the thermoelastic bending problem of FG sandwich plates based on the refined trigonometric shear deformation theory. Bessaim *et al*. (2013) presented a novel higher-order shear and normal deformation theory for the static and free vibration responses of sandwich plates with FG isotropic face sheets. Houari *et al*. (2013) studied the thermoelastic bending behavior of FG sandwich plates via a new higher order shear and normal deformation theory. Xiang *et al*. (2013) analyzed the free vibration response of FG sandwich plates by employing an *nth*-order shear deformation theory and a meshless method, while Ait Amar Meziane *et al*. (2014) investigated the buckling and free vibration responses of FG sandwich plates using a simple refined shear deformation theory. Khalfi *et al*. (2014) proposed a refined and simple shear deformation theory for thermal buckling of solar FG plates resting on elastic foundation. Attia *et al*. (2015) discussed the free vibration analysis of FG plates with temperature-dependent properties using various four variable refined plate theories. Ait Yahia *et al*. (2015) investigated the wave propagation in FG plates with porosities using various higher-order shear deformation plate theories. Al-Basyouni *et al*. (2015) proposed a novel unified beam formulation and a modified couple stress theory that considers a variable length scale parameter in conjunction with the neutral axis concept to study bending and dynamic behaviors of FG micro beam Recently, the thickness stretching effect on mechanical response of FG structures is demonstrated by Bousahla *et al*. (2014), Fekrar *et al*. (2014), Belabed *et al*. (2014), Hebali *et al*. (2014), Houari *et al*. (2013), Bessaim *et al*. (2013), Saidi *et al*. (2013), Hamidi *et al*. (2015), Bourada *et al*. (2015), Larbi Chaht *et al*. (2015).

Hyperbolic shear deformation theories have been applied to various problems in literature such as bending and vibration of beams (Ghugal and Sharma 2009, Li *et al*. 2013, Sayyad and Ghugal 2011, Berrabah *et al*. 2013, Ould Larbi *et al*. 2013); bending, vibration and buckling of plates (Ghugal and Pawar 2011, Ghugal 2011); bending, vibration and buckling of laminated composite plates (Nedri *et al*. 2014, Grover *et al*. 2013, Akavci and Tanrikulu 2008, Akavci 2010) and bending, vibration and buckling of FG plates (Akavci 2014a, b, Hebali *et al*. 2014, Mahi *et al*. 2015, Belkorissat *et al*. 2015, El Meiche *et al*. 2011). Noting the fact that hyperbolic shear deformation theories have been utilized earlier in flexure, vibration and buckling analysis of beams and plates; and also taking a cue from exact three dimensional theory of elasticity solutions of plate, hyperbolic functions are used in the present work, for describing displacement variation across plate thickness.

In this work, a new hyperbolic shear and normal deformation beam theory is presented to study the vibration and buckling responses of FG sandwich beams under boundary conditions. By dividing the transverse displacement into bending, shear and thickness stretching parts, the motion equations of the functionally graded sandwich beams are obtained together with Hamilton's principle. Material properties of the sandwich beam faces are assumed to vary in the thickness direction only according to power-law form distribution in terms of the volume fractions of the constituents. The core layer is still homogeneous and made of an isotropic material. Numerical examples are given to show the effects of varying gradients, thickness stretching, boundary conditions, and thickness to length ratios on the bending, free vibration and buckling of FG sandwich beams.

2. Refined plate theory for functionally graded sandwich beams

2.1 Problem formulation

Consider a sandwich beam with homogeneous core and FG face layers, with total height (h), length (L), and width (b) referred to the Cartesian coordinates (x, y, z) as shown in Fig. 1. The top and bottom faces of the beam are at $z = \pm h/2$, and the horizontal edges of the beam are parallel to axes x and y.

The sandwich beam is composed of three layers, namely, "Layer 1", "Layer 2", and "Layer 3", from bottom to top of the beam. The vertical ordinates of the bottom, the two interfaces, and the top are denoted by $h_1 = -h/2$, h_2, h_3, $h_4 = h/2$, respectively.

The face layers of the sandwich beam are made of an isotropic material with material properties varying smoothly in the z-direction only. The core layer is made of an isotropic homogeneous material. For the brevity, the ratio of the height of each layer from bottom to top is denoted by the combination of three numbers, i.e., ''1-0-1'', ''2-1-2'' and so on. As shown in Fig. 1.

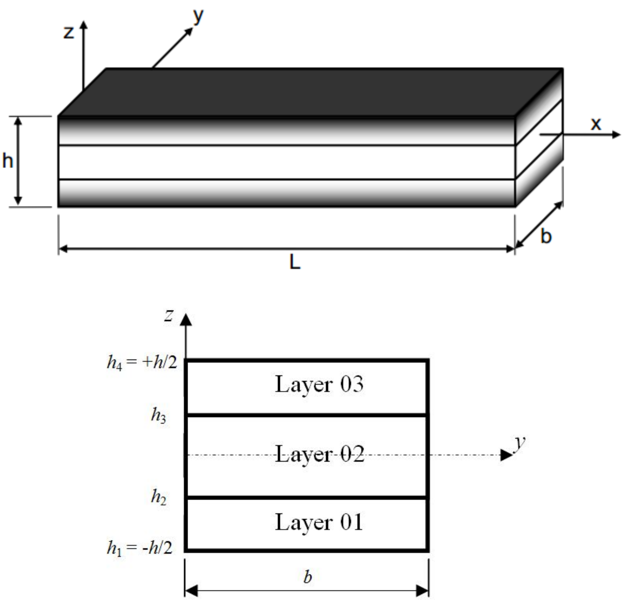

Fig. 1 Coordinate and geometry of a FG sandwich beam in the rectangular Cartesian coordinates

2.2 Material properties

The properties of FGM vary continuously due to gradually changing the volume fraction of the constituent materials, usually in the height direction only. Power-law function is used to describe these variations of materials properties

$$V^{(1)}=\left(\frac{z-h_1}{h_2-h_1}\right)^k, \quad z\in[h_1,h_2] \tag{1a}$$

$$V^{(2)}=1, \quad z\in[h_2,h_3] \tag{1b}$$

$$V^{(3)}=\left(\frac{z-h_4}{h_3-h_4}\right)^k, \quad z\in[h_3,h_4] \tag{1c}$$

where $V^{(n)}$, $(n = 1, 2, 3)$ denotes the volume fraction function of layer n; k is the volume fraction index $(0\le k\le+\infty)$, which dictates the material variation profile through the height of beam.

The effective material properties, like Young's modulus E, Poisson's ratio v, and mass density ρ, then can be expressed by the rule of mixture (Benachour *et al.* 2011, Bachir Bouiadjra *et al.* 2012, Tounsi *et al.* 2013, Bouderba *et al.* 2013, Bachir Bouiadjra *et al.* 2013, Zidi *et al.* 2014) as follows

$$P^{(n)}(z)=P_2+\left(P_1-P_2\right)V^{(n)} \tag{2}$$

where $P^{(n)}$ is the effective material property of FGM of layer n. Where, P_1 and P_2 are the properties of the top and bottom faces of layer 1, respectively, and vice versa for layer 3 depending on the volume fraction $V^{(n)}, (n = 1, 2, 3)$.

For simplicity, Poisson's ratio of plate is assumed to be constant in this study for that the effect of Poisson's ratio on the deformation is much less than that of Young's modulus (Dellal and Erdogan 1983).

2.3 Basic assumptions

The assumptions of the present theory are as follows:

(i) The displacements are small in comparison with the beam thickness and, therefore, strains involved are infinitesimal.

(ii) The transverse displacement w includes three components of bending w_b, shear w_s, and stretching effect w_{st}. The two first components are functions of coordinate x only and the third one is function of x and z.

$$w(x,z,t)=w_b(x,t)+w_s(x,t)+w_{st}(x,z,t) \tag{3}$$

(iii) The displacements u in x-direction consist of extension, bending, and shear components.

$$u=u_0+u_b+u_s, \tag{4}$$

The bending component u_b is assumed to be similar to the displacement given by the classical beam theory. Therefore, the expression for u_b can be given as

$$u_b = -z\frac{\partial w_b}{\partial x}, \tag{5}$$

The shear component u_s gives rise, in conjunction with w_s, to the sinusoidal variations of shear strain γ_{xz} and hence to shear stress τ_{xz} through the thickness of the beam in such a way that shear stress τ_{xz} is zero at the top and bottom faces of the beam. Consequently, the expression for u_s can be given as

$$u_s = -f(z)\frac{\partial w_s}{\partial x}, \tag{6}$$

where

$$f(z) = \frac{\frac{h}{\pi}\sinh\left(\frac{\pi}{h}z\right) - z}{\left[\cosh\left(\frac{\pi}{2}\right) - 1\right]} \tag{7}$$

The component due to the stretching effect w_{st} can be given as

$$w_{st}(x,z,t) = g(z)\,\varphi(x,t) \tag{8}$$

The additional displacement φ accounts for the effect of normal stress is included and $g(z)$ is given as follows

$$g(z) = 1 - \frac{\cosh\left(\frac{\pi}{h}z\right) - 1}{\cosh\left(\frac{\pi}{2}\right) - 1}; \tag{9}$$

2.4 Kinematicsand constitutive equations

Based on the assumptions made in the preceding section, the displacement field can be obtained using Eqs. (3)-(9) as

$$u(x,z,t) = u_0(x,t) - z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x} \tag{10a}$$

$$w(x,z,t) = w_b(x,t) + w_s(x,t) + g(z)\,\varphi(x,t) \tag{10b}$$

The strains associated with the displacements in Eq. (10) are

$$\varepsilon_x = \varepsilon_x^0 + z\,k_x^b + f(z)\,k_x^s \tag{11a}$$

$$\gamma_{xz} = g(z)\,\gamma_{xz}^0 \tag{11b}$$

$$\varepsilon_z = g'(z)\,\varepsilon_z^0 \tag{11c}$$

where

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x},\ k_x^b = -\frac{\partial^2 w_b}{\partial x^2},\ k_x^s = -\frac{\partial^2 w_s}{\partial x^2},\ \gamma_{xz}^0 = \frac{\partial w_s}{\partial x} + \frac{\partial \varphi}{\partial x},\ \varepsilon_z^0 = \varphi \tag{11d}$$

and

$$g'(z) = \frac{dg(z)}{dz} \tag{11e}$$

By assuming that the material of FG beam obeys Hooke's law, the stresses in the beam become

$$\sigma_x = Q_{11}(z)\varepsilon_x + Q_{13}(z)\varepsilon_z, \quad \tau_{xz} = Q_{55}(z)\,\gamma_{xz}, \quad \text{and} \quad \sigma_z = Q_{13}(z)\varepsilon_x + Q_{33}(z)\varepsilon_z \tag{12a}$$

where

$$Q_{11}(z) = Q_{33}(z) = \frac{E(z)}{(1-\nu^2)}, \quad Q_{13}(z) = \nu\, Q_{11}(z), \quad \text{and} \quad Q_{55}(z) = \frac{E(z)}{2(1+\nu)} \tag{12b}$$

2.5 Equations of motion

Hamilton's principle is used here in to derive the equations of motion. The principle can be stated in analytical form as (Reddy 2002, Draiche *et al*. 2014)

$$\int_{t_1}^{t_2} (\delta U + \delta V - \delta K)dt = 0 \tag{13}$$

where t is the time; t_1 and t_2 are the initial and end time, respectively; δU is the virtual variation of the strain energy; $\delta V$is the variation of work done by external forces; and δK is the virtual variation of the kinetic energy. The variation of the strain energy of the beam can be stated as

$$\begin{aligned}\delta U &= \int_0^L \int_{-\frac{h}{2}-C}^{\frac{h}{2}} (\sigma_x \delta\, \varepsilon_x + \sigma_z \delta\, \varepsilon_z + \tau_{xz} \delta\, \gamma_{xz})dzdx \\ &= \int_0^L \left(N\frac{d\delta\, u_0}{dx} + N_z \delta\, \varphi - M_b \frac{d^2\delta\, w_b}{dx^2} - M_s \frac{d^2 \delta\, w_s}{dx^2} + Q\left[\frac{d\delta\, w_s}{dx} + \frac{d\delta\, \varphi}{dx}\right]\right)dx\end{aligned} \tag{14}$$

where N, M_b, M_s, N_z and Q are the stress resultants defined as

$$(N, M_b, M_s) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1, z, f(z))\,\sigma_x dz, \quad N_z = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_z g'(z) dz \quad \text{and} \quad Q = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} g(z) dz \tag{15}$$

The variation of work done by externally transverse loads q and axial force $\overline{N}$ can be expressed as

$$\delta V = -\int_0^L q\delta(w_b + w_s)dx + \int_0^L \overline{N}\frac{d(w_b + w_s + g_0\varphi)}{dx}\frac{d\delta(w_b + w_s + g_0\varphi)}{dx}dx \tag{16}$$

The variation of the kinetic energy can be expressed as

$$\begin{aligned}
\delta K &= \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(z)\left[\dot{u}\delta\,\dot{u} + \dot{w}\delta\,\dot{w}\right]dzdx \\
&= \int_0^L \Big\{ I_0\left[\dot{u}_0\delta\dot{u}_0 + (\dot{w}_b + \dot{w}_s)(\delta\dot{w}_b + \delta\dot{w}_s)\right] + J_0\left[(\dot{w}_b + \dot{w}_s)\delta\,\dot{\varphi} + \dot{\varphi}\,\delta(\dot{w}_b + \dot{w}_s)\right] \\
&\quad - I_1\left(\dot{u}_0\frac{d\delta\dot{w}_b}{dx} + \frac{d\,\dot{w}_b}{dx}\delta\,\dot{u}_0\right) + I_2\left(\frac{d\dot{w}_b}{dx}\frac{d\delta\,\dot{w}_b}{dx}\right) - J_1\left(\dot{u}_0\frac{d\delta\dot{w}_s}{dx} + \frac{d\,\dot{w}_s}{dx}\delta\,\dot{u}_0\right) \\
&\quad + K_2\left(\frac{d\dot{w}_s}{dx}\frac{d\delta\,\dot{w}_s}{dx}\right) + J_2\left(\frac{d\dot{w}_b}{dx}\frac{d\delta\,\dot{w}_s}{dx} + \frac{d\dot{w}_s}{dx}\frac{d\delta\,\dot{w}_b}{dx}\right) + K_0\dot{\varphi}\,\delta\,\dot{\varphi}\Big\}dx
\end{aligned} \tag{17}$$

where dot-superscript convention indicates the differentiation with respect to the time variable t; and (I_i, J_i, K_i) are mass inertias defined as

$$(I_0, I_1, I_2) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1, z, z^2)\rho(z)dz \tag{18a}$$

$$(J_0, J_1, J_2) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (g, f, z\,f)\rho(z)dz \tag{18b}$$

$$(K_0, K_2) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (g^2, f^2)\rho(z)dz \tag{18c}$$

Substituting the expressions for δU, δV, and δK from Eqs. (14), (16), and (17) into Eq. (13) and integrating by parts, and collecting the coefficients of δu_0, δw_b, δw_s and $\delta\varphi$, the following equations of motion of the FG beam are obtained

$$\delta u_0: \ \frac{dN}{dx} = I_0\ddot{u}_0 - I_1\frac{d\ddot{w}_b}{dx} - J_1\frac{d\ddot{w}_s}{dx} \tag{19a}$$

$$\delta w_b:\ \frac{d^2M_b}{dx^2}+q+\overline{N}\left(\frac{d^2\left(w_b+w_s+g_0\varphi\right)}{dx^2}\right)$$
$$=I_0\left(\ddot{w}_b+\ddot{w}_s\right)+J_0\ddot{\varphi}+I_1\frac{d\ddot{u}_0}{dx}-I_2\frac{d^2\ddot{w}_b}{dx^2}-J_2\frac{d^2\ddot{w}_s}{dx^2} \tag{19b}$$

$$\delta w_s:\ \frac{d^2M_s}{dx^2}+\frac{dQ}{\partial x}+q+\overline{N}\left(\frac{d^2\left(w_b+w_s+g_0\varphi\right)}{dx^2}\right)$$
$$=I_0\left(\ddot{w}_b+\ddot{w}_s\right)+J_0\ddot{\varphi}+J_1\frac{d\ddot{u}_0}{dx}-J_2\frac{d^2\ddot{w}_b}{dx^2}-K_2\frac{d^2\ddot{w}_s}{dx^2} \tag{19c}$$

$$\delta\varphi:\ \frac{dQ}{dx}-N_z+\overline{N}g_0\left(\frac{d^2\left(w_b+w_s+g_0\varphi\right)}{dx^2}\right)=J_0\left(\ddot{w}_b+\ddot{w}_s\right)+K_0\ddot{\varphi} \tag{19d}$$

Eqs. (19a)-(19d) can be expressed in terms of displacements (u_0, w_b, w_s and φ) by using Eqs. (10), (11), (12) and (15) as follows

$$A_{11}\frac{d^2u_0}{dx^2}-B_{11}\frac{d^3w_b}{dx^3}-B_{11}^s\frac{d^3w_s}{dx^3}+L\frac{d\varphi}{dx}=I_0\ddot{u}_0-I_1\frac{d\ddot{w}_b}{dx}-J_1\frac{d\ddot{w}_s}{dx} \tag{20a}$$

$$B_{11}\frac{d^3u_0}{dx^3}-D_{11}\frac{d^4w_b}{dx^4}-D_{11}^s\frac{d^4w_s}{dx^4}+L^a\frac{d^2\varphi}{dx^2}+q+\overline{N}\left(\frac{d^2\left(w_b+w_s+g_0\varphi\right)}{dx^2}\right)$$
$$=I_0\left(\ddot{w}_b+\ddot{w}_s\right)+J_0\ddot{\varphi}+I_1\frac{d\ddot{u}_0}{dx}-I_2\frac{d^2\ddot{w}_b}{dx^2}-J_2\frac{d^2\ddot{w}_s}{dx^2} \tag{20b}$$

$$B_{11}^s\frac{d^3u_0}{dx^3}-D_{11}^s\frac{d^4w_b}{dx^4}-H_{11}^s\frac{d^4w_s}{dx^4}+A_{55}^s\frac{d^2w_s}{dx^2}+(R+A_{55}^s)\frac{d^2\varphi}{dx^2}+q+\overline{N}\left(\frac{d^2\left(w_b+w_s+g_0\varphi\right)}{dx^2}\right)$$
$$=I_0(\ddot{w}_b+\ddot{w}_s)+J_0\ddot{\varphi}+J_1\frac{d\ddot{u}_0}{dx}-J_2\frac{d^2\ddot{w}_b}{dx^2}-K_2\frac{d^2\ddot{w}_s}{dx^2} \tag{20c}$$

$$B_{11}\frac{d^3u_0}{dx^3}-D_{11}\frac{d^4w_b}{dx^4}-D_{11}^s\frac{d^4w_s}{dx^4}+L^a\frac{d^2\varphi}{dx^2}+q+\overline{N}\left(\frac{d^2\left(w_b+w_s+g_0\varphi\right)}{dx^2}\right)$$
$$=I_0\left(\ddot{w}_b+\ddot{w}_s\right)+J_0\ddot{\varphi}+I_1\frac{d\ddot{u}_0}{dx}-I_2\frac{d^2\ddot{w}_b}{dx^2}-J_2\frac{d^2\ddot{w}_s}{dx^2} \tag{20b}$$

$$B_{11}^s\frac{d^3u_0}{dx^3}-D_{11}^s\frac{d^4w_b}{dx^4}-H_{11}^s\frac{d^4w_s}{dx^4}+A_{55}^s\frac{d^2w_s}{dx^2}+(R+A_{55}^s)\frac{d^2\varphi}{dx^2}+q+\overline{N}\left(\frac{d^2\left(w_b+w_s+g_0\varphi\right)}{dx^2}\right)$$
$$=I_0(\ddot{w}_b+\ddot{w}_s)+J_0\ddot{\varphi}+J_1\frac{d\ddot{u}_0}{dx}-J_2\frac{d^2\ddot{w}_b}{dx^2}-K_2\frac{d^2\ddot{w}_s}{dx^2} \tag{20c}$$

$$L\frac{du_0}{dx}-L^a\frac{d^2w_b}{dx^2}-\left(R+A_{55}^s\right)\frac{d^2w_s}{dx^2}+R^a\varphi-A_{55}^s\frac{d^2\varphi}{dx^2}+\overline{N}g_0\left(\frac{d^2\left(w_b+w_s+g_0\varphi\right)}{dx^2}\right) \\ =J_0\left(\ddot{w}_b+\ddot{w}_s\right)+K_0\ddot{\varphi} \tag{20d}$$

where A_{11}, B_{11}, etc., are the beam stiffness, defined by

$$\left(A_{11},B_{11},D_{11},B_{11}^s,D_{11}^s,H_{11}^s\right)=\int_{-\frac{h}{2}}^{\frac{h}{2}}Q_{11}\left(1,z,z^2,f(z),z\,f(z),f^2(z)\right)dz \tag{21a}$$

and

$$A_{55}^s=\int_{-\frac{h}{2}}^{\frac{h}{2}}Q_{55}\left[g(z)\right]^2dz,\quad \left[L,L^a,R,R^a\right]=\int_{-\frac{h}{2}}^{\frac{h}{2}}Q_{11}\left[\nu,\nu.z,\nu.f(z),g'(z)\right]g'(z)dz \tag{21b}$$

3. Analytical solution

The exact solution of Eqs. (20a)-(20d) for the FG sandwich beam under various boundary conditions can be constructed. The boundary conditions for an arbitrary edge with simply supported and clamped edge conditions are

- Clamped (C):

$$u_0=w_b=\partial w_b/\partial x=w_s=\partial w_s/\partial x=\varphi=\partial\varphi/\partial x=0 \quad \text{at} \quad x=0,L \tag{22a}$$

- and simply supported (S)

$$w_b=w_s=\varphi=0 \quad \text{at} \quad x=0,L \tag{22b}$$

The following representation for the displacement quantities, that satisfy the above boundary conditions, is appropriate in the case of our problem

$$\begin{Bmatrix}u_0\\w_b\\w_s\\\varphi\end{Bmatrix}=\begin{Bmatrix}U_mX_m'\,e^{i\omega t}\\W_{bm}X_me^{i\omega t}\\W_{sm}X_m\,e^{i\omega t}\\\Phi_{stm}X_m\,e^{i\omega t}\end{Bmatrix} \tag{23}$$

where U_m, W_{bm}, W_{sm} and Φ_{stm} are arbitrary parameters to be determined, ω is the eigenfrequency associated with mth eigenmode, and $\lambda = m\pi / L$. The function $X_m(x)$ is suggested by Reddy (2004) to satisfy at least the geometric boundary conditions given in Eqs. (22a)-(22b) and represents approximate shapes of the deflected beam. These functions, for the different cases of boundary conditions, are listed in Table 1.

Table 1 The admissible functions $X_m(x)$ (Reddy 2004)

	Boundary conditions at x = 0, a	The functions $X_m(x)$
SS	$X_m(0) = X''_m(0) = 0$	$\sin(\lambda x)$ $\lambda = m\pi / a$
CC	$X_m(0) = X'_m(0) = 0$	$\sin(\lambda_m x) - \sinh(\lambda_m x) - \xi_m [\cos(\lambda_m x) - \cosh(\lambda_m x)]$ $\xi_m = [\sin(\lambda_m a) - \sinh(\lambda_m a)] / [\cos(\lambda_m a) - \cosh(\lambda_m a)]$ $\lambda_m = (m + 0{,}5)\pi / a$

The transverse load q is also expanded in Fourier series as

$$q(x) = \sum_{m=1}^{\infty} Q_m \sin(\lambda\, x) \tag{24}$$

where Q_m is the load amplitude calculated from

$$Q_m = \frac{2}{L}\int_0^L q(x)\sin(\lambda\, x)dx \tag{25}$$

The coefficients Q_m are given below for some typical loads. For the case of a sinusoidally distributed load, we have

$$m = 1 \quad \text{and} \quad Q_1 = q_0 \tag{26a}$$

and for the case of uniform distributed load, we have

$$Q_m = \frac{4q_0}{m\pi}, \ \ (m = 1, 3, 5....) \tag{26a}$$

Substituting Eqs. (23) and (24) into Eq. (20), the analytical solutions can be obtained; for free vibration problem the load parameters vanish and the free vibrating solution can be obtained as eigenvalue problem for any fixed value of m, as

$$\left([K] - \omega^2[M]\right)\{\Delta\} = \{0\} \tag{27}$$

For buckling problems, the natural frequency vanishes and the buckling equation can be expressed as the following eigenvalue problem

$$\left([K] - \Lambda[N]\right)\{\Delta\} = \{0\} \tag{28}$$

In the case of static problems, we obtain the following operator equation

$$[K]\{\Delta\} = \{F\} \tag{29}$$

where

$$[K]=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \quad [M]=\begin{bmatrix} m_{11} & m_{12} & m_{13} & 0 \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & m_{42} & m_{43} & m_{44} \end{bmatrix} \tag{30}$$

and

$$[N]=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \overline{N}\alpha_3 & \overline{N}\alpha_3 & \overline{N}g_0\alpha_3 \\ 0 & \overline{N}\alpha_3 & \overline{N}\alpha_3 & \overline{N}g_0\alpha_3 \\ 0 & \overline{N}g_0\alpha_3 & \overline{N}g_0\alpha_3 & \overline{N}g_0^2\alpha_3 \end{bmatrix}, \quad \{\Delta\}=\begin{Bmatrix} U_m \\ W_{bm} \\ W_{sm} \\ \Phi_{stm} \end{Bmatrix}, \quad \{F\}=\begin{Bmatrix} 0 \\ Q_m \\ Q_m \\ 0 \end{Bmatrix} \tag{31}$$

where

$$a_{11}=A_{11}\alpha_4, \quad a_{12}=-B_{11}\alpha_4, \quad a_{13}=-B_{11}^s\alpha_4, \quad a_{14}=L\alpha_2, \tag{32a}$$

$$a_{21}=B_{11}\alpha_5, \quad a_{22}=-D_{11}\alpha_5, \quad a_{23}=-D_{11}^s\alpha_5, \quad a_{24}=L^a\alpha_3, \tag{32b}$$

$$a_{31}=B_{11}^s\alpha_5, \quad a_{23}=-D_{11}^s\alpha_5, \quad a_{33}=-H_{11}^s\alpha_5+A_{55}^s\alpha_3, \quad a_{34}=\left(A_{55}^s+R\right)\alpha_3, \tag{32c}$$

$$a_{41}=L\alpha_3, \quad a_{42}=-L^a\alpha_3, \quad a_{43}=-\left(A_{55}^s+R\right)\alpha_3, \quad a_{44}=R^a\alpha_1-A_{55}^s\alpha_3, \tag{32d}$$

$$m_{11}=-I_0\alpha_2, \quad m_{12}=I_1\alpha_2, \quad m_{13}=J_1\alpha_2, \quad m_{14}=0 \tag{32e}$$

$$m_{21}=-I_1\alpha_3, \quad m_{22}=-I_0\alpha_1+I_2\alpha_3, \quad m_{23}=-I_0\alpha_1+J_2\alpha_3, \quad m_{24}=-J_0\alpha_1 \tag{32f}$$

$$m_{31}=-J_1\alpha_3, \quad m_{32}=-I_0\alpha_1+J_2\alpha_3, \quad m_{33}=-I_0\alpha_1+K_2\alpha_3, \quad m_{34}=-J_0\alpha_1 \tag{32g}$$

$$m_{41}=0, \quad m_{42}=J_0\alpha_1, \quad m_{43}=J_0\alpha_1, \quad m_{44}=K_0\alpha_1 \tag{32h}$$

with

$$\left(\alpha_1,\alpha_3,\alpha_5\right)=\int_0^L\left(X_m, X_m'', X_m''''\right)\ X_m dx \tag{32i}$$

$$\left(\alpha_2,\alpha_4\right)=\int_0^L\left(X_m', X_m'''\right)\ X_m' dx \tag{32j}$$

For non-trivial solutions of eigenvalue problem of Eqs. (27), (28) the following determinants should be zero

$$\left|[K]-\omega^2[M]\right|=\{0\} \tag{33a}$$

$$\left|[K]-\Lambda[N]\right|=\{0\} \tag{33b}$$

The Eqs. (33a)-(33b) give the natural frequencies ω and buckling loads Λ of the FG sandwich beam.

4. Results and discussion

In this section, various numerical examples are presented and discussed to check the accuracy of the present theory. For verification purpose, the fundamental natural frequencies and critical buckling loads of FG sandwich beams predicted by the present model are compared in Tables 2-13 with those Vo *et al*. (2014) and (2015). The material properties adopted here are as follows

- Ceramic (P_c: Alumina, Al_2O_3): E_c = 380 GPa; ν = 0.3; ρ_c = 3960 kg/m^3.
- Metal (P_m: Aluminium, Al): E_m = 70 GPa; ν = 0.3; ρ_m = 2707 kg/m^3.

Table 2 Non-dimensional fundamental natural frequencies of simply supported FG sandwich beams with homogeneous hardcore (L / h = 5)

K	Theory	1-0-1	2-1-2	1-1-1	1-2-1	1-8-1
0	Present ($\varepsilon_z \neq 0$)	5.1629	5.1629	5.1629	5.1629	5.1629
	Present ($\varepsilon_z = 0$)	5.1529	5.1529	5.1529	5.1529	5.1529
	Vo *et al*. (2014)	5.1528	5.1528	5.1528	5.1528	5.1528
	Vo *et al*. (2015)	5.1618	5.1618	5.1618	5.1618	5.1618
0.5	Present ($\varepsilon_z \neq 0$)	4.1352	4.2438	4.3394	4.4895	4.8526
	Present ($\varepsilon_z = 0$)	4.1270	4.2353	4.3305	4.4799	4.8425
	Vo *et al*. (2014)	4.1268	4.2351	4.3303	4.4798	4.8422
	Vo *et al*. (2015)	4.1344	4.2429	4.3383	4.4881	4.8511
1	Present ($\varepsilon_z \neq 0$)	3.5809	3.7376	3.8840	4.1199	4.6900
	Present ($\varepsilon_z = 0$)	3.5730	3.7302	3.8754	4.1108	4.6091
	Vo *et al*. (2014)	3.5735	3.7298	3.8755	4.1105	4.6084
	Vo *et al*. (2015)	3.5803	3.7369	3.8830	4.1185	4.6884
2	Present ($\varepsilon_z \neq 0$)	3.0741	3.2433	3.4267	3.7424	4.5248
	Present ($\varepsilon_z = 0$)	3.0672	3.2368	3.4187	3.7336	4.5151
	Vo *et al*. (2014)	3.0680	3.2365	3.4190	3.7334	4.5142
	Vo *et al*. (2015)	3.0737	3.2427	3.4257	3.7410	4.5231
5	Present ($\varepsilon_z \neq 0$)	2.7490	2.8492	3.0247	3.3854	4.3607
	Present ($\varepsilon_z = 0$)	2.7433	2.8436	3.0178	3.3770	4.3511
	Vo *et al*. (2014)	2.7446	2.8439	3.0181	3.3771	4.3501
	Vo *et al*. (2015)	2.7493	2.8489	3.0238	3.3840	4.3589
10	Present ($\varepsilon_z \neq 0$)	2.6971	2.7399	2.8867	3.2438	4.2882
	Present ($\varepsilon_z = 0$)	2.6918	2.7353	2.8806	3.2353	4. 2782
	Vo *et al*. (2014)	2.6932	2.7355	2.8808	3.2356	4.2776
	Vo *et al*. (2015)	2.6978	2.7400	2.8860	3.2422	4.2864

Table 3 Non-dimensional fundamental natural frequencies of simply supported FG sandwich beams with homogeneous hardcore ($L / h = 20$)

K	Theory	1-0-1	2-1-2	1-1-1	1-2-1	1-8-1
0	Present ($\varepsilon_z \neq 0$)	5.4634	5.4634	5.4634	5.4634	5.4634
	Present ($\varepsilon_z = 0$)	5.4603	5.4603	5.4603	5.4603	5.4603
	Vo *et al*. (2014)	5.4603	5.4603	5.4603	5.4603	5.4603
	Vo *et al*. (2015)	5.4610	5.4610	5.4610	5.4610	5.4610
0.5	Present ($\varepsilon_z \neq 0$)	4.3173	4.4316	4.5351	4.7006	5.1095
	Present ($\varepsilon_z = 0$)	4.3148	4.4290	4.5324	4.6979	5.1067
	Vo *et al*. (2014)	4.3148	4.4290	4.5324	4.6979	5.1067
	Vo *et al*. (2015)	4.3153	4.4296	4.5330	4.6985	5.1073
1	Present ($\varepsilon_z \neq 0$)	3.7169	3.8791	4.0352	4.2914	4.9259
	Present ($\varepsilon_z = 0$)	3.7146	3.8768	4.0328	4.2889	4.9233
	Vo *et al*. (2014)	3.7147	3.8768	4.0328	4.2889	4.9233
	Vo *et al*. (2015)	3.7152	3.8773	4.0333	4.2895	4.9239
2	Present ($\varepsilon_z \neq 0$)	3.1785	3.3488	3.5413	3.8793	4.7406
	Present ($\varepsilon_z = 0$)	3.1763	3.3465	3.5389	3.8769	4.7382
	Vo *et al*. (2014)	3.1764	3.3465	3.5389	3.8769	4.7382
	Vo *et al*. (2015)	3.1768	3.3469	3.5394	3.8774	4.7388
5	Present ($\varepsilon_z \neq 0$)	2.8457	2.9331	3.1134	3.4946	4.5577
	Present ($\varepsilon_z = 0$)	2.8438	2.9310	3.1110	3.4921	4.5554
	Vo *et al*. (2014)	2.8439	2.9310	3.1111	3.4921	4.5554
	Vo *et al*. (2015)	2.8443	2.9314	3.1115	3.4926	4.5560
10	Present ($\varepsilon_z \neq 0$)	2.8057	2.8207	2.9685	3.3434	4.4772
	Present ($\varepsilon_z = 0$)	2.8040	2.8188	2.9661	3.3406	4.4749
	Vo *et al*. (2014)	2.8041	2.8188	2.9662	3.3406	4.4749
	Vo *et al*. (2015)	2.8045	2.8191	2.9665	3.3411	4.4755

Two cases of FG sandwich beams with two values of span-to-height ratio, $L / h = 5$ and 20, are examined:

- Hardcore: homogeneous core with Al_2O_3 ($E_1 = E_c$; $\nu_1 = \nu_c$; $\rho_1 = \rho_c$) and FG faces with top and bottom surfaces made of Al ($E_2 = E_m$; $\nu_2 = \nu_m$; $\rho_2 = \rho_m$).
- Softcore: homogeneous core with Al ($E_1 = E_m$; $\nu_1 = \nu_m$; $\rho_1 = \rho_m$) and FG faces with top and bottom surfaces made of Al_2O_3 ($E_2 = E_c$; $\nu_2 = \nu_c$; $\rho_2 = \rho_c$).

For convenience, the following dimensionless forms are used.

$$\overline{\omega} = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}, \quad P_{cr} = \overline{N}\frac{12 L^2}{E_m h^3}$$

4.1 Results for free vibration analysis

For free vibration analysis, different types of FG sandwich beams are considered. Tables 2-5 contain dimensionless fundamental natural frequencies for simply supported FG sandwich beams with both hardcore and softcore. The obtained results are compared with those of Vo *et al*. (2014) and Vo *et al*. (2015) for various values of material parameter k. It is observed that the proposed theory without the thickness stretching effect (ε_z = 0) and the model of Vo *et al*. (2014) give solutions close to each other, and these solutions are in an excellent agreement with both the present theory and Vo *et al*. (2015) that considers the thickness stretching effect ($\varepsilon_z \neq 0$) for moderately thick beams (L/h = 20). However, the present theory without the thickness stretching effect (ε_z = 0) and the model of Vo *et al*. (2014) slightly overestimate the frequency for thick beams (L/h = 5) due to ignoring the thickness stretching effect.

Table 4 Non-dimensional fundamental natural frequencies of simply supported FG sandwich beams with homogeneous softcore (L/h = 20)

K	Theory	1-0-1	2-1-2	1-1-1	1-2-1	1-8-1
0	Present ($\varepsilon_z \neq 0$)	2.6826	2.6826	2.6826	2.6826	2.6826
	Present ($\varepsilon_z = 0$)	2.6774	2.6774	2.6774	2.6774	2.6774
	Vo *et al*. (2014)	2.6773	2.6773	2.6773	2.6773	2.6773
	Vo *et al*. (2015)	4.4557	4.3184	4.1968	4.0016	3.4379
0.5	Present ($\varepsilon_z \neq 0$)	4.4446	4.3067	4.1924	3.9959	3.4374
	Present ($\varepsilon_z = 0$)	4.4427	4.3046	4.1839	3.9921	3.4342
	Vo *et al*. (2014)	4.8683	4.7368	4.6050	4.3814	3.7101
	Vo *et al*. (2015)	4.8593	4.7254	4.5922	4.3726	3.7052
1	Present ($\varepsilon_z \neq 0$)	4.8525	4.7178	4.5858	4.3663	3.7065
	Present ($\varepsilon_z = 0$)	5.1108	5.0190	4.8984	4.6677	3.9344
	Vo *et al*. (2014)	5.1002	5.0012	4.8815	4.6512	3.9296
	Vo *et al*. (2015)	5.0945	4.9970	4.8740	4.6459	3.9303
2	Present ($\varepsilon_z \neq 0$)	5.2022	5.1818	5.0968	4.8841	4.1194
	Present ($\varepsilon_z = 0$)	5.1916	5.1644	5.0769	4.8646	4.1140
	Vo *et al*. (2014)	5.1880	5.1603	5.0703	4.8564	4.1139
	Vo *et al*. (2015)	5.1973	5.2165	5.1561	4.9622	4.1920
5	Present ($\varepsilon_z \neq 0$)	5.1869	5.2057	5.1376	4.9408	4.1857
	Present ($\varepsilon_z = 0$)	5.1848	5.1966	5.1301	4.9326	4.1855
	Vo *et al*. (2014)	2.6826	2.6826	2.6826	2.6826	2.6826
	Vo *et al*. (2015)	2.6774	2.6774	2.6774	2.6774	2.6774
10	Present ($\varepsilon_z \neq 0$)	2.6773	2.6773	2.6773	2.6773	2.6773
	Present ($\varepsilon_z = 0$)	4.4557	4.3184	4.1968	4.0016	3.4379
	Vo *et al*. (2014)	4.4446	4.3067	4.1924	3.9959	3.4374
	Vo *et al*. (2015)	4.4427	4.3046	4.1839	3.9921	3.4342

Table 5 Non-dimensional fundamental natural frequencies of simply supported FG sandwich beams with homogeneous softcore ($L / h = 20$)

K	Theory	1-0-1	2-1-2	1-1-1	1-2-1	1-8-1
0	Present ($\varepsilon_z \neq 0$)	2.8387	2.8387	2.8387	2.8387	2.8387
	Present ($\varepsilon_z = 0$)	2.8371	2.8371	2.8371	2.8371	2.8371
	Vo *et al*. (2014)	2.8371	2.8371	2.8371	2.8371	2.8371
0.5	Present ($\varepsilon_z \neq 0$)	4.8607	4.7488	4.6320	4.4181	3.7271
	Present ($\varepsilon_z = 0$)	4.8582	4.7465	4.6297	4.4161	3.7257
	Vo *et al*. (2014)	4.8579	4.7460	4.6294	4.4160	3.7255
1	Present ($\varepsilon_z \neq 0$)	5.3023	5.2250	5.1193	4.8965	4.0664
	Present ($\varepsilon_z = 0$)	5.2996	5.2220	5.1165	4.8941	4.0647
	Vo *et al*. (2014)	5.2990	5.2217	5.1160	4.8938	4.0648
2	Present ($\varepsilon_z \neq 0$)	5.5273	5.5150	5.4449	5.2479	4.3558
	Present ($\varepsilon_z = 0$)	5.5244	5.5118	5.4415	5.2448	4.3541
	Vo *et al*. (2014)	5.5239	5.5113	5.4410	5.2445	4.3542
5	Present ($\varepsilon_z \neq 0$)	5.5679	5.6422	5.6284	5.4884	4.6007
	Present ($\varepsilon_z = 0$)	5.5648	5.6387	5.6247	5.4847	4.5991
	Vo *et al*. (2014)	5.5645	5.6382	5.6242	5.4843	4.5991
10	Present ($\varepsilon_z \neq 0$)	5.5335	5.6491	5.6663	5.5617	4.6977
	Present ($\varepsilon_z = 0$)	5.5303	5.6459	5.6627	5.5579	4.6961
	Vo *et al*. (2014)	5.5302	5.6452	5.6621	5.5575	4.6960

Tables 6 and 7 show the dimensionless fundamental natural frequencies for clamped – clamped FG sandwich beam with homogenous hardcore for both thick and moderately thick beams, respectively. In general, a good agreement between present results and previous solutions can be remarked.

Table 6 Non-dimensional fundamental natural frequencies of clamped – clamped FG sandwich beams with homogeneous hardcore ($L / h = 5$)

K	Theory	1-0-1	2-1-2	1-1-1	1-2-1	1-8-1
0	Present ($\varepsilon_z \neq 0$)	10.5190	10.5190	10.5190	10.5190	10.5190
	Present ($\varepsilon_z = 0$)	10.2571	10.2571	10.2571	10.0306	10.2571
	Vo *et al*. (2014)	10.0678	10.0678	10.0678	10.0678	10.0678
	Vo *et al*. (2015)	10.1851	10.1851	10.1851	10.1851	10.1851
0.5	Present ($\varepsilon_z \neq 0$)	8.7065	8.9289	9.1100	9.3800	9.9996
	Present ($\varepsilon_z = 0$)	8.4689	8.6857	8.8632	9.1291	9.7420
	Vo *et al*. (2014)	8.3600	8.5720	8.7423	8.9942	9.5731
	Vo *et al*. (2015)	8.4635	8.6780	8.8498	9.1036	9.6857

Table 6 Continued

K	Theory	1-0-1	2-1-2	1-1-1	1-2-1	1-8-1
1	Present ($\varepsilon_z \neq 0$)	7.6568	7.9915	8.2792	8.7196	9.7207
	Present ($\varepsilon_z = 0$)	7.4386	7.7638	8.0450	8.4774	9.4658
	Vo *et al*. (2014)	7.3661	7.6865	7.9580	8.3705	9.3076
	Vo *et al*. (2015)	7.4611	7.7854	8.0595	8.4752	9.4174
2	Present ($\varepsilon_z \neq 0$)	6.6494	7.0366	7.4111	8.0218	9.4333
	Present ($\varepsilon_z = 0$)	6.4542	6.8282	7.1929	7.7906	9.1815
	Vo *et al*. (2014)	6.4095	6.7826	7.1373	7.7114	9.0343
	Vo *et al*. (2015)	6.4952	6.8740	7.2328	7.8114	9.1415
5	Present ($\varepsilon_z \neq 0$)	5.9358	6.2420	6.6199	7.3413	9.1444
	Present ($\varepsilon_z = 0$)	5.7633	6.0534	6.4193	7.1229	8.8959
	Vo *et al*. (2014)	5.7264	6.0293	6.3889	7.0691	8.7605
	Vo *et al*. (2015)	5.8016	6.1124	6.4780	7.1652	8.8653
10	Present ($\varepsilon_z \neq 0$)	5.7474	6.0072	6.3400	7.0645	9.0158
	Present ($\varepsilon_z = 0$)	5.5863	5.8260	6.1467	6.6869	8.5622
	Vo *et al*. (2014)	5.5375	5.8059	6.1240	6.8087	8.6391
	Vo *et al*. (2015)	5.6074	5.8848	6.2099	6.9030	8.7430

Table 7 Non-dimensional fundamental natural frequencies of clamped – clamped FG sandwich beams with homogeneous hardcore ($L / h = 20$)

K	Theory	1-0-1	2-1-2	1-1-1	1-2-1	1-8-1
0	Present ($\varepsilon_z \neq 0$)	12.5691	12.5691	12.5691	12.5691	12.5691
	Present ($\varepsilon_z = 0$)	12.1642	12.1642	12.1642	12.1642	12.1642
	Vo *et al*. (2014)	12.2228	12.2228	12.2228	12.2228	12.2228
	Vo *et al*. (2015)	12.2660	12.2660	12.2660	12.2660	12.2660
0.5	Present ($\varepsilon_z \neq 0$)	9.9635	10.2267	10.4635	10.8407	11.7685
	Present ($\varepsilon_z = 0$)	9.6400	9.8947	10.1240	10.4893	11.3883
	Vo *et al*. (2014)	9.6942	9.9501	10.1800	10.5460	11.4459
	Vo *et al*. (2015)	9.7297	9.9865	10.2172	10.5842	11.4867
1	Present ($\varepsilon_z \neq 0$)	8.5895	8.9644	9.3227	9.9087	11.3523
	Present ($\varepsilon_z = 0$)	8.3096	8.6722	9.0191	9.5865	10.9851
	Vo *et al*. (2014)	8.3594	8.7241	9.0722	9.6411	11.0421
	Vo *et al*. (2015)	8.3908	8.7569	9.1061	9.6768	11.0815
2	Present ($\varepsilon_z \neq 0$)	7.3519	7.7478	8.1912	8.9668	10.9316
	Present ($\varepsilon_z = 0$)	7.1117	7.4944	7.9234	8.6743	10.5775
	Vo *et al*. (2014)	7.1563	7.5417	7.9727	8.7262	10.6336
	Vo *et al*. (2015)	7.1839	7.5711	8.0035	8.7593	10.6719

Table 7 Continued

K	Theory	1-0-1	2-1-2	1-1-1	1-2-1	1-8-1
5	Present ($\varepsilon_z \neq 0$)	6.5812	6.7911	7.2080	8.0849	10.5156
	Present ($\varepsilon_z = 0$)	6.3663	6.5685	6.9716	7.8203	10.1746
	Vo *et al*. (2014)	6.4064	6.6116	7.0170	7.8692	10.2298
	Vo *et al*. (2015)	6.4308	6.6379	7.0451	7.9000	10.2669
10	Present ($\varepsilon_z \neq 0$)	6.4817	6.5313	6.8741	7.7371	10.3323
	Present ($\varepsilon_z = 0$)	6.2707	6.3173	6.6485	7.4834	9.8974
	Vo *et al*. (2014)	6.3086	6.3590	6.6924	7.5311	10.0519
	Vo *et al*. (2015)	6.3319	6.3841	6.7194	7.5609	10.0884

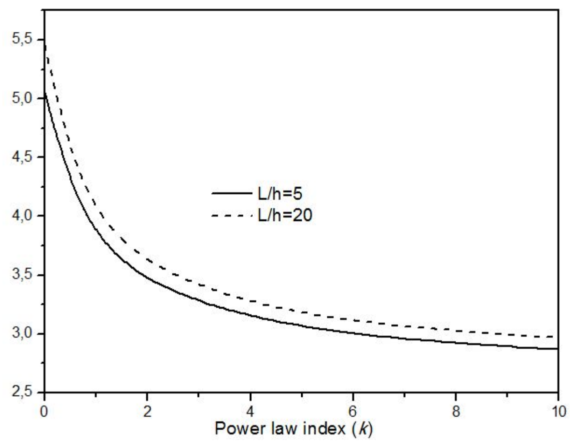

Fig. 2 Variation of fundamental frequencies $\overline{\omega}$ versus the material parameter k for (1-1-1) simply supported FG sandwich beams with homogeneous hardcore

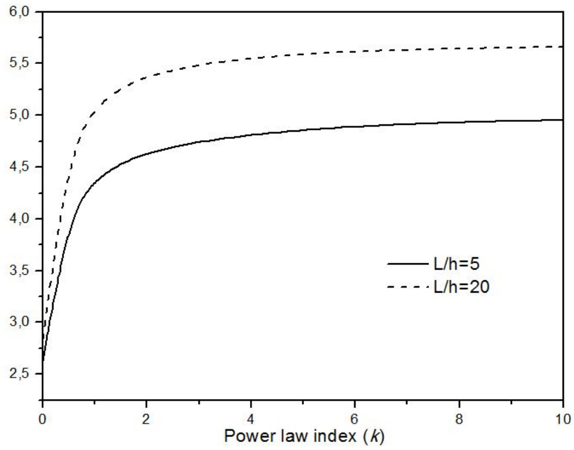

Fig. 3 Variation of fundamental frequencies $\overline{\omega}$ versus the material parameter k for (1-1-1) simply supported FG sandwich beams with homogeneous softcore

Figs. 2 and 3 depict the fundamental frequencies parameters versus the material parameter k of simply supported power-law (1-1-1) FG sandwich beams with both homogeneous hardcore and softcore, respectively. It can be seen, that as the material parameter increases, the natural frequencies decrease for sandwich beams with hardcore and increase for sandwich beams with softcore.

4.2 Results for buckling analysis

The accuracy of the proposed hyperbolic shear and normal deformation beam theory is also verified for the buckling analysis of a simply supported FG sandwich beams. Tables 8-11 show the nondimensional critical buckling loads for different types of FG sandwich beams with both hardcore and softcore. The results are obtained via the proposed theory and compared to those of

Table 8 Non-dimensional critical buckling loads of simply supported FG sandwich beams with homogeneous hardcore ($L/h = 5$)

K	Theory	1-0-1	2-1-2	1-1-1	1-2-1	1-8-1
0	Present ($\varepsilon_z \neq 0$)	49.7122	49.7122	49.7122	49.7122	49.7122
	Present ($\varepsilon_z = 0$)	48.5991	48.5991	48.5991	48.5991	48.5991
	Vo *et al*. (2014)	48.5959	48.5959	48.5959	48.5959	48.5959
	Vo *et al*. (2015)	49.5906	49.5906	49.5906	49.5906	49.5906
0.5	Present ($\varepsilon_z \neq 0$)	28.5487	30.7776	32.6737	35.6322	43.0068
	Present ($\varepsilon_z = 0$)	27.8534	30.0274	31.8783	34.7701	42.0004
	Vo *et al*. (2014)	27.8574	30.0301	31.8784	34.7653	41.9897
	Vo *et al*. (2015)	28.4624	30.6825	32.5699	35.5156	42.8751
1	Present ($\varepsilon_z \neq 0$)	20.1540	22.7837	25.1971	29.1839	39.7489
	Present ($\varepsilon_z = 0$)	19.6467	22.2062	24.5578	28.4487	38.7970
	Vo *et al*. (2014)	19.6525	22.2108	24.5596	28.4447	38.7838
	Vo *et al*. (2015)	20.7425	22.7065	25.1075	29.0755	39.6144
2	Present ($\varepsilon_z \neq 0$)	13.9254	16.3324	18.8488	23.4004	36.6032
	Present ($\varepsilon_z = 0$)	13.5728	15.9090	18.3551	22.7885	35.7061
	Vo *et al*. (2014)	13.5801	15.9152	18.3587	22.7863	35.6914
	Vo *et al*. (2015)	13.8839	16.2761	18.7772	23.3042	36.4677
5	Present ($\varepsilon_z \neq 0$)	10.3878	11.9641	14.0859	18.5891	33.6301
	Present ($\varepsilon_z = 0$)	10.1363	11.6596	13.7155	18.0909	32.7877
	Vo *et al*. (2014)	10.1460	11.6676	13.7212	18.0914	32.7725
	Vo *et al*. (2015)	10.3673	11.9301	14.0353	18.5092	33.4958
10	Present ($\varepsilon_z \neq 0$)	9.6681	10.7939	12.5805	16.8308	32.3594
	Present ($\varepsilon_z = 0$)	9.4411	10.5256	12.2536	16.3754	31.3243
	Vo *et al*. (2014)	9.4515	10.5348	12.2605	16.3783	31.5265
	Vo *et al*. (2015)	9.6535	10.7689	12.5393	16.7574	32.2264

Table 9 Non-dimensional critical buckling loads of simply supported FG sandwich beams with homogeneous hardcore ($L / h = 20$)

K	Theory	1-0-1	2-1-2	1-1-1	1-2-1	1-8-1
0	Present ($\varepsilon_z \neq 0$)	53.3621	53.3621	53.3621	53.3621	53.3621
	Present ($\varepsilon_z = 0$)	53.2366	53.2366	53.2366	53.2366	53.2366
	Vo *et al*. (2014)	53.2364	53.2364	53.2364	53.2364	53.2364
	Vo *et al*. (2015)	53.3145	53.3145	53.3145	53.3145	53.3145
0.5	Present ($\varepsilon_z \neq 0$)	29.7909	32.1328	34.1705	37.4075	45.6821
	Present ($\varepsilon_z = 0$)	29.7172	32.2533	34.0862	37.3162	45.5749
	Vo *et al*. (2014)	29.7175	32.2629	34.0862	37.3159	45.5742
	Vo *et al*. (2015)	29.7626	32.1022	34.1380	41.8227	45.6424
1	Present ($\varepsilon_z \neq 0$)	20.7741	23.4811	26.0251	30.3066	41.9993
	Present ($\varepsilon_z = 0$)	20.7208	23.4208	25.9586	30.2309	41.9014
	Vo *et al*. (2014)	20.7212	23.4211	25.9588	30.2307	41.9004
	Vo *et al*. (2015)	20.7530	23.4572	25.9989	30.2774	41.9639
2	Present ($\varepsilon_z \neq 0$)	14.2348	16.6499	19.2518	24.0527	38.4734
	Present ($\varepsilon_z = 0$)	14.1967	16.6045	19. 3122	23.9901	38.3841
	Vo *et al*. (2014)	14.1973	16.6050	19.3116	23.9900	38.3831
	Vo *et al*. (2015)	14.2190	16.6307	19.2299	24.0276	38.4419
5	Present ($\varepsilon_z \neq 0$)	10.6436	12.1215	14.2694	18.9403	35.16836
	Present ($\varepsilon_z = 0$)	10.6164	12.0877	14.2280	18.8873	35.0867
	Vo *et al*. (2014)	10.6171	12.0883	14.2284	18.8874	35.0856
	Vo *et al*. (2015)	10.6330	12.1068	14.2505	18.9172	35.1400
10	Present ($\varepsilon_z \neq 0$)	10.0082	10.9361	12.7186	17.0966	33.7641
	Present ($\varepsilon_z = 0$)	9.9840	10.9068	12.6814	17.0441	33.5708
	Vo *et al*. (2014)	9.9847	10.9075	12.6819	17.0443	33.6843
	Vo *et al*. (2015)	9.9995	10.9239	12.7014	17.0712	33.7367

Vo *et al*. (2014) and (2015). Noted that the results given by Vo *et al*. (2014) and (2015), are obtained based on finite element model. Again, the proposed theory without the thickness stretching effect ($\varepsilon_z = 0$) and the model of Vo *et al*. (2014) give solutions close to each other, and these solutions are in an excellent agreement with the present theory that considers the thickness stretching effect ($\varepsilon_z \neq 0$) for moderately thick beams ($L / h = 20$). However, the present theory without the thickness stretching effect ($\varepsilon_z = 0$) and the model of Vo *et al*. (2014) slightly overestimate the critical loading loads for thick beams ($L / h = 5$) due to ignoring the thickness stretching effect.

Tables 12 and 13 contain dimensionless critical buckling loads for clamped – clamped FG sandwich beam with homogenous hardcore for both thick and moderately thick beams, respectively. Also, a good agreement between present results and other previous solutions is demonstrated from this comparison.

Table 10 Non-dimensional critical buckling loads of simply supported FG sandwich beams with homogeneous softcore ($L/h = 5$)

K	Theory	1-0-1	2-1-2	1-1-1	1-2-1	1-8-1
0	Present ($\varepsilon_z \neq 0$)	9.1575	9.1575	9.1575	9.1575	9.1575
	Present ($\varepsilon_z = 0$)	8.9524	8.9524	8.9524	8.9524	8.9524
	Vo *et al*. (2014)	8.9519	8.9519	8.9519	8.9519	8.9519
	Vo *et al*. (2015)	29.0755	26.5470	24.5952	21.8303	15.4469
0.5	Present ($\varepsilon_z \neq 0$)	28.4995	25.9956	24.1306	21.4239	15.1495
	Present ($\varepsilon_z = 0$)	28.4280	25.9503	24.0540	21.3821	15.1589
	Vo *et al*. (2014)	37.0638	33.7041	30.9841	27.0786	18.2334
	Vo *et al*. (2015)	36.3172	32.9786	30.3046	26.5884	17.8974
1	Present ($\varepsilon_z \neq 0$)	36.2103	32.8974	30.2449	26.4801	17.9093
	Present ($\varepsilon_z = 0$)	43.4657	39.8517	36.6422	31.7878	20.7839
	Vo *et al*. (2014)	42.5601	38.9753	35.8463	31.2146	20.4130
	Vo *et al*. (2015)	42.4501	38.8589	35.7058	31.0152	20.4222
2	Present ($\varepsilon_z \neq 0$)	47.7566	44.6508	41.4173	35.9875	23.0939
	Present ($\varepsilon_z = 0$)	46.7208	43.6303	40.4032	35.3273	22.6907
	Vo *et al*. (2014)	46.6504	43.5338	40.3235	35.0357	22.6881
	Vo *et al*. (2015)	48.9012	46.2561	43.2072	37.7129	24.0628
5	Present ($\varepsilon_z \neq 0$)	47.8231	45.2333	42.1412	36.7453	23.6402
	Present ($\varepsilon_z = 0$)	47.7825	45.1141	42.0693	36.6874	23.6329
	Vo *et al*. (2014)	9.1575	9.1575	9.1575	9.1575	9.1575
	Vo *et al*. (2015)	8.9524	8.9524	8.9524	8.9524	8.9524
10	Present ($\varepsilon_z \neq 0$)	8.9519	8.9519	8.9519	8.9519	8.9519
	Present ($\varepsilon_z = 0$)	29.0755	26.5470	24.5952	21.8303	15.4469
	Vo *et al*. (2014)	28.4995	25.9956	24.1306	21.4239	15.1495
	Vo *et al*. (2015)	28.4280	25.9503	24.0540	21.3821	15.1589

Table 11 Non-dimensional critical buckling loads of simply supported FG sandwich beams with homogeneous softcore ($L/h = 20$)

K	Theory	1-0-1	2-1-2	1-1-1	1-2-1	1-8-1
0	Present ($\varepsilon_z \neq 0$)	9.8299	9.8299	9.8299	9.8299	9.8299
	Present ($\varepsilon_z = 0$)	9.8067	9.8067	9.8067	9.8067	9.8067
	Vo *et al*. (2014)	9.8067	9.8067	9.8067	9.8067	9.8067
0.5	Present ($\varepsilon_z \neq 0$)	33.2953	30.9259	28.8816	25.6595	17.4709
	Present ($\varepsilon_z = 0$)	33.2249	30.8623	28.8236	25.6094	17.4347
	Vo *et al*. (2014)	33.2187	30.8546	28.8167	25.6086	17.4355

Table 11 Continued

K	Theory	1-0-1	2-1-2	1-1-1	1-2-1	1-8-1
1	Present ($\varepsilon_z \neq 0$)	42.2809	39.5081	36.9329	32.6526	21.1098
	Present ($\varepsilon_z = 0$)	42.1902	39.4266	36.8594	32.5906	21.0688
	Vo *et al*. (2014)	42.1810	39.4124	36.8445	32.5803	21.0698
2	Present ($\varepsilon_z \neq 0$)	48.8388	46.3198	43.6517	38.8121	24.5803
	Present ($\varepsilon_z = 0$)	48.7307	46.2216	43.5632	38.7387	24.5348
	Vo *et al*. (2014)	48.7215	46.2035	43.5408	38.7192	24.5356
5	Present ($\varepsilon_z \neq 0$)	52.4921	50.8904	48.6442	43.8770	27.8236
	Present ($\varepsilon_z = 0$)	52.3711	50.7776	48.5416	43.7922	27.7739
	Vo *et al*. (2014)	52.3655	50.7608	48.5163	43.7637	27.7736
10	Present ($\varepsilon_z \neq 0$)	53.1605	52.1127	50.2228	45.7224	29.1999
	Present ($\varepsilon_z = 0$)	53.0363	51.9945	50.1143	45.6354	29.1492
	Vo *et al*. (2014)	53.0331	51.9804	50.0902	45.6040	29.1471

Table 12 Non-dimensional critical buckling loads of clamped – clamped FG sandwich beams with homogeneous hardcore ($L / h = 5$)

K	Theory	1-0-1	2-1-2	1-1-1	1-2-1	1-8-1
0	Present ($\varepsilon_z \neq 0$)	165.7137	165.7137	165.7137	165.7137	165.7137
	Present ($\varepsilon_z = 0$)	154.5978	154.5978	154.5978	154.5978	154.5978
	Vo *et al*. (2014)	152.1470	152.1470	152.1470	152.1470	152.1470
	Vo *et al*. (2015)	160.2780	160.2780	160.2780	160.2780	160.2780
0.5	Present ($\varepsilon_z \neq 0$)	102.0793	109.8888	116.1127	125.3367	146.8777
	Present ($\varepsilon_z = 0$)	94.4652	101.7035	107.5141	116.1804	136.6076
	Vo *et al*. (2014)	92.8833	99.9860	105.6790	114.1710	134.2870
	Vo *et al*. (2015)	98.4559	105.9750	111.9680	120.8630	141.7880
1	Present ($\varepsilon_z \neq 0$)	74.4910	84.2031	92.5165	105.5315	137.4443
	Present ($\varepsilon_z = 0$)	68.6650	77.6050	85.3186	97.4780	127.6257
	Vo *et al*. (2014)	67.4983	76.2634	83.8177	95.7287	125.3860
	Vo *et al*. (2015)	71.7654	81.0936	89.0834	101.6130	132.5510
2	Present ($\varepsilon_z \neq 0$)	52.7606	62.2868	71.4009	86.9540	128.1608
	Present ($\varepsilon_z = 0$)	48.5114	57.2000	65.5961	80.0364	118.8061
	Vo *et al*. (2014)	47.7010	56.2057	64.4229	78.5608	116.6580
	Vo *et al*. (2015)	50.8183	59.9354	68.6743	83.6159	123.4770
5	Present ($\varepsilon_z \neq 0$)	39.2032	46.5945	54.7453	70.8229	119.2304
	Present ($\varepsilon_z = 0$)	36.0974	42.7203	50.1642	64.9913	110.3426
	Vo *et al*. (2014)	35.5493	42.0033	49.2763	63.7824	108.2970
	Vo *et al*. (2015)	37.8295	44.8488	52.6395	68.0510	114.7700

Table 12 Continued

K	Theory	1-0-1	2-1-2	1-1-1	1-2-1	1-8-1
10	Present ($\varepsilon_z \neq 0$)	35.4627	42.1045	49.2629	64.7226	115.3670
	Present ($\varepsilon_z = 0$)	32.7650	38.6182	45.1208	59.3329	106.6885
	Vo *et al*. (2014)	32.3019	37.9944	44.3374	58.2461	104.6920
	Vo *et al*. (2015)	34.2824	40.5544	47.3804	62.1959	111.0120

Table 13 Non-dimensional critical buckling loads of clamped – clamped FG sandwich beams with homogeneous hardcore ($L / h = 20$)

K	Theory	1-0-1	2-1-2	1-1-1	1-2-1	1-8-1
0	Present ($\varepsilon_z \neq 0$)	228.3777	228.3777	228.3777	228.3777	228.3777
	Present ($\varepsilon_z = 0$)	213.5879	213.5879	213.5879	213.5879	213.5879
	Vo *et al*. (2014)	208.9510	208.9510	208.9510	208.9510	208.9510
	Vo *et al*. (2015)	210.7420	210.7420	210.7420	210.7420	210.7420
0.5	Present ($\varepsilon_z \neq 0$)	128.3061	138.3757	147.0931	160.8884	195.9677
	Present ($\varepsilon_z = 0$)	119.9188	129.3312	137.4835	150.3902	183.2318
	Vo *et al*. (2014)	117.3030	126.5080	134.4810	147.1040	179.2350
	Vo *et al*. (2015)	118.3530	127.6410	135.6840	148.4130	180.8010
1	Present ($\varepsilon_z \neq 0$)	89.7126	101.4032	112.3328	130.6555	180.3802
	Present ($\varepsilon_z = 0$)	83.8241	94.7462	104.9626	122.0979	168.6366
	Vo *et al*. (2014)	81.9927	92.6741	102.6650	119.4220	164.9490
	Vo *et al*. (2015)	82.7434	93.5248	103.6060	120.5090	166.4060
2	Present ($\varepsilon_z \neq 0$)	61.5867	72.0716	83.2946	103.9198	165.4276
	Present ($\varepsilon_z = 0$)	57.5336	67.3225	77.8079	97.0883	154.6386
	Vo *et al*. (2014)	56.2773	65.8489	76.1020	94.9563	151.2500
	Vo *et al*. (2015)	56.7986	66.4664	76.8166	95.8403	152.6000
5	Present ($\varepsilon_z \neq 0$)	46.0361	52.54716	61.8472	81.9806	151.3849
	Present ($\varepsilon_z = 0$)	43.0119	49.0795	57.7618	76.5732	141.4945
	Vo *et al*. (2014)	42.0775	48.0070	56.4958	74.8903	138.3880
	Vo *et al*. (2015)	42.4596	48.4588	57.0343	75.6019	139.6370
10	Present ($\varepsilon_z \neq 0$)	43.1931	47.4158	55.1536	74.0377	145.4102
	Present ($\varepsilon_z = 0$)	40.3659	44.2893	51.5090	69.1453	135.9027
	Vo *et al*. (2014)	39.4930	43.3233	50.3811	67.6270	132.9170
	Vo *et al*. (2015)	39.8436	43.7273	50.8611	68.2737	134.1220

Figs. 4 and 5 plot the critical buckling loads parameters versus the material parameter k of simply supported power-law (1-1-1) FG sandwich beams with both homogeneous hardcore and softcore, respectively. It can be seen, that as the material parameter increases, the critical buckling loads decrease for sandwich beams with hardcore and increase for sandwich beams with softcore.

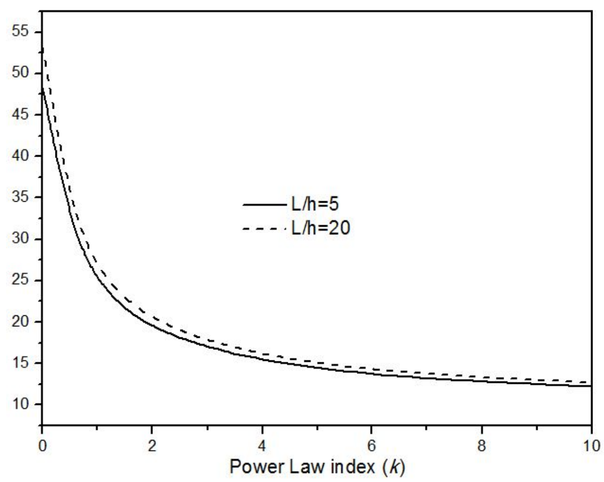

Fig. 4 Variation of critical buckling load P_{cr} versus the material parameter k for (1-1-1) simply supported FG sandwich beams with homogeneous hardcore

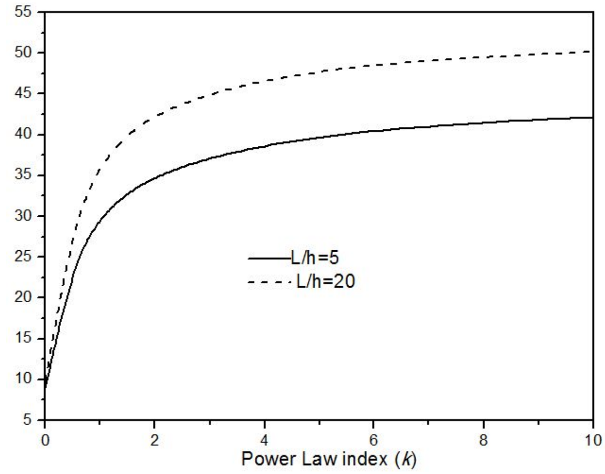

Fig. 5 Variation of critical buckling load P_{cr} versus the material parameter k for (1-1-1) simply supported FG sandwich beams with homogeneous softcore

5. Conclusions

A simple hyperbolic shear deformation beam theory with thickness stretching effect for the buckling and vibration analysis of FG sandwich beams is developed. The equations of motion are obtained by utilizing the Hamilton's principle. Results prove that the present model is able to introduce the thickness stretching effect and providing very accurate results compared with the other existing higher-order beam theories.

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