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A computational shear displacement model for vibrational analysis of functionally graded beams with porosities

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Abstract. This work presents a free vibration analysis of functionally graded metal-ceramic (FG) beams with considering porosities that may possibly occur inside the functionally graded materials (FGMs) during their fabrication. For this purpose, a simple displacement field based on higher order shear deformation theory is implemented. The proposed theory is based on the assumption that the transverse displacements consist of bending and shear components in which the bending components do not contribute toward shear forces and, likewise, the shear components do not contribute toward bending moments. The most interesting feature of this theory is that it accounts for a quadratic variation of the transverse shear strains across the thickness, and satisfies the zero traction boundary conditions on the top and bottom surfaces of the beam without using shear correction factors. In addition, it has strong similarities with Euler-Bernoulli beam theory in some aspects such as equations of motion, boundary conditions, and stress resultant expressions. The rule of mixture is modified to describe and approximate material properties of the FG beams with porosity phases. By employing the Hamilton's principle, governing equations of motion for coupled axial-shear-flexural response are determined. The validity of the present theory is investigated by comparing some of the present results with those of the first-order and the other higher-order theories reported in the literature. Illustrative examples are given also to show the effects of varying gradients, porosity volume fraction, aspect ratios, and thickness to length ratios on the free vibration of the FG beams.

Keywords: FG beam; shear deformation theory; vibration; porosity

1. Introduction

It is well-known that classical Euler-Bernoulli theory of beam bending, also known as elementary theory of bending (ETB), disregards the effects of the shear deformation. The theory is

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suitable for slender beams and is not suitable for thick or deep beams since it is based on the assumption that the transverse normal to the neutral axis remains so during bending and after bending, implying that the transverse shear strain is zero. Since the theory neglects the transverse shear deformation, it underestimates deflections and overestimates the natural frequencies in case of thick beams, where shear deformation effects are significant. The other well-known beam theory is Timoshenko beam theory or the first-order shear deformation beam theory (FSDBT) in which straight lines perpendicular to the mid-plane before bending remain straight, but no longer remain perpendicular to the mid-plane after bending. Bresse (1859), Rayleigh (1880), and Timoshenko (1921) were the pioneer investigators to include refined effects such as the rotatory inertia and shear deformation in the beam theory. Timoshenko showed that the effect of transverse shear is much greater than that of rotatory inertia on the response of transverse vibration of prismatic bars. In FSDBT, the distribution of the transverse shear stress with respect to the thickness coordinate is assumed constant. Thus, a shear correction factor is required to compensate for the error because of this assumption in FSDBT. Cowper (1966) has given refined expressions for the shear correction factor for different cross-sections of the beam. The accuracy of Timoshenko beam theory for transverse vibrations of simply supported beam with respect to the fundamental frequency is verified by Cowper (1966) with a plane stress exact elasticity solution. To remove the discrepancies in the classical theory and FSDBT, higher-order or equivalent refined shear deformation theories were developed and are available in the open literature for static and vibration analysis of beams. The higher-order shear deformation beam theories consider the warping of the cross-sections and satisfy the zero transverse shear stress condition of the upper and lower fibers of the cross-section without a shear correction factor. In the literature, various higher-order shear deformation theories which satisfy the above-mentioned conditions are proposed by several researchers. The well-known higher-order beam theories are as follows: (i) parabolic shear deformation beam theory (PSDBT) (Reddy 1984); (ii) trigonometric shear deformation beam theory (TSDBT) (Touratier 1991); (iii) exponential shear deformation beam theory (ESDBT) (Karama *et al*. 2003).

Recently, advanced composite materials known as functionally graded material have attracted much attention in many engineering applications due to their advantages of being able to resist to high temperature gradients while maintaining structural integrity (Koizumi 1997). The functionally graded materials (FGMs) are microscopically inhomogeneous, in which the mechanical properties vary smoothly and continuously from one surface to the other. They are usually made from a mixture of ceramics and metals to attain the significant requirement of material properties.

Due to the increased relevance of the FGMs structural components in the design of engineering structures, their static and vibration characteristics have attracted the attention of many scientists in recent years (Tounsi *et al*. 2013a, Bouderba *et al*. 2013, Bachir Bouiadjra *et al*. 2013, Saidi *et al*. 2013, Hebali *et al*. 2014, Fekrar *et al*. 2014, Belabed *et al*. 2014, Ait Amar Meziane *et al*. 2014, Chakraverty and Pradhan 2014, Zidi *et al*. 2014, Bousahla *et al*. 2014, Khalfi *et al*. 2014, Swaminathan and Naveenkumar 2014, Najafov *et al*. 2014, Ait Yahia *et al*. 2015, Hamidi *et al*. 2015, Attia *et al*. 2015, Mahi *et al*. 2015, Duc and Thang 2015). It is observed from the literature that the amount of such work carried out for isotropic beams are considerable, and limited literature is available on composite beams. However, very few literatures on the analysis of the FGMs beam exist. Furthermore, the literature on the static and the dynamic analysis of FG beams is still limited in numbers compared to plates and shells. Sankar (2001) gave an elasticity solution based on the Euler-Bernoulli beam theory for functionally graded beam subjected to static transverse loads by assuming that Young's modulus of the beam vary exponentially through the thickness. Zhong and Yu (2007) presented an analytical solution of a cantilever FG beam with arbitrary graded variations of material property distribution based on two-dimensional elasticity theory. Li (2008) proposed a new unified approach to investigate the static and the free vibration behavior of Euler-Bernoulli and Timoshenko beams. Kadoli *et al*. (2008) studied the static behavior of an FG beam by using higher-order shear deformation theory and finite element method. Benatta *et al*. (2008) proposed an analytical solution to the bending problem of a symmetric FG beam by including warping of the cross-section and shear deformation effect. Sallai *et al*. (2009) investigated the static responses of a sigmoid FG thick beam by using different beam theories. Benatta *et al*. (2009) presented a mathematical solution for bending of short hybrid composite beams with variable fibers spacing. Bedjilili *et al*. (2009) presented exact solutions for shear flexible symmetric composite beams with a variable fiber volume fraction through thickness. Ould Larbi *et al*. (2013) presented an efficient shear deformation beam theory based on neutral surface position for bending and free vibration of FG beams. Yaghoobi and Torabi (2013a) investigated the post-buckling and nonlinear vibration of imperfect FG beams. Yaghoobi and Torabi (2013b) examined analytically the large amplitude vibration and post-buckling of FG beams resting on non-linear elastic foundations. Yaghoobi *et al*. (2014) studied also the post-buckling and nonlinear free vibration response of FG beams resting on nonlinear elastic foundation under thermo-mechanical loading using the variational iteration method (VIM). A simple refined nth-order shear deformation theory is presented by Yaghoobi and Fereidoon (2014) to discuss the mechanical and thermal buckling behaviors of FG plates supported by elastic foundation. Hadji *et al*. (2014) studied the bending and vibration responses of FG beams via a higher shear deformation beam theory. Bourada *et al*. (2015) used the concept of the neutral surface position to develop a simple and refined trigonometric higher-order beam theory for bending and vibration behavior of FG beams. Al-Basyouni *et al*. (2015) proposed a novel unified beam formulation and a modified couple stress theory for bending and dynamic behaviours of FG micro-beam. Recently, Larbi Chaht *et al*. (2015) studied the bending and buckling behaviors of size-dependent FG nanobeams including the thickness stretching effect. In this work, the size-dependent FG nanobeam is investigated on the basis of the nonlocal continuum model (Heireche *et al*. 2008, Benzair *et al*. 2008, Amara *et al*. 2010, Tounsi *et al*. 2013b, c, d, Berrabah *et al*. 2013, Semmah *et al*. 2014, Benguediab *et al*. 2014).

However, in FGM fabrication, micro voids or porosities can occur within the materials during the process of sintering. This is because of the large difference in solidification temperatures between material constituents (Zhu *et al*. 2001). Wattanasakulpong *et al*. (2012) also gave the discussion on porosities happening inside FGM samples fabricated by a multi-step sequential infiltration technique. Therefore, it is important to take in to account the porosity effect when designing FGM structures subjected to dynamic loadings. Recently, Wattanasakulpong and Ungbhakorn (2014) studied linear and nonlinear vibration problems of elastically end restrained FG beams having porosities.

In this paper, a variationally consistent shear deformation theory is developed using a new displacement field for thick FG beams having porosities. The displacement field of the proposed theory is chosen based on the following assumptions: (1) the axial and transverse displacements consist of bending and shear components in which the bending components do not contribute toward shear forces and, likewise, the shear components do not contribute toward bending moments; (2) the bending component of axial displacement is similar to that given by the Euler-Bernoulli beam theory; and (3) the shear component of axial displacement gives rise to the parabolic variation of shear strain and hence to shear stress through the thickness of the beam in such a way that shear stress vanishes on the top and bottom surfaces of the beam without using shear correction factors. The model is applied to simply supported FGM beam of rectangular cross-section for both static and free vibration analysis. A general solution is obtained. The results obtained are compared with those of elementary, refined, and exact beam theories available in the literature.

2. Theoretical formulation

The theoretical formulation of a uniform FG thick beam having porosities based on certain kinematical and physical assumptions is presented. The variationally correct forms of differential equations and boundary conditions, based on the assumed displacement field are obtained using the principle of virtual work. As it is seen in Fig. 1, the beam under consideration occupies the region

$$0 \le x \le L; \quad -b/2 \le y \le b/2; \quad -h/2 \le z \le h/2 \tag{1}$$

where x, y, z are Cartesian coordinates, L is the length, b is the width, and h is the total depth of beam. The beam is subjected to transverse load of intensity $q(x)$ per unit length of the beam. The beam can have any meaningful boundary conditions.

2.1 Effective material properties of metal ceramic functionally graded beams

A FG beam made from a mixture of two material phases, for example, a metal and a ceramic. The material properties of FG beams are assumed to vary continuously through the thickness of the beam. In this investigation, the imperfect beam is assumed to have porosities spreading within the thickness due to defect during production. Consider an imperfect FGM with a porosity volume fraction, α ($\alpha \ll 1$), distributed evenly among the metal and ceramic, the modified rule of mixture proposed by Wattanasakulpong and Ungbhakorn (2014) is used as

$$P = P_m\left(V_m - \frac{\alpha}{2}\right) + P_c\left(V_c - \frac{\alpha}{2}\right) \tag{2}$$

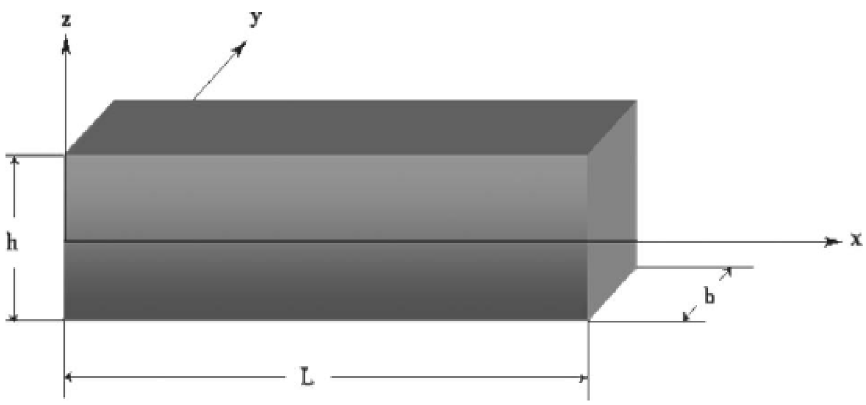

Fig. 1 Co-ordinates and geometry of functionally graded beam

Now, the total volume fraction of the metal and ceramic is: $V_m + V_c = 1$, and the power law of volume fraction of the ceramic is described as

$$V_c = \left(\frac{z}{h} + \frac{1}{2}\right)^k \tag{3}$$

Hence, all properties of the imperfect FGM can be written as

$$P = (P_c - P_m)\left(\frac{z}{h} + \frac{1}{2}\right)^k + P_m - (P_c + P_m)\frac{\alpha}{2} \tag{4}$$

It is noted that the positive real number k ($0 \leq k \leq \infty$) is the power law or volume fraction index, and z is the distance from the mid-plane of the FG plate. The FG beam becomes a fully ceramic plate when k is set to zero and fully metal for large value of k.

Thus, the Young's modulus (E) and material density (ρ) equations of the imperfect FGM beam can be expressed as

$$E(z) = (E_c - E_m)\left(\frac{z}{h} + \frac{1}{2}\right)^k + E_m - (E_c + E_m)\frac{\alpha}{2} \tag{5}$$

$$\rho(z) = (\rho_c - \rho_m)\left(\frac{z}{h} + \frac{1}{2}\right)^k + \rho_m - (\rho_c + \rho_m)\frac{\alpha}{2} \tag{6}$$

However, Poisson's ratio (v) is assumed to be constant. The material properties of a perfect FG beam can be obtained when α is set to zero.

In addition, for another scenario of porosity distribution, it is possible to obtain imperfect FGM samples which have almost porosities spreading around the middle zone of the cross-section and the amount of porosity seems to be on the decrease to zero at the top and bottom of the cross-section. Based on the principle of the multi-step sequential infiltration technique that can be employed to fabricate FGM samples (Wattanasakulpong *et al*. 2012), the porosities mostly occur at the middle zone. At this zone, it is difficult to infiltrate the materials completely, while at the top and bottom zones, the process of material infiltration can be performed easier and leaves less porosity. Consider this scenario, the equations of Young's modulus (E) and material density (ρ) in Eqs. (5)-(6) are replaced by the following forms

$$E(z) = (E_c - E_m)\left(\frac{z}{h} + \frac{1}{2}\right)^k + E_m - (E_c + E_m)\frac{\alpha}{2}\left(1 - \frac{2|z|}{h}\right) \tag{7}$$

$$\rho(z) = (\rho_c - \rho_m)\left(\frac{z}{h} + \frac{1}{2}\right)^k + \rho_m - (\rho_c + \rho_m)\frac{\alpha}{2}\left(1 - \frac{2|z|}{h}\right) \tag{8}$$

2.2 Assumptions made in the theoretical formulation

(i) The displacements are small in comparison with the beam thickness and, therefore, strains involved are infinitesimal.

(ii) The transverse displacement w includes two components of bending w_b, and shear w_s.

These components are functions of coordinate x only.

$$w(x,z) = w_b(x) + w_s(x) \tag{9}$$

(iii) The transverse normal stress σ_z is negligible in comparison with in-plane stress σ_x.
(iv) The axial displacement u consists of extension, bending, and shear components.

$$u = u_0 + u_b + u_s, \tag{10}$$

The bending components u_b and v_b are assumed to be similar to the displacements given by the classical beam theory (ETB). Therefore, the expression for u_b can be given as

$$u_b = -z\frac{\partial w_b}{\partial x}, \tag{11}$$

The shear component u_s gives rise, in conjunction with w_s, to the parabolic variations of shear strain γ_{xz} and hence to the shear stress τ_{xz} through the thickness of the beam in such a way that the shear stress τ_{xz} is zero at the top and bottom faces of the beam. Consequently, the expression for u_s can be given as (Shimpi and Patel 2006a, b, Tounsi *et al*. 2013b)

$$u_s = z\left[\frac{1}{4} - \frac{5}{3}\left(\frac{z}{h}\right)^2\right]\frac{\partial w_s}{\partial x}, \tag{12}$$

2.3 Kinematics and constitutive equations

Based on the above-mentioned assumptions, the displacement field of the present beam theory is given by

$$\begin{aligned} u(x,z,t) &= u_0(x,t) - z\frac{\partial w_b}{\partial x} + z\left[\frac{1}{4} - \frac{5}{3}\left(\frac{z}{h}\right)^2\right]\frac{\partial w_s}{\partial x} \\ w(x,z,t) &= w_b(x,t) + w_s(x,t) \end{aligned} \tag{13}$$

The strains associated with the displacements in Eq. (13) are

$$\begin{aligned} \varepsilon_x &= \varepsilon_x^0 + z\, k_x^b + f\, k_x^s \\ \gamma_{xz} &= g\, \gamma_{xz}^s \end{aligned} \tag{14}$$

where

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x}, \quad k_x^b = -\frac{\partial^2 w_b}{\partial x^2}, \quad k_x^s = -\frac{\partial^2 w_s}{\partial x^2}, \quad \gamma_{xz}^s = \frac{\partial w_s}{\partial x}, \quad f = \frac{-1}{4}z + \frac{5}{3}z\left(\frac{z}{h}\right)^2, \quad g = \frac{5}{4} - 5\left(\frac{z}{h}\right)^2 \tag{15}$$

By assuming that the material of FGM beam obeys Hooke's law, the stresses in the beam become

$$\sigma_x = E(z)\,\varepsilon_x \quad \text{and} \quad \tau_{xz} = G(z)\,\gamma_{xz} \tag{16}$$

2.4 Governing equations and boundary conditions

Using Eqs. (14)-(16) for strains and stresses and dynamic version of principle of virtual work (Draiche *et al*. 2014, Nedri *et al*. 2014), variationally consistent governing differential equations and boundary conditions for the beam under consideration are obtained. The principle of virtual work when applied to the beam leads to

$$\int_0^L \int_{-h/2}^{h/2} \left(\sigma_x \delta\,\varepsilon_x + \tau_{zx}\delta\,\gamma_{zx}\right) dz\,dx - \int_0^L \int_{-h/2}^{h/2} \rho\left[\ddot{u}_0 \delta\,u_0 + \ddot{v}_0 \delta\,v_0 + (\ddot{w}_b + \ddot{w}_s)\delta\,(w_b + w_s)\right] dz\,dx = 0 \tag{17}$$

Collecting the coefficients δu_0, δw_b and δw_s in Eq. (17), equations of motion are obtained as

$$\delta\,u_0: \ \frac{dN}{dx} = I_0 \ddot{u}_0 \tag{18a}$$

$$\delta\,w_b: \ \frac{d^2 M_b}{dx^2} = I_0(\ddot{w}_b + \ddot{w}_s) - I_2 \frac{d^2 \ddot{w}_b}{dx^2} \tag{18b}$$

$$\delta\,w_s: \ \frac{d^2 M_s}{dx^2} + \frac{dQ}{dx} = I_0(\ddot{w}_b + \ddot{w}_s) - \frac{I_2}{84}\frac{d^2 \ddot{w}_s}{dx^2} \tag{18c}$$

where N, M_b, M_s and Q are the stress resultants defined as

$$(N, M_b, M_s) = \int_{-h/2}^{h/2} (1, z, f)\,\sigma_x dz \quad \text{and} \quad Q = \int_{-h/2}^{h/2} g\,\tau_{xz} dz \tag{19a}$$

and (I_0, I_2) are mass inertias defined as

$$\left(I_0, I_2\right) = \int_{-h/2}^{h/2} \left(1, z^2\right) \rho(z) dz \tag{19b}$$

Eq. (18) can be expressed in terms of displacements (u_0, w_b, w_s) by using Eqs. (13), (14), (15) and (19) as follows

$$A_{11}\frac{\partial^2 u_0}{\partial x^2} - B_{11}\frac{\partial^3 w_b}{\partial x^3} - B_{11}^s \frac{\partial^3 w_s}{\partial x^3} = I_0 \ddot{u}_0 \tag{20a}$$

$$B_{11}\frac{\partial^3 u_0}{\partial x^3} - D_{11}\frac{\partial^4 w_b}{\partial x^4} - D_{11}^s \frac{\partial^4 w_s}{\partial x^4} = I_0(\ddot{w}_b + \ddot{w}_s) - I_2 \frac{\partial^2 \ddot{w}_b}{\partial x^2} \tag{20b}$$

$$B_{11}^s\frac{\partial^3 u_0}{\partial x^3} - D_{11}^s\frac{\partial^4 w_b}{\partial x^4} - H_{11}^s \frac{\partial^4 w_s}{\partial x^4} + A_{55}^s \frac{\partial^2 w_s}{\partial x^2} = I_0(\ddot{w}_b + \ddot{w}_s) - \frac{I_2}{84}\frac{\partial^2 \ddot{w}_b}{\partial x^2} \tag{20c}$$

where the stiffness components and inertias are given as

$$
\begin{aligned}
&\{A_{11}, B_{11}, D_{11}, E_{11}, F_{11}, H_{11}\} = \int_{-h/2}^{h/2} \{1, z, z^2, z^3, z^4, z^6\} E(z)dz\,, \\
&B_{11}^s = -\frac{1}{4}B_{11} + \frac{5}{3h^2}E_{11}, \\
&D_{11}^s = -\frac{1}{4}D_{11} + \frac{5}{3h^2}F_{11}, \\
&H_{11}^s = \frac{1}{16}D_{11} - \frac{5}{6h^2}F_{11} + \frac{25}{9h^4}H_{11}, \\
&\{A_{55}, D_{55}, F_{55}\} = \int_{-h/2}^{h/2} \{1, z^2, z^4\} G(z)dz, \\
&A_{55}^s = \frac{25}{16}A_{55} - \frac{25}{2h^2}D_{55} + \frac{25}{h^4}F_{55},
\end{aligned}
\tag{21}
$$

3. Analytical solution

The equations of motion admit the Navier solutions for simply supported beams. The variables u_0, w_b, w_s can be written by assuming the following variations

$$
\begin{Bmatrix} u_0 \\ w_b \\ w_s \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} U_m \cos(\lambda\, x)\, e^{i\,\omega\, t} \\ W_{bm} \sin(\lambda\, x)\, e^{i\,\omega\, t} \\ W_{sm} \sin(\lambda\, x)\, e^{i\,\omega\, t} \end{Bmatrix} \tag{22}
$$

where U_m, W_{bm}, and W_{sm} are arbitrary parameters to be determined, ω is the eigenfrequency associated with mth eigenmode, and $\lambda = m\pi / L$.

Substituting Eqs. (21) and (22) into equations of motion (20) enables to get below eigenvalue equations for any fixed value of m, for free vibration problem

$$
\left([K] - \omega^2 [M]\right)\{\Delta\} = \{0\} \tag{23}
$$

where $\{\Delta\}$ denotes the column

$$
\{\Delta\}^T = \{U_m, W_{bm}, W_{sm}\}, \tag{24}
$$

and

$$
[K] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{bmatrix}, \quad [M] = \begin{bmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & m_{23} \\ 0 & m_{23} & m_{33} \end{bmatrix} \tag{25}
$$

in which

$$
\begin{aligned}
k_{11} &= A_{11}.\alpha^2 \\
k_{12} &= -B_{11}.\alpha^3 \\
k_{13} &= -B_{11}^s.\alpha^3 \\
k_{22} &= D_{11}.\alpha^4 \\
k_{23} &= D_{11}^s.\alpha^4 \\
k_{33} &= H_{11}^s.\alpha^4 + A_{55}^s.\alpha^2 \\
m_{11} &= m_{23} = I_0 \\
m_{22} &= I_0 + I_2.\alpha^2 \\
m_{33} &= I_0 + \frac{I_2}{84}\alpha^2
\end{aligned}
\tag{26}
$$

4. Numerical results and discussion

In numerical analysis, fundamental frequencies of simply supported perfect and imperfect FG beams are evaluated. The FG beams are made of aluminium (Al; E_m = 701 GPa, ρ_m = 2700 kg/m^3, ν_m = 0.23) and alumina (Al_2O_3; E_c = 380 GPa, ρ_c = 3800 kg/m^3, ν_c = 0.23) and their properties change through the thickness of the beam according to power-law. The bottom surfaces of the FG beams are aluminium rich, whereas the top surfaces of the FG beams are alumina rich.

To validate accuracy of the proposed theory, the comparisons between the present results and the available results obtained by Koochaki (2011) and Sina *et al*. (2009) are shown in Tables 1

Table 1 Comparison of non-dimensional fundamental frequencies of homogenous beams

$$\left(\overline{\omega} = (\omega L^2/h)\sqrt{I_0 \Big/ \int_{-h/2}^{h/2} E(z)dz}\right)$$

h/L	ETB (Reddy 1999)	FSDBT (Koochaki 2011)	PSDBT (Koochaki 2011)	Present
0.01	2.985526	2.986137	2.9861380	2.9861344
0.0125	2.985232	2.985827	2.9858280	2.9858287
0.0142	2.984340	2.985556	2.9855680	2.9855821
0.0166	2.984865	2.985155	2.9851680	2.9851807
0.02	2.983701	2.984505	2.9845054	2.9845054
0.025	2.982588	2.983285	2.9832858	2.9832858
0.033	2.979668	2.980657	2.9806572	2.9807765
0.04	2.976570	2.978020	2.9780220	2.9780222
0.05	2.971688	2.973193	2.9731941	2.9731941
0.066	2.962858	2.962858	2.9628610	2.9633287
0.1	2.931568	2.934044	2.9340570	2.9340576

Table 2 Non-dimensional fundamental frequency of FG beam: $\left(\overline{\omega} = (\omega L^2 / h) \sqrt{I_0 \Big/ \int_{-h/2}^{h/2} E(z) dz} \right)$

L / h	k	Present	Sina *et al*. (2009)
10	0	2.879551	2.879
	0.3	2.774811	2.774
30	0	2.922108	2.922
	0.3	2.813328	2.813
100	0	2.927100	2.927
	0.3	2.817838	2.817

and 2. Indeed, in Table 1, the nondimensional natural frequency of a simply supported homogenous beam (k = 0) obtained from the present new beam theory are compared with other beams theories results (Koochaki 2011) for three different values of thickness-to-length ratio. As can be seen the results of three shear deformation theories are in good agreement with the Euler-Bernoulli beam theory results. Also, the frequencies predicted by the present beam theory and the other two shear deformation theories are very close to each other.

Table 2 shows non-dimensional natural frequencies for the perfect FG beam with k = 0 and 0.3 for different length-to-height ratios. The results of the present formulation are compared with those of Sina *et al*. (2009) and the agreement is very satisfactory.

Fig. 2 shows the non-dimensional fundamental natural frequency versus length-to-height ratio for simply supported perfect FG beam based on PSDBT and the present beam theory. As it can

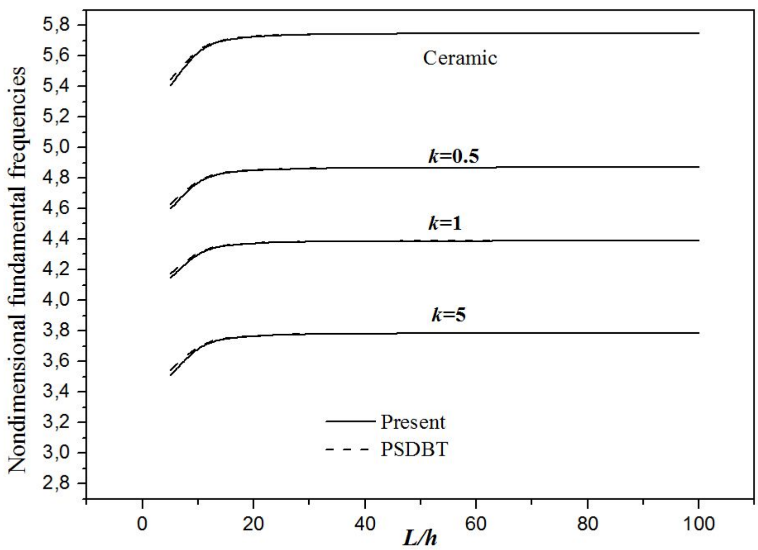

Fig. 2 Variation of the fundamental frequency $(\overline{\omega} = (\omega L^2 / h)\,(\rho_M / E_M))$ of FG beams with L / h ratio for various values of the power-law exponent

be seen, the power-law exponent plays an important role on the fundamental frequency of the perfect FG beam and the frequencies are increased when the value of L / h is increased. In addition, the comparisons show that the agreement between the present results and those obtained using PSDBT is satisfactory.

Table 3 Six first Non-dimensional frequencies $\left(\overline{\omega} = (\omega L^2 / h)\sqrt{I_0 \Big/ \int_{-h/2}^{h/2} E(z)dz}\right)$ of FG beam ($L / h = 5$)

K	α	$\overline{\omega}_1$	$\overline{\omega}_2$	$\overline{\omega}_3$	$\overline{\omega}_4$	$\overline{\omega}_5$	$\overline{\omega}_6$
	0.0	2.652071106	9.227581431	17.69478543	26.95936800	36.56997080	46.35346951
0.5	0.1	2.629935615	9.163948853	17.59574226	26.83563397	36.43006594	46.20291888
	0.2	2.601215142	9.080767423	17.46517379	26.67110200	36.24244879	45.99940418
	0.0	2.581070224	8.998837152	17.29734773	26.41699106	35.91550139	45.61898733
1.0	0.1	2.527515127	8.838878481	17.03790125	26.08073189	35.52391636	45.18878939
	0.2	2.450084220	8.604783550	16.65295765	25.57471517	34.92630325	44.52326210
	0.0	2.586406159	8.950808821	17.11088871	26.04796748	35.35579761	44.88145525
2.0	0.1	2.487791307	8.651090665	16.61507660	25.39454980	34.58585936	44.03039853
	0.2	2.316541140	8.123771581	15.72853020	24.20495086	33.15759214	42.42204192
	0.0	2.792915979	9.395559685	17.55065355	26.27845939	35.25569424	44.39249113
5.0	0.1	2.694743432	9.069598562	16.96555175	25.44949237	34.21042237	43.15861543
	0.2	2.450980911	8.307099364	15.65914662	23.66287326	32.02232974	40.64091433

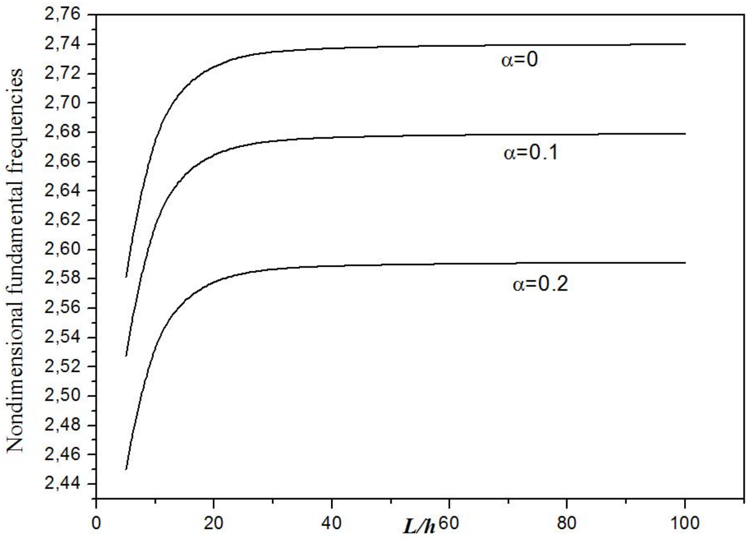

Fig. 3 Variation of the fundamental frequency $\left(\overline{\omega} = (\omega L^2 / h)\sqrt{I_0 \Big/ \int_{-h/2}^{h/2} E(z)dz}\right)$ of FG beams ($k = 1$) with L / h ratio for various values of the porosity volume fraction by considering the first solution

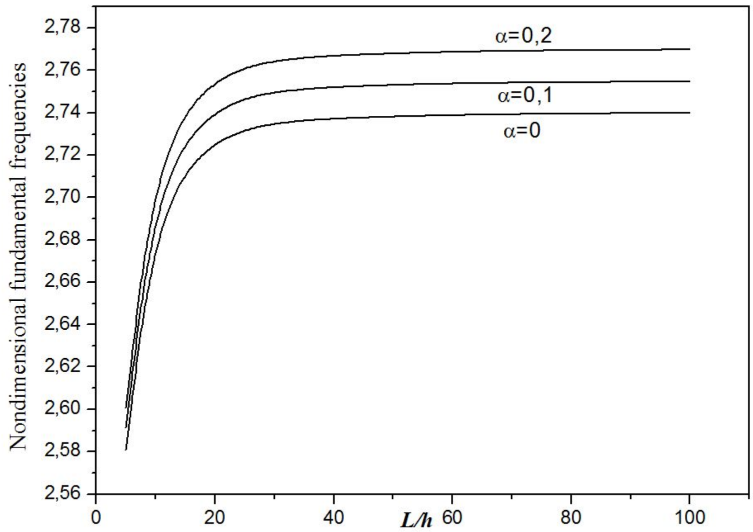

Fig. 4 Variation of the fundamental frequency $\left(\overline{\omega} = (\omega L^2/h)\sqrt{I_0 \Big/ \int_{-h/2}^{h/2} E(z)dz}\right)$ of FG beams (k = 1) with L / h ratio for various values of the porosity volume fraction by considering the second solution

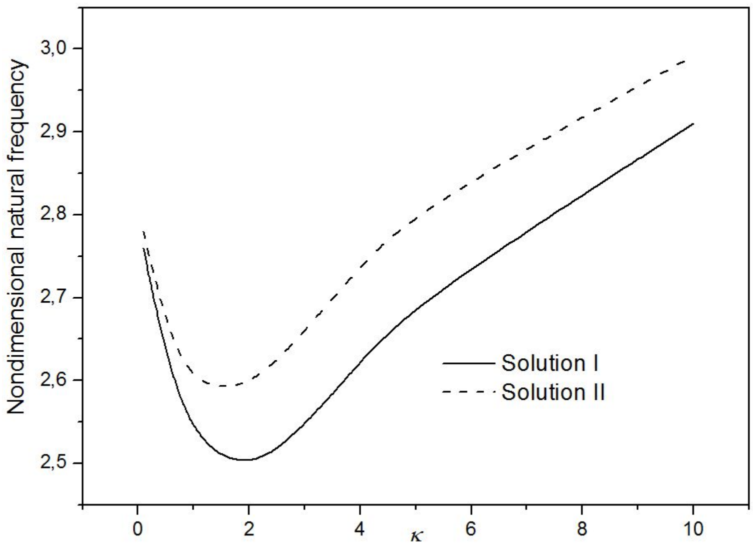

Fig. 5 Variation of the fundamental frequency $\left(\overline{\omega} = (\omega L^2/h)\sqrt{I_0 \Big/ \int_{-h/2}^{h/2} E(z)dz}\right)$ of imperfect FG beams with power-law exponent k. (L / h = 5, α = 0.1)

The first six dimensionless frequencies of perfect and imperfect FG beams are provided in Table 3. It should be noted that the materials properties are predicted using Eqs. (4)-(5). The results reveal that the frequency results decrease as the volume fraction of porosity (α) increases.

In Figs. 3 and 4, the effect of the porosity the fundamental frequencies of FG beams with two different types of porosity distribution is illustrated. It is noted that Solution I refers to the result of imperfect FG beams with evenly distributed porosities using Eqs. (4)-(5), while, Solution II is for the beams with another type of porosity distribution using Eqs. (6)-(7). It can be seen from Fig. 3 that the porosity leads to an increase of frequency and hence this type of porosity distribution (Solution I) makes the beam stiffer. However, the effect of porosity on fundamental frequencies (Fig. 4) using Solution II is reversed and this type of porosity distribution makes the beam flexible.

It is interesting to compare the free vibration results obtained from different types of porosity distribution. Thus, the next numerical examples are given for this purpose. In Fig. 5, the fundamental frequencies of imperfect FG beams with two different types of porosity distribution are plotted versus the power-law exponent (k). As observed, Solution II provides higher frequencies than those of Solution I; moreover, the frequencies increase with the increase of the power-law exponent (k) when this latter takes values more than 2.

5. Conclusions

A new shear deformation beam theory is proposed for free vibration of perfect and imperfect FG beams. The theory accounts for a quadratic variation of the shear strains across the thickness, and satisfies the zero traction boundary conditions on the top and bottom surfaces of the beam without using shear correction factor. The modified rule of mixture covering porosity phases is used to describe and approximate material properties of the imperfect FG beams. Numerical examples show that the proposed theory gives solutions which are almost identical with those obtained using other shear deformation theories. The influence of the porosities on natural frequencies is then discussed. The formulation lends itself particularly well to wave propagation in orthotropic non-homogeneous medium (Mahmoud *et al*. 2014), which will be considered in the near future.

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