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Cross-sectional analysis of arbitrary sections allowing for residual stresses

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Abstract. The method of cross-section analysis for different sections in a structural frame has been widely investigated since the 1960s for determination of sectional capacities of beam-columns. Many hand-calculated equations and design graphs were proposed for the specific shape and type of sections in pre-computer age decades ago. In design of many practical sections, these equations may be uneconomical and inapplicable for sections with irregular shapes, leading to the high construction cost or inadequate safety. This paper not only proposes a versatile numerical procedure for sectional analysis of beam-columns, but also suggests a method to account for residual stress and geometric imperfections separately and the approach is applied to design of high strength steels requiring axial force-moment interaction for advanced analysis or direct analysis. A cross-section analysis technique that provides interaction curves of arbitrary welded sections with consideration of the effects of residual stress by meshing the entire section into small triangular fibers is formulated. In this study, two doubly symmetric sections (box-section and H-section) fabricated by high-strength steel is utilized to validate the accuracy and efficiency of the proposed method against a hand-calculation procedure. The effects of residual stress are mostly not considered explicitly in previous works and they are considered in an explicit manner in this paper which further discusses the basis of the yield surface theory for design of structures made of high strength steels.

Keywords: cross-sectional analysis; high-strength steel; residual stress

1. Introduction

Research on sectional analysis has been conducted extensively since 1960s. The information is essential in design of framed and buildings as the safety of a beam-column is checked by locating the moment-axial force coordinate inside or outside the yield or plastic surfaces. In the pre-computer age, numerous proposed equations and design graphs for hand calculation were derived for the specific cross-sections. However, for members under the combined action of axial

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loads and biaxial bending, these equations consisted of the over-simplified iterative relationships between two flexural axes may bring about uneconomical or unsafe design.

Santathadaporn and Chen (1968) discussed the interaction curves of steel sections involving rectangle and wide-flange I-section under biaxial bending and axial load. Based on the equilibrium method, they proposed design equations to compute the upper and lower limits of moment capacity. Chen and Atsuta (1972) studied the interaction relationships for biaxially loaded sections, and proposed simple analytical equations. These sections include a wide-flange section, a box-section and a circular section. However, the effect of residual stress was not considered when generating the interaction curves and the neutral axis of sections also needs to be determined in advance. Yen (1991) presented a quasi-Newton iterative approach applicable in computer application for analysis and design of reinforced-concrete cross-sections subjected to uniaxial or biaxial forces. Yau *et al*. (1993) proposed a numerical method for analyzing arbitrarily shaped reinforced-concrete cross-sections under combined axial load and biaxial bending. In this research, an iterative technique is utilized to determine the location of neutral axis. Thus, this method is also suited to irregular built-up sections. Chiorean (2010) presented an incremental-iterative procedure for determining interaction diagrams and moment capacity contours of composite steel and concrete cross sections. A tangent stiffness strategy was utilized to solve the nonlinear equilibrium equations. Recently, Liu *et al*. (2012a) proposed a rigorous cross section analysis approach for arbitrarily-shaped composite steel-concrete cross-sections.

With the development of metallurgical technology, high-strength steels are produced and it is required to investigate the interaction relations of sections made of high-strength steel. In this study, two types of interaction curves, i.e., initial yield and plastic failure surfaces, are constructed. As two fundamental criteria of strength control, being the failure surface for full plastic limits and initial yield surface for the elastic boundary conditions, are formulated for elasto-plastic analysis and design of structures, i.e., the spring modeling the plastic hinge starts to deteriorate from infinite spring stiffness upon reaching the initial yield surface and becomes frictionless without any spring stiffness at fully plastic stage. A quasi-Newton is utilized to determine the location of neutral axis through the orientation angle and depth of the axis, and generate the failure surface of arbitrary welded sections with consideration of residual stresses. The cross-section of steel members is meshed into small triangular fibers with the force and moment resultants of section obtained by stress integration. All fibers of section are monitored and the outset of the strain exceeding the elastic limit is checked when generating the initial yield surface.

At the end of this paper, two numerical examples are studied and discussed to verify the accuracy and efficiency of the proposed sectional analysis method by comparison with a hand calculation method. Based on the yield surface theory, the effects of residual stress are further presented. The concept of separated consideration of geometrical imperfections and residual stresses is elaborated and demonstrated to be more flexible in dealing with cross sectional analysis of steel members.

2. Sectional analysis technique allowing for residual stresses

The method for sectional analysis is suitable for analysis and design of arbitrary steel sections considering the effect of residual stress of which the models presented by Wang *et al*. (2012a, b) are employed to quantify the influence of residual stress on the capacity of welded box-section and H-section.

2.1 Residual stress models

Wang *et al*. (2012a, b) present a straight-line model for residual stress distribution shown in Fig. 1. It is suitable for width-to-thickness ratio ranging from 8 to 18 for box-section and from 3 to 7 for H-section. The residual stress distributions and magnitude can be written as,

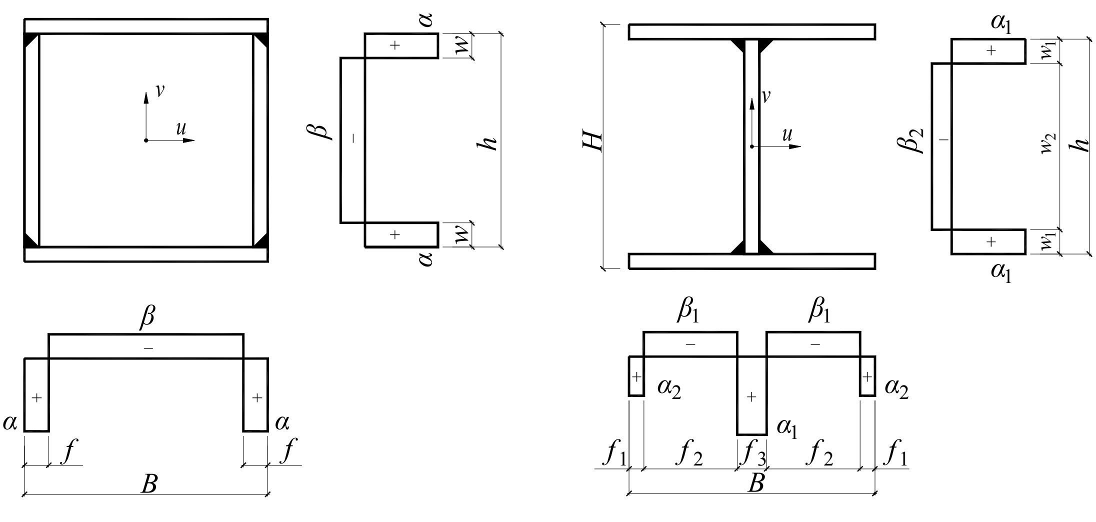
For box-section flange

$$\sigma_{fr}(u)=\begin{cases}\sigma_{rt}=\alpha\cdot f_y, & -\dfrac{B}{2}\le u<-\dfrac{B}{2}+f\\ \sigma_{rc}=\beta\cdot f_y, & -\dfrac{B}{2}+f\le u\le\dfrac{B}{2}-f\\ \sigma_{rt}=\alpha\cdot f_y, & \dfrac{B}{2}-f<u\le\dfrac{B}{2}\end{cases}\tag{1}$$

and for web

$$\sigma_{wr}(v)=\begin{cases}\sigma_{rt}=\alpha\cdot f_y, & -\dfrac{B}{2}+t\le v<-\dfrac{B}{2}+t+w\\ \sigma_{rc}=\beta\cdot f_y, & -\dfrac{B}{2}+t+w\le v\le\dfrac{B}{2}-t-w\\ \sigma_{rt}=\alpha\cdot f_y, & \dfrac{B}{2}-t-w<v\le\dfrac{B}{2}-t\end{cases}\tag{2}$$

where α and β listed in Table 1 are the residual stress ratios on tensile area and compressive area, respectively; f_y is the yield strength of materials.



(a) Pattern for box-section (b) Pattern for H-section

Fig. 1 Residual-stress model

Table 1 Residual stress ratios of box-section specimens

Width-to-thickness ratio	α	β
8	0.555	-0.255
12	0.678	-0.195
18	0.787	-0.142

$$f = w - 0.35t \tag{3}$$

$$w = \frac{\beta(B+h)}{2(\beta-\alpha)} - f \tag{4}$$

From Eqs. (3)-(4), the width of the tensile area on web and flange, w and f, can be determined as

$$w = \frac{\beta(B+h)}{4(\beta-\alpha)} + 0.175t \tag{5}$$

$$f = \frac{\beta(B+h)}{4(\beta-\alpha)} - 0.175t \tag{6}$$

For H-section flange

$$\sigma_{fr}(u) = \begin{cases} \sigma_{frte} = \alpha_2 \cdot f_y, & -\frac{B}{2} \le u < -\frac{B}{2} + f_1 \\ \sigma_{frc} = \beta_1 \cdot f_y, & -\frac{B}{2} + f_1 \le u < -\frac{f_3}{2} \\ \sigma_{frt} = \alpha_1 \cdot f_y, & -\frac{f_3}{2} \le u \le \frac{f_3}{2} \\ \sigma_{frc} = \beta_1 \cdot f_y, & \frac{f_3}{2} < u \le \frac{f_3}{2} + f_2 \\ \sigma_{frte} = \alpha_2 \cdot f_y, & \frac{f_3}{2} + f_2 < u \le \frac{B}{2} \end{cases} \tag{7}$$

and for web

$$\sigma_{wr}(v) = \begin{cases} \sigma_{wrt} = \alpha_1 \cdot f_y, & -w_1 - \frac{w_2}{2} \le v < -\frac{w_2}{2} \\ \sigma_{wrc} = \beta_2 \cdot f_y, & -\frac{w_2}{2} \le v \le \frac{w_2}{2} \\ \sigma_{wrt} = \alpha_1 \cdot f_y, & \frac{w_2}{2} < v \le \frac{w_2}{2} + w_1 \end{cases} \tag{8}$$

Table 2 Residual stress ratios of H-section specimens

Width-to-thickness ratio	α_1	α_2	β_1	β_2
3	1.039	0.080	-0.408	-0.152
5	0.900	0.243	-0.271	-0.235
7	0.731	0.488	-0195	-0.131

where α_1, α_2 and β_1 listed in Table 2 are the residual stress ratio of major tension zone in flange and web, marginal tension zone and compression zone on flange; β_2 is the residual stress ratio of compression zone in web; f_y is the yield strength of materials. The width of each tension and compression zone on flange and web are given by

$$w_1 = 0.12h; \quad w_2 = 0.76h; \quad f_1 = 0.05B \tag{9}$$

$$f_3 = \frac{\beta_1(2f_1 - B) + \beta_2\left(\dfrac{h}{2} - w_1\right) - \alpha_1 w_1 - 2\alpha_2 f_1}{\alpha_1 - \beta_1} \tag{10}$$

$$f_2 = \frac{\beta - 2f_1 - f_3}{2} \tag{11}$$

2.2 Loading axes

It can be seen from Fig. 2 that rotation angle θ_n and the depth d_n of the neutral axis can be used to determine the position of neutral axis. Brondum-Nielsen (1985) and Yen (1991) discussed the applicability of the quasi-Newton method for determining θ_n and d_n of regular section. For steel cross-sections consisting of numerous steel sheets, the origin of reference loading axes chooses the geometric centroid as

$$Z_{gc} = \frac{\sum_{i=1}^{n} A_i Z_i}{\sum_{i=1}^{n} A_i} \tag{12}$$

$$Y_{gc} = \frac{\sum_{i=1}^{n} A_i Y_i}{\sum_{i=1}^{n} A_i} \tag{13}$$

in which A_i is the area of the i-th plate strips; Z_i and Y_i are the centroidal coordinates of the i-th plate strips; Z_{gc} and Y_{gc} are the coordinates of the geometrical centroid of a whole section.

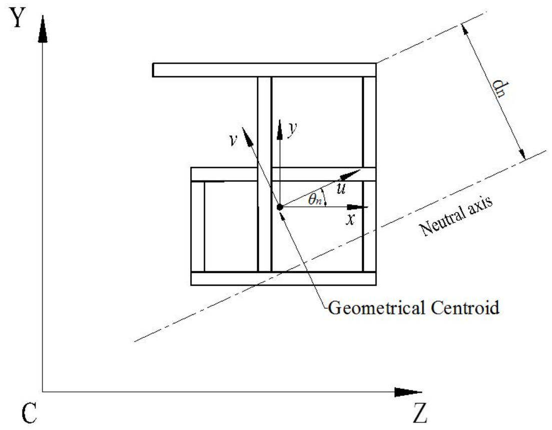


Fig. 2 Arbitrarily welded section

2.3 Co-ordinate systems

Three co-ordinate systems, including ZCY, zoy and uov, are utilized during analysis and the cross-section is also referred to the ZCY system. There are two transformations of co-ordinates in the whole iterative process. The global ZCY system is transformed to the reference loading zoy system and the transmutation of the zoy system is to the uov system, in which the u-axis is parallel to the neutral axis.

The conversion equations with respect to the three coordinate systems are given by

$$z = Z - Z_{gc} \tag{14}$$

$$y = Y - Y_{gc} \tag{15}$$

$$u = z\cos\theta_n + y\sin\theta_n \tag{16}$$

$$v = y\cos\theta_n - z\sin\theta_n \tag{17}$$

in which z and y, Z and Y and u and v are the coordinates in zoy, ZOY and uov systems, respectively

2.4 Stress expressions in local and global systems

The whole section is meshed made of triangular fibers and its stress outcomes can be written as

$$N_x = \sum_{i=1}^{n} \sigma_i A_i \tag{18}$$

$$M_u = \sum_{i=1}^{n} \sigma_i A_i v_i \tag{19}$$

$$M_v = \sum_{i=1}^{n} \sigma_i A_i u_i \tag{20}$$

where A_i is the area of individual fibers; σ_i is the related stresses considering residual stresses. The stress σ_i imposed on the i-th element is a function of the strain ε_i involving three parts as

$$\sigma_i = f(\varepsilon_i) \tag{21}$$

$$\varepsilon_i = \varepsilon_{ai} + \varepsilon_{ri} + \phi_i v_i \tag{22}$$

in which ε_{ai} is the axial strain; ε_{ri} is the residual strain; φ_i is the curvature.

The bending moments obtained from Eqs. (19)-(20) need to be summed and converted to the xoy system through the following transmutations. The total moments on the cross-section are computed as

$$M_z = M_u \cos\theta_n - M_v \sin\theta_n = -\left(\sum_{i=1}^{n} \sigma_i A_i v_i \cdot \cos\theta_n + \sum_{i=1}^{n} \sigma_i A_i u_i \cdot \sin\theta_n \right) \tag{23}$$

$$M_y = M_u \sin\theta_n + M_v \cos\theta_n = -\sum_{i=1}^{n} \sigma_i A_i v_i \cdot \sin\theta_n + \sum_{i=1}^{n} \sigma_i A_i u_i \cdot \cos\theta_n \tag{24}$$

$$N_x = \sum_{i=1}^{n} \sigma_i A_i \tag{25}$$

in which M_u, M_v and N_x are the bending moments and axial force with respect to uov system; M_z and M_y are the global moments referred to the geometrical centroid.

2.5 Iterative strategy

In order to accurately compute section capacity, the determination of depth d_n of neutral axis can be carried out by rotating orientation θ_n from 0° to 360°. The application of the Regula-Falsi method is then used to derive equilibrium, compatibility and constitutive relationships. Fig. 3 illuminates the procedure of such a cross-sectional analysis.

The iteration for axial force capacity N_x in regard to d_n is conducted by the following equation with the constant θ_n.

$$d_{nk} = d_{ns} + \frac{d_{ng} - d_{ns}}{N_{xg} - N_{xs}} \left(N_{xd} - N_{xs} \right) \tag{26}$$

in which d_{nk} is the depth of the updated neutral axis; N_{xs} and N_{xg} are the axial force capacity computed at d_{ns} and d_{ng}, and N_{xd} is the present design axial load; d_{ns} and d_{ng} are the depths of neutral axis corresponding to the axial capacity being smaller and greater than the design axial load.

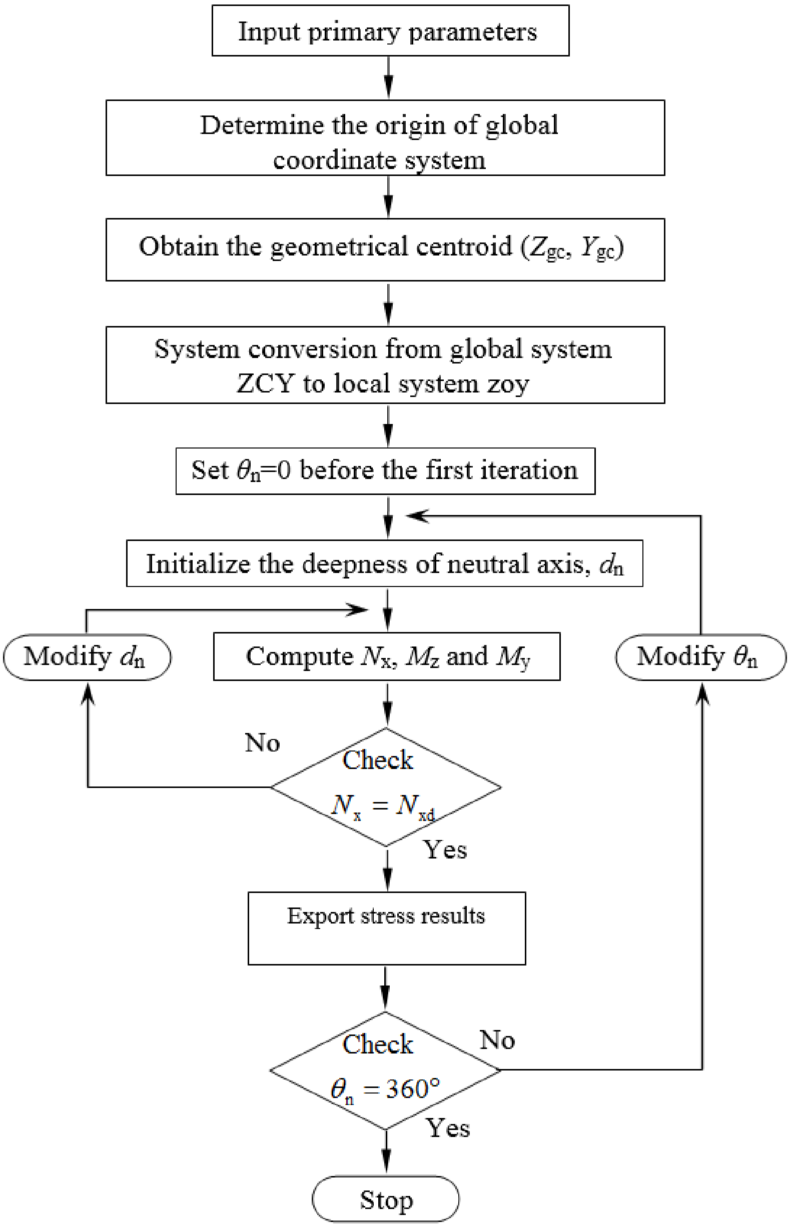


Fig. 3 Flowchart for sectional analysis under given axial load

2.6 Constitutive relation

The constitutive relations of steels in the present study is assumed to be elastic-perfectly-plastic (see Fig. 4). Mathematically they are written as

$$\sigma = f_y, \qquad \varepsilon_e \le \varepsilon \le \varepsilon_u \tag{27}$$

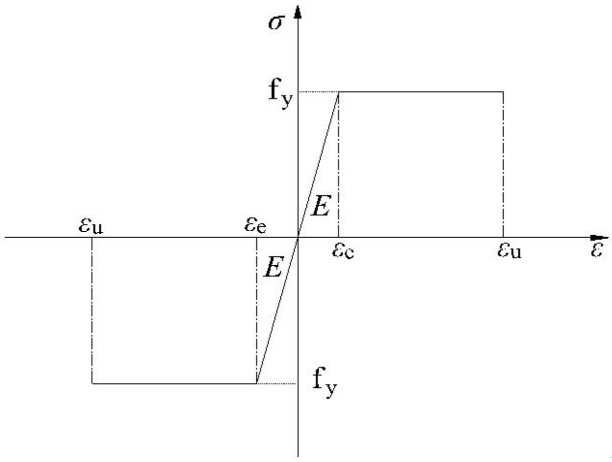


Fig. 4 Constitutive relations of steel

$$\sigma = E \cdot \varepsilon, \qquad -\varepsilon_e \le \varepsilon \le \varepsilon_e \tag{28}$$

$$\sigma = -f_y, \qquad -\varepsilon_u \le \varepsilon \le -\varepsilon_e \tag{29}$$

where, f_y is the yield strength of steels; E is the Young's modulus of steels; ε_e and ε_u are the strain limits for elasticity and plasticity, respectively.

2.7 Strength interaction surfaces

For any steel section under a given axial load, the corresponding moments can be computed by the presented sectional analysis method. The strength interaction surfaces, adopted to describe the axial force and related moments of the section, is plotted below (see Fig. 5).

2.7.1 Failure surface

The outermost boundary in Fig. 5 is a failure surface. The ultimate limit state of a section is described by these data points that constitute the failure surface of steel section. The surface is necessary in determining the individual load bearing capacity in traditional design of steel structures.

Many codes and specifications provide fundamental equations based upon typical load conditions. The failure surface obtained from a uniaxial bending case is a special plane that crosses the strength interaction surface with the moment about an axis kept zero and, in many scenarios, its use is inadequate. In order to improve computational efficiency, indexing method proposed by Liu *et al*. (2012b) is applied in this study.

2.7.2 Initial yield surface

Initial yield surface is also a basic strength control criterion for plastic analysis, as shown in Fig. 5. Within this surface generated from a specific load combination, the corresponding section maintains elastic. The elastic limit ε_e may bring about slight violation for tracing the load-displacement path in the gradual yield range due to the finite load increment that could not exactly

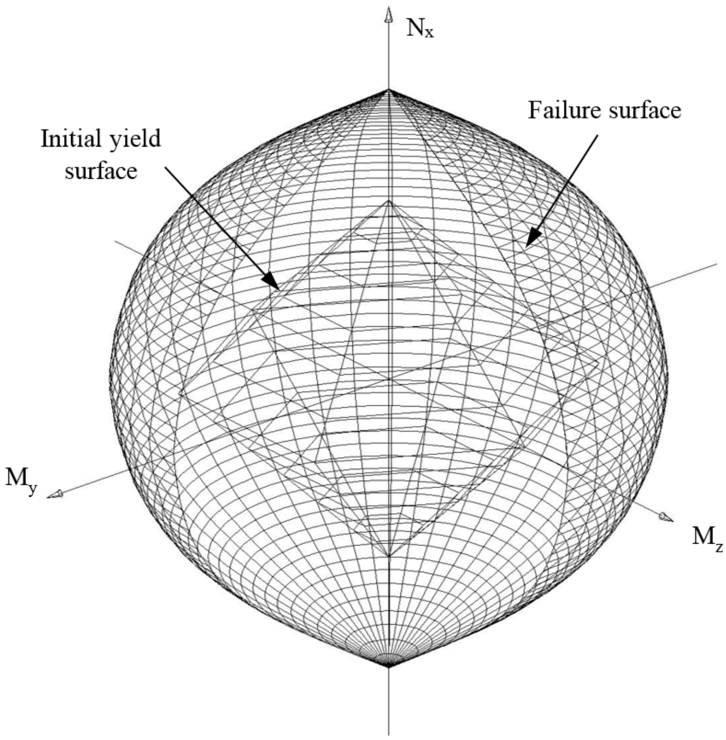
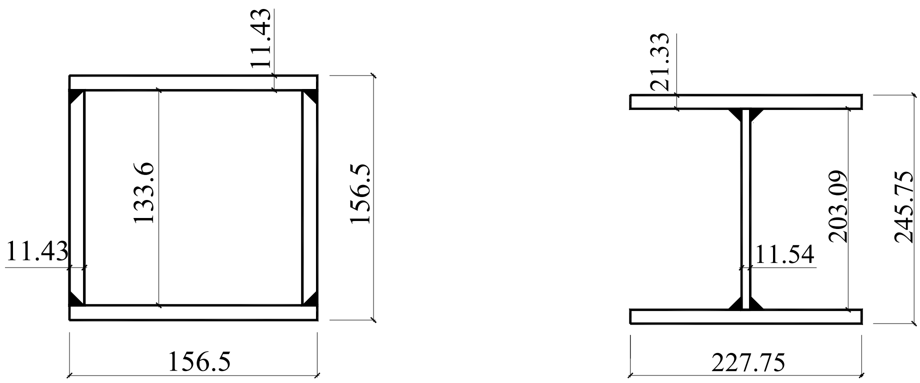


Fig. 5 Envelop for initial yield and failure surfaces of steel section

fit onto the yield surface.

When generating initial yield surfaces, all fibers of a section will be monitored and the onset of strains beyond the elastic limit is detected. The internal force and moments are calculated through the integration of stresses along the cross-section. Subsequently, sectional capacity can be obtained from the initial yield condition.



(a) Box-section

(b) H-section

Fig. 6 Geometrical dimensions

3. Verification examples

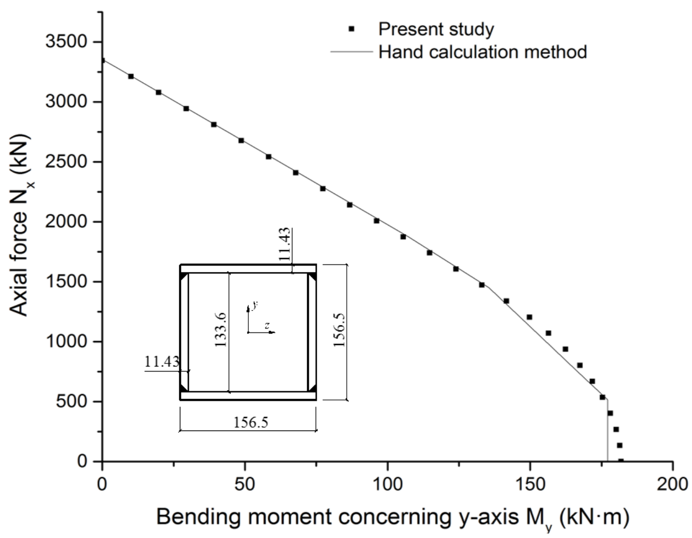
Two examples are selected to validate the accuracy and efficiency of the section analysis method. The welded box-section and H-section in these examples are analyzed under uniaxial loads. The numerical results allowing and ignoring the effects of residual stress will be compared with a simplified hand calculation method.

3.1 Box-section

The yield strength of box-section steel is 505.8 MPa, and the geometric dimensions are shown in Fig. 6(a). The residual stress model presented by Wang *et al*. (2012a) is utilized in the current study. A quarter of initial yield and failure surfaces will be considered because of double symmetry of the box-section.

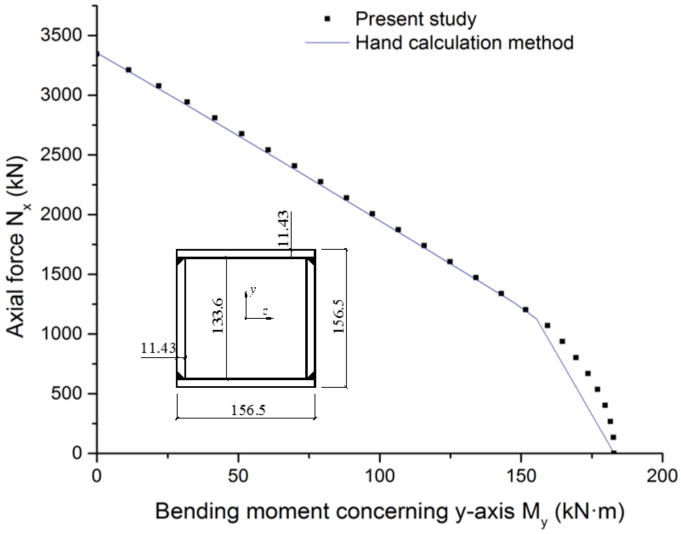
Figs. 7(a)-(b) indicate the accuracy and efficiency of the proposed method of section analysis. In these figures, solid lines represent the results by hand calculation method and the dotted lines are obtained from the present study. The failure surfaces are plotted in Figs. 7(a)-(b) in terms of box-section with and without consideration of residual stress. There are slight discrepancies in the results by the two methods due to the assumption and simplification in the hand calculation.

For box-sections, residual stress has an obvious effect on the initial yield surface whereas the effect is significant in terms of failure surfaces (see Fig. 7(c)). This is because this stress contributes to the development of yielding on a section, which affects the initial yielding state more than the ultimate plastic state of sections. It can be seen from Fig. 7(c) that the stress reduces the capacity of the box-section when under bending about y-axis. When the sectional capacity is controlled by bending moment, the zone of residual tensile stress on upper flange and compressive stress zone on lower flange have a remarkable effect on the reduction of section moment (see

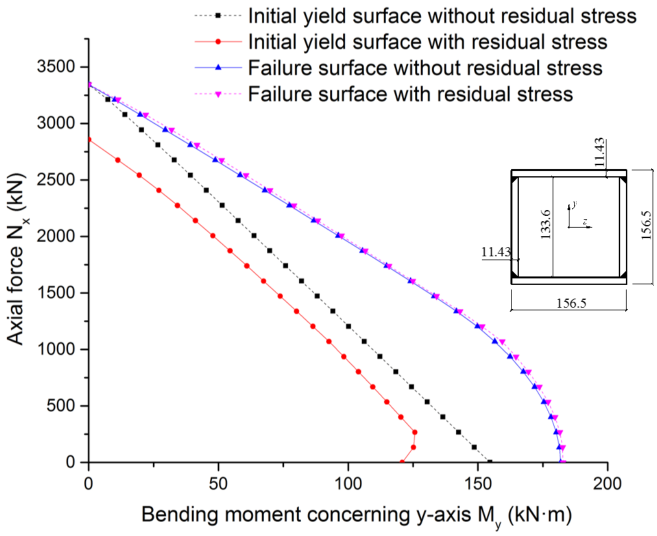


(a) Failure surface without residual stress

Fig. 7 Comparison results of box-section



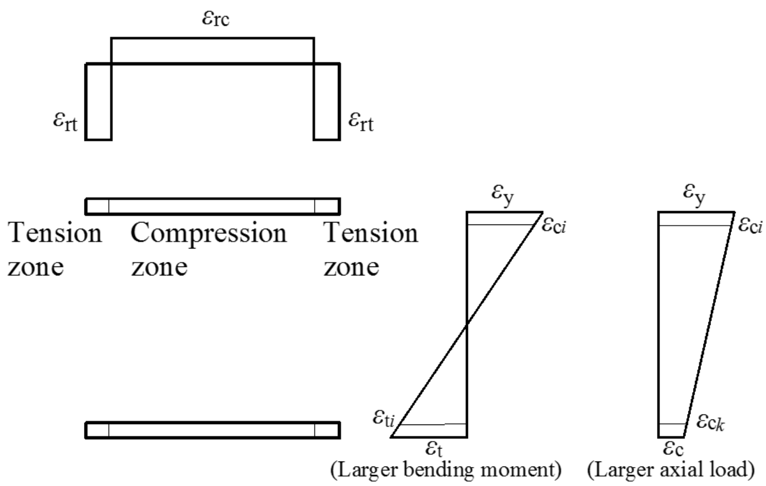
(b) Failure surface with residual stress



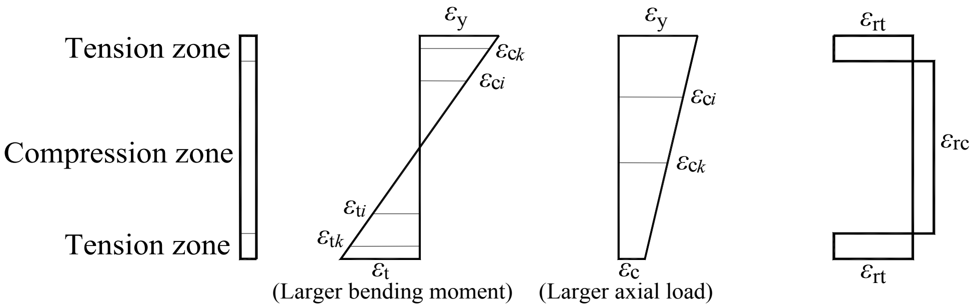
(c) Residual stress effects on initial yield and failure surfaces

Fig. 7 Continued

Figs. 8(a)-(b)); If the axial load controls the sectional capacity, the tensile zone on upper flange and web zone greatly reduces the sectional moment. It indicates that the early yielding of box-section will occur due to the existence of residual stress. Normally, initial yield surfaces are symmetrical about the *y*- and *z*-axes. The surface in Fig. 7 is not ideally symmetrical about the horizontal axis because the adopted model of residual stress is not ideally and perfectly self-equilibrating due to numerical truncation error which, however, is insignificant in practical design.



(a) Residual stress influence on flanges



(b) Residual stress influence on a web

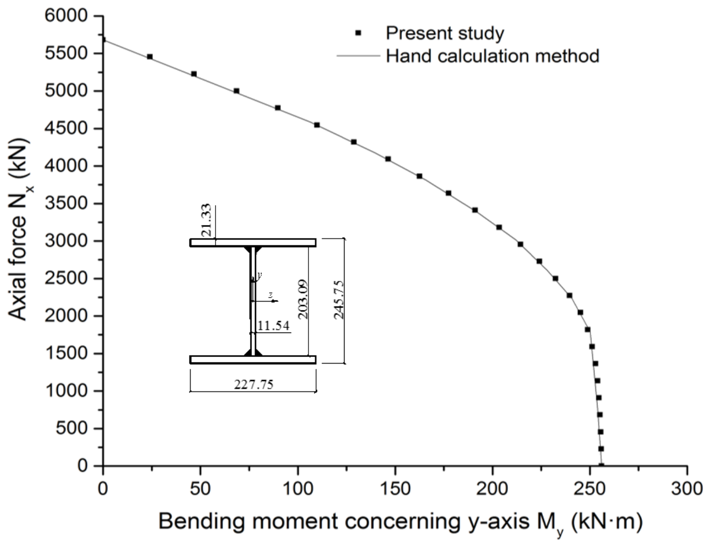
Fig. 8 Residual stress influence on section

3.2 H-section

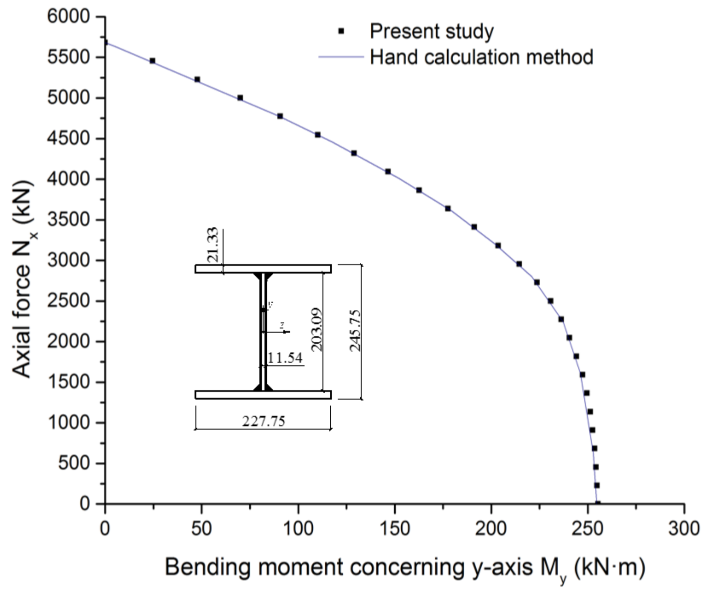
The thicknesses of flange and web of the H-section adopted in the analysis are different. The yield strength of flange and web are 464 MPa and 505.8 MPa respectively. The geometric dimensions are shown in Fig. 6(b). The residual stress model by Wang *et al*. (2012b) is employed in the current study. Only a quarter of initial yield and failure surfaces are plotted because of the doubly symmetrical property of H-section.

As shown in Figs. 9(a)-(b), the results validate the accuracy of the sectional analysis method. The failure surfaces are drawn in Figs. 9(a)-(b) in terms of H-section with and without residual stress. Some slight differences appear in the results between the two methods due to the finite element size in manual calculation.

For initial yield surfaces, when the H-section is bent about minor y-axis as shown in Fig. 9(c), the residual stress has a beneficial effect on the capacity of the H-section subjected to larger axial load while it reduces the capacity of this section under large bending moment. Under the condition of dominant axial load, the zones of residual compressive stress on lower flanges decreases considerably the section moment capacity, as illustrated in Figs. 8(a)-(b). Under the scenario controlled by bending moment, residual stress at the tensile zones on the bottom of flanges greatly increases the sectional moment. For failure surfaces, it does not have a noticeable effect. These findings suggest that the residual stress has an advantage and disadvantage on the section capacity in different situation for H-section when bent about the minor axis. Based on the yield surface theory, it verifies the fact that residual stress in the tensile zone is beneficial to the capacity of H-section when bent under minor axis.

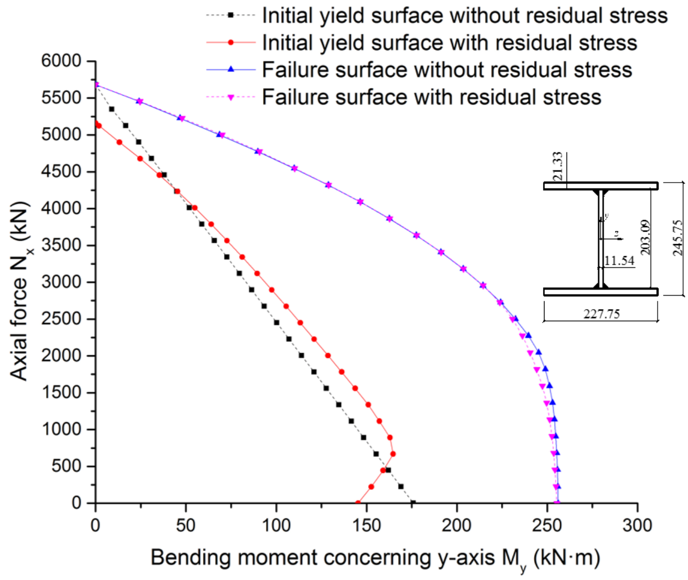


(a) Failure surface without residual stress



(b) Failure surface with residual stress

Fig. 9 Comparison results of H-section



(c) Residual stress effects on initial yield and failure surfaces

Fig. 9 Comparison results of H-section

4. Conclusions

Construction of initial yield and full plastic failure surfaces is indispensable in advanced or direct analysis for first yield and plastic moment capacities of sections so their research is crucial in the advanced design method. In general, the section strength commonly modeled by spring-stiffness begins to deteriorate when yielding is attained and attains its full capacity when plastic moment is reached. The consideration of residual stresses by constructing interaction curves could avoid the scenario that the effects of residual stress are over or under-estimated in practical design owing to the simplified code consideration for residual stress. From the examples studied in this paper, the proposed method for sectional analysis is suitable for analysis of welded box-section and H-section under axial and uniaxial bending. For box-sections, the residual stress reduces the elastic capacity of section but it enhances the plastic capacity of H-section under dominant bending moment. In contrast, it reduces the section capacity when axial load is dominant. However, as seen in Figs. 7(c) and 9(c), the failure surfaces for full yield or plastic capacities of cross-section are unaffected. This information may be less important for plastic analysis, but it is required for an elastic analysis which may be required in some limit states such as analysis under unfactored working load.

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