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Elastic solution of a curved beam made of functionally graded materials with different cross sections

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Abstract. This research deals with the analytical solution of a curved beam with different shapes made of functionally graded materials (FGM's). It was assumed that modulus of elasticity is graded along the thickness direction of curved beam based on a power function. The beam was loaded under pure bending. Using the linear theory of elasticity, the general relation for radial distribution of radial and circumferential stresses of arbitrary cross section was derived. The effect of nonhomogeneity was considered on the radial distribution of circumferential stress. This behavior can be investigated for positive and negative values of nonhomogeneity index. The novelty of this study is application of the obtained results for different combination of material properties and cross sections. Achieved results indicate that employing different nonhomogeneity index and selection of various types of cross sections (rectangular, triangular or circular) can control the distribution of radial and circumferential stresses as designer want and propose new solutions by these options. Increasing the nonhomogeneity index for positive or negative values of nonhomogeneity index and for various cross sections presents different behaviors along the thickness direction. In order to validate the present research, the results of this research can be compared with previous result for reachable cross sections and non homogeneity index.

Keywords: curved beam; functionally graded beams (FGBs); nonhomogeneity index; stress; bending

1. Introduction

One of the most applicable structures in the scope of mechanical engineering and analysis of the structures are beams. There are two main theories to analyze the beam under specific loads. Euler Bernoulli and Timoshenko theories are two mentioned. Euler-Bernoulli theory considers the bending deformation and does not consider shear deformation. Timoshenko beam theory is the other theory that can be applied for wide beams and those classes of the beams that shear deformation is important in those analyses. The mentioned theories mostly applied for prismatic and straight beam with no curvature. A complete review on the analysis of the beam can be performed in this stage.

Dryden (2007) presented stress analysis of a circular beam subjected to pure bending. A slight generalization for the form of the elastic stiffness was used by the author. The obtained

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approximate results have been compared with exact results. Ying *et al*. (2008) investigated on the bending and free vibration of (FGBs) resting on a Winkler-Pasternak elastic foundation. They used two-dimensional theory of elasticity for description of deformations. Exponentially function has been used for variation of material properties along the thickness direction.

Hu *et al*. (2008) developed a nonlinear mathematical model for large deformation analysis of beams with discontinuity conditions and initial displacements. The differential quadrature method (DQM) was applied to discretize the nonlinear mathematical model. A new beam theory has been developed by Sina *et al*. (2008) in order to analyze free vibration of FGBs. The beam properties are assumed to be varied through the thickness following a simple power law distribution in terms of volume fraction of material constituents. Hamilton's principle has been used for derivation of governing equations of motion. The effects of boundary conditions, volume fraction and shear deformation on natural frequencies and mode shapes were investigated. Xiang and Yang (2008) investigated on the free and forced vibration of laminated FGBs of variable thickness under thermal load. The beam was made of a homogeneous substrate and two non-homogeneous FG layers. A two dimensional analysis has been used and therefore, both the axial and rotary inertia of the beam were considered in that analysis. Vibrations of axially moving flexible beams made of FGMs was studied by Piovan and Sampaio (2008). The used model was a thin-walled beam with annular cross-section. A finite element scheme was employed to obtain numerical approximations to the variational equation of the problem.

Mohammadi and Dryden (2008) developed thermo elastic analysis of a functionally graded beam that graded along the radial direction. They used a fairy general from of functionality for variation of stiffness. Li (2008) presented a new approach for analyzing the static and dynamic behaviors of FGBs with the rotary inertia and shear deformation included. All material properties were arbitrary functions along the beam thickness. A single fourth-order governing partial differential equation was derived and all physical quantities were expressed in terms of the solution of the resulting equation. As a case study, the Euler–Bernoulli and Rayleigh beam theories have been derived by reducing the Timoshenko beam theory. Benatta *et al*. (2008) used high-order flexural theories for short FG symmetric beams under three-point bending. The governing equations were obtained using the principle of virtual work (PVW). Kadoli *et al.* (2008) presented static behavior of functionally graded metal-ceramic beams under ambient temperature by using a higher order shear deformation theory. Using the principle of stationary potential energy, the finite element form of static equilibrium equation for FGM beam was presented. The effect of power law exponent for various combination of metal-ceramic FGBs on the deflection and stresses were investigated. A rectangular and simply supported FGBs with thick thickness under transverse loading has been investigated by Ben-Oumrane *et al*. (2009). First order and higher order shear deformation theory have been used as two methods of derivation. They assumed that Young's modulus vary continuously throughout the thickness direction according to the volume fraction of constituents. Pradhan and Murmu (2009) presented thermo-mechanical vibration analysis of functionally graded beams (FGB's) and functionally graded sandwich beams (FGSW). Both beams have been considered to be resting on two various foundations. Functionalities have been considered along the thickness direction. The effect of different parameters such as temperature distribution, power-law index and parameters of foundation has been considered on the vibration characteristics of the beam. The out-of-plane free vibration analysis of thin and thick FG circular curved beams on two-parameter elastic foundation was presented by Malekzadeh *et al*. (2010). They used first-order shear deformation theory (FSDT) in order to account the effects of shear deformation and rotary inertia due to both torsional and flexural vibrations. The material properties were assumed to be graded in the radial direction of the beam curvature. Filipich and Piovan (2010) presented a technical theory for dynamic analysis of thick curved beams made of functionally graded materials. The concept of material neutral-axis shifting was employed in the deduction procedure in order to reduce the algebraic handling and complexity of the motion equations. Filipich *et al*. (2011) developed a general model for transient dynamic analysis of FGM thick arch/ring. Timoshenko and other first-order shear deformation theories have been employed for analysis of the beam structure. The material properties such as Young's modulus, shear modulus, density were considered arbitrarily along the generally-shaped cross-section.

The free vibration analysis of FG curved beams was presented by Yousefi and Rastgoo (2011). Natural frequencies have been derived using first-order shear deformation theory (FSDT) and Ritz method. Wang and Liu (2013) presented elasticity solution for curved beams with *n* orthotropic FG layers by means of the Airy stress function method. The beams subjected to a uniform load on the outer surface.

Yaghoobi and Torabi (2013) presented non-linear vibration and post-buckling analysis of beams made of FGMs rest on a non-linear elastic foundation subjected to an axial force. Based on Euler-Bernoulli beam theory and von-Karman geometric non-linearity, the partial differential equation of motion is derived. Arefi and Rahimi (2013) presented nonlinear analysis of a FG beam. The beam has been subjected to uniform loading. Functionality has been considered along the longitudinal direction. Shyma and Rajendran (2014) developed a mathematical model for evaluation of deflection of beam made of FGM. They used the principle of minimum potential energy. Functionality has been considered along the thickness direction. Lee *et al*. (2014) presented vibration analysis of a horizontally curved beam. The cross section of each beam was considered a solid regular polygon whose depth is varied in a functional fashion. Three shapes of beam were considered. Arefi (2014) developed nonlinear analysis of a FG beam resting on a nonlinear foundation.

A comprehensive investigation on the literature indicates that most recent studies focused on different analysis of straight FGBs. The effect of curvature of the beam and various cross sections has been ignored in those analyses. The present paper considers the effect of both curvature and various cross sections on the stress analysis of a FGB under applied loads. This study tries to propose the various distributions of stress in the beam in terms of variation of non homogeneous index and employing the different cross sections.

2. Formulation

This section presents the fundamental equations based on the linear theory of elasticity for a curved beam made of FGM. The polar coordinate is used for derivation of equations. As mentioned previously, the beam is loaded under pure bending. Under this loading, the distribution of circumferential and radial stress must be derived. Due to symmetry, it is assumed that all components such as radial and circumferential stress are only radially distributed.

Fig. 1 shows a curved beam under bending loadings. By considering an element in the radial coordinate system and applying the equilibrium equations for that, we have resultant of force and moment as follows

$$N = \int \sigma_{\theta\theta} dA, \quad M = \int (R - r)\sigma_{\theta\theta} dA \tag{1}$$

where, R is the curvature radius of mid-plane of beam and r is the radial location of element dA. In order to evaluate two integrals in Eq. (1), the functional relation between circumferential stress and radial component must be derived using linear theory of elasticity.

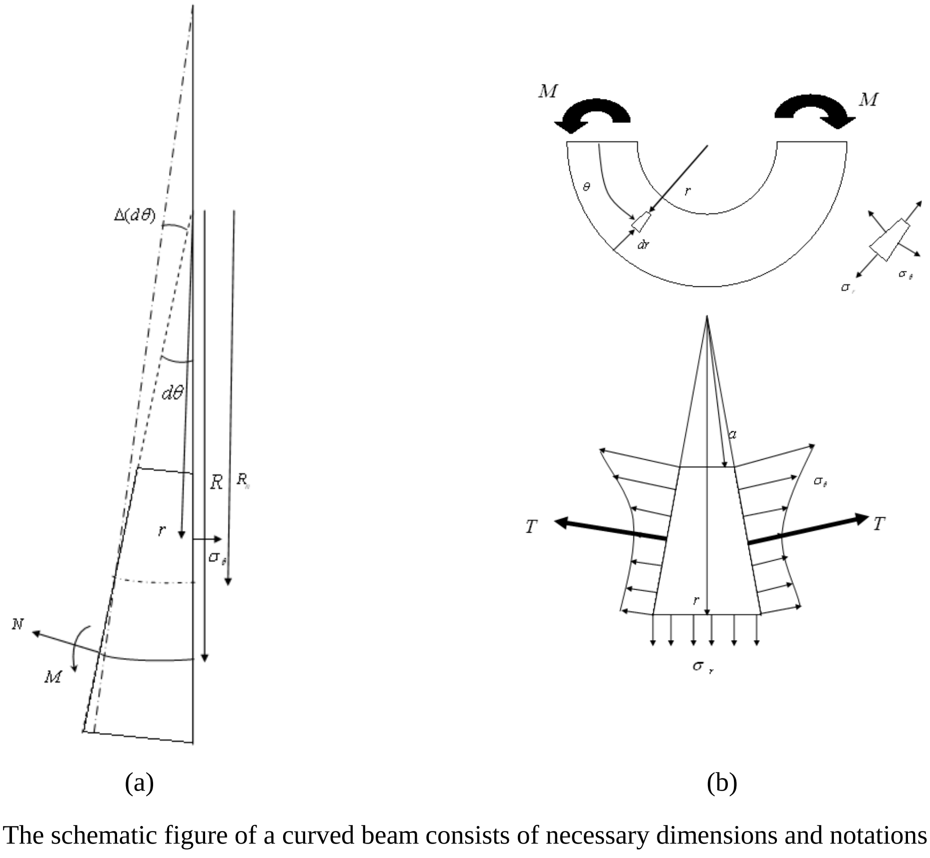
For evaluation of the stress distribution, deformation of assumed element before and after loading must be considered. Shown in Fig. 1 is original and deformed shapes of mentioned element. The deformed configuration indicates that the center of rotation has been changed. Due to curvature of beam, the beam after deformation rotates through a new line that locates with R_n. Using this figure and notation, we can determine the increasing or decreasing the length of a line located at location r. If rotation of line is defined with $\Delta(d\theta)$, the value of increasing length of element ($de_{\theta\theta}$) at location r is

$$de_{\theta\theta} = (R_n - r)\Delta(d\theta) \tag{2}$$

where, $d\theta$ is initial angle of used element. As depicted in Fig. 1, the value of strain is equal to ratio of circumferential change of an element with respect to initial length of that element as follows

$$\varepsilon_{\theta\theta} = \frac{de_{\theta\theta}}{rd\theta} = \frac{(R_n - r)\Delta(d\theta)}{rd\theta} = \left(\frac{R_n}{r} - 1\right)\frac{\Delta(d\theta)}{d\theta} = \left(\frac{R_n}{r} - 1\right)\omega \tag{3}$$

where, $\omega = \dfrac{\Delta(d\theta)}{d\theta}$. The obtained radial distribution for circumferential strain indicates that this



(a) (b)

Fig. 1 The schematic figure of a curved beam consists of necessary dimensions and notations distribution is not linear. This nonlinear distribution indicates the fundamental difference between analysis of straight and curved beam where the strain distribution for straight beam was linear. Using this definition and using Hooke's law for circumferential stress and strain, we have circumferential component of stress as follows

$$\sigma_{\theta\theta} = E\varepsilon_{\theta\theta} = \frac{E\omega R_n}{r} - E\omega \tag{4}$$

where, E is modulus of elasticity. The effect of radial stress has been neglected on the circumferential stress. For a material that graded along the radial direction, this property is variable in terms of radial component. Substitution of circumferential stress from Eq. (4) into Eq. (1) gives

$$\begin{aligned} N &= \int \left(\frac{E\omega R_n}{r} - E\omega \right) dA = \omega R_n \int \frac{E}{r} dA - \omega \int E dA \\ M &= \int (R-r) \left(\frac{E\omega R_n}{r} - E\omega \right) dA = RR_n\omega \int \frac{E}{r} dA - \omega R \int E dA - \omega R_n \int E dA + \omega \int rE dA \end{aligned} \tag{5}$$

where, N, M are resultant of force and moments. For a beam under pure bending, we have $N = 0$. This assumption gives the location of neutral axis as follows

$$N = \omega R_n \int \frac{E}{r} dA - \omega \int E dA = 0 \quad \rightarrow \quad R_n = \frac{\int E dA}{\int \frac{E}{r} dA} \tag{6}$$

Substituting R_n from above equation (Eq. (6)) into second equations of Eq. (5) presents the required moment for bending in terms of property and geometry of the structure.

$$M = \omega \left[\int rE dA - \frac{\left(\int E dA \right)^2}{\int \frac{E}{r} dA} \right] \tag{7}$$

Concurrently considering Eqs. (4), (7) and elimination of ω between them yield circumferential stress as follows

$$\sigma_{\theta\theta} = \frac{ME\left[\frac{R_n}{r} - 1 \right]}{\left[\int rE dA - \frac{\left(\int E dA \right)^2}{\int \frac{E}{r} dA} \right]} \tag{8}$$

Eq. (8) is the radial distribution of circumferential stress of a curved beam subjected to pure bending. Before derivation of this distribution, the location of center of rotation of plane R_n must be determined from Eq. (6).

The radial distribution of radial stress can be obtained by considering equilibrium of used element in Fig. 1. For an element from $r = a$ till r, we can write the equilibrium equation along the radial direction. If Γ be summation of circumferential stress form inner surface ($r = a$) till location r, the equilibrium equation along the radial direction gives

$$\sum F_r = 0 \quad \rightarrow \quad -2\Gamma \times \sin\left(\frac{d\theta}{2}\right) + \sigma_{rr} t r d\theta = 0 \quad \rightarrow \quad \sigma_{rr} = \frac{\Gamma}{tr} \tag{9}$$

where, Γ is summation of circumferential stress from inner surface ($r = a$) till location r presented as follows

$$\Gamma = \int_{r=a}^{r} \sigma_{\theta\theta} dA \tag{10}$$

where, t in Eq. (9) is depth of section at location r.

In order to evaluate the responses of the system, we have to define functionality of the used material.

$$E = E_0 \left(\frac{r}{a}\right)^n \tag{11}$$

where, E is distribution of modulus of elasticity, E_0 is modulus of elasticity at inner surface ($r = a$) and n is nonhomogeneity index.

3. Results and discussion

This section presents the important results of this research. The obtained results consist of distribution of radial and circumferential stresses along the thickness direction for different values of nonhomogeneity indexes. Furthermore, the obtained results can be presented for various cross sections such as rectangular, triangle and circular.

3.1 Circumferential stress

3.1.1 Rectangular section

The radial distribution of circumferential stress can be presented for various values of non homogeneous index and cross sections. Due to different responses of the system for negative and positive nonhomogeneity index, the obtained results are presented separately for positive and negative values of nonhomogeneity index. Shown in Fig. 2 is the radial distribution of circumferential stress along the thickness direction of a FGB with rectangular cross section for positive values of nonhomogeneity index. The same parameter can be presented in Fig.3 for negative values of nonhomogeneity index.

It observed that the value of circumferential stress at inner radius is positive and with increasing the radial coordinate, decreases continuously. This monotonically decreasing is reserve for different values of nonhomogeneity index lower than $n = 5$. For nonhomogeneity index greater than mentioned $n \geq 5$, the location of maximum positive values of circumferential stress moves to medium of thickness. Furthermore, it can be concluded for large values of nonhomogeneity index,

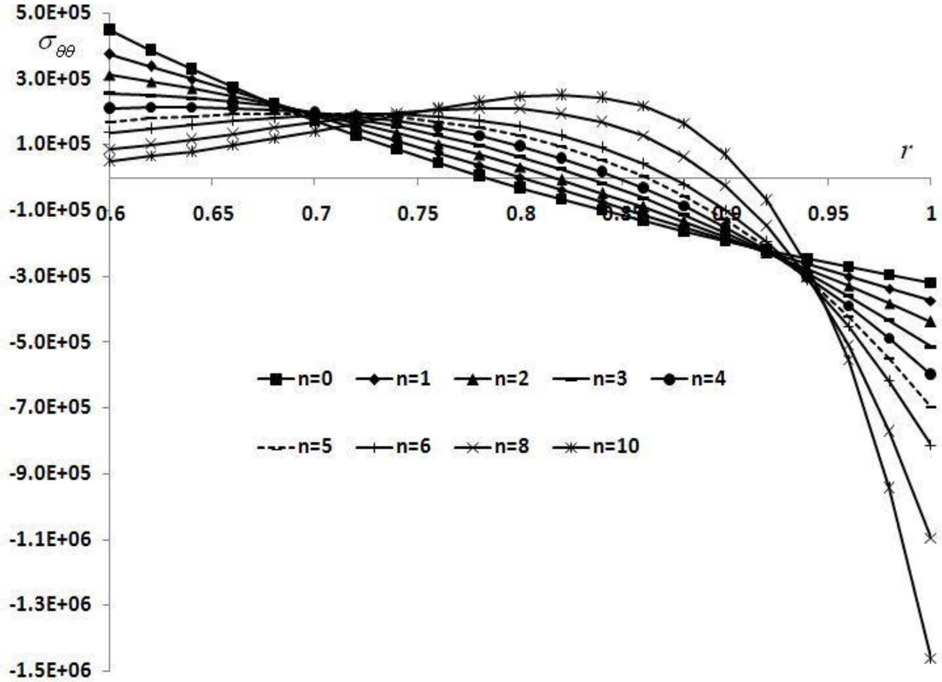


Fig. 2 The radial distribution of circumferential stress along the thickness direction of a rectangular FG beam for positive values of nonhomogeneity index

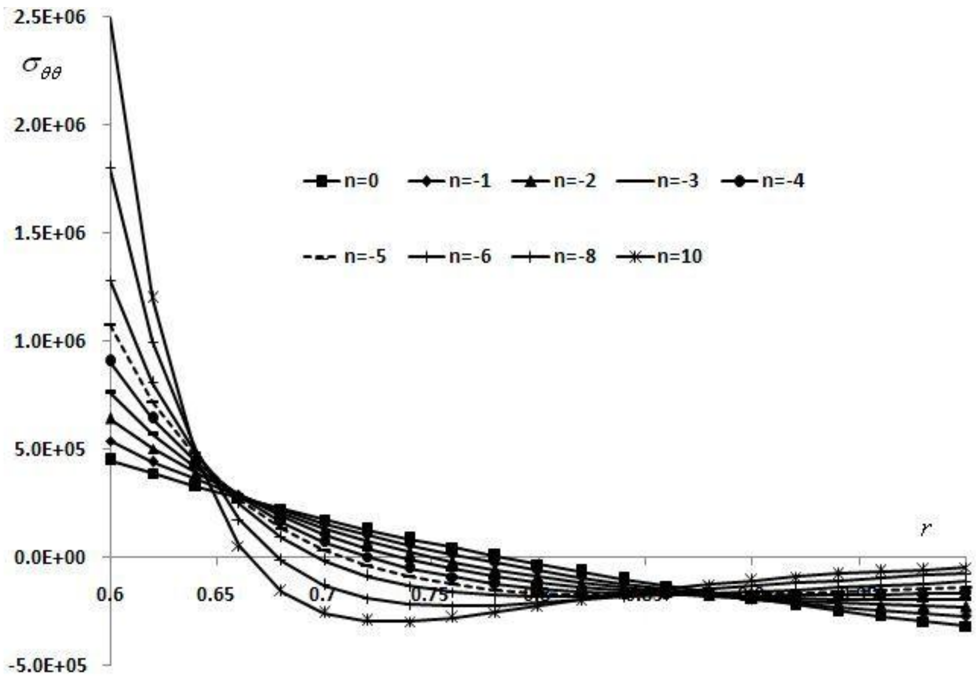


Fig. 3 The radial distribution of circumferential stress along the thickness direction of a rectangular FG beam for negative values of nonhomogeneity index

the maximum negative values of circumferential stress increases considerably at outer surface of the curved beam. The reason for these high values of circumferential stress is increasing the modulus of elasticity due to increasing the non homogeneity index.

The same trend can be observed in Fig. 3 for negative values of nonhomogeneity index except the location of maximum positive and negative values of circumferential stress. As depicted in Fig.3, the maximum value of circumferential stress is located at inner surface of the curved beam. This is due to high value of modulus of elasticity at inner surface.

Figs. 2 and 3 show that for $|n| \leq 5$ and both positive and negative values of nonhomogeneity index, there are two radii which the value of circumferential stress is identical for all values of nonhomogeneity index.

3.1.2 Triangle section

Shown in Fig. 4 is the radial distribution of circumferential stress along the thickness direction of a FG beam with triangular cross section for positive values of nonhomogeneities. The obtained results indicate that the maximum value of stress is located at inner surface of the curved beam. This maximum value increases with increasing the nonhomogeneity index. Increasing the positive values of nonhomogeneous index tends to increasing stiffness and consequently increasing the stress at inner surface.

The same results can be presented for negative values of nonhomogeneities of a FG triangular curved beam in Fig. 5. It is observed that for large values of nonhomogeneity index, the maximum negative value of circumferential stress is located at outer surface of the cylinder.

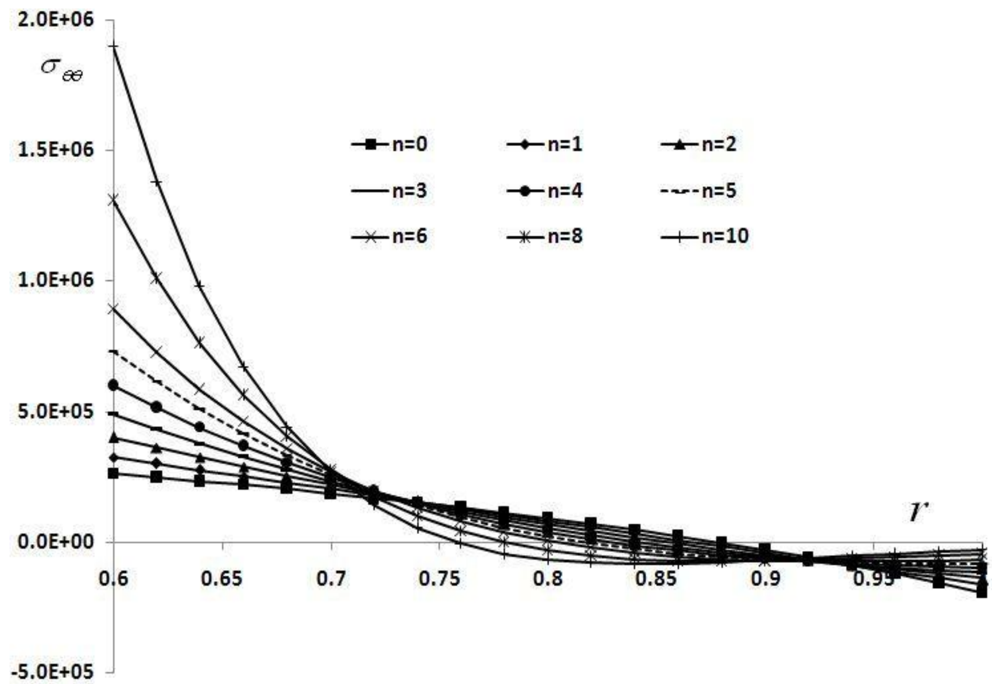


Fig. 4 The radial distribution of circumferential stress along the thickness direction of a triangular FG beam for positive values of nonhomogeneity index

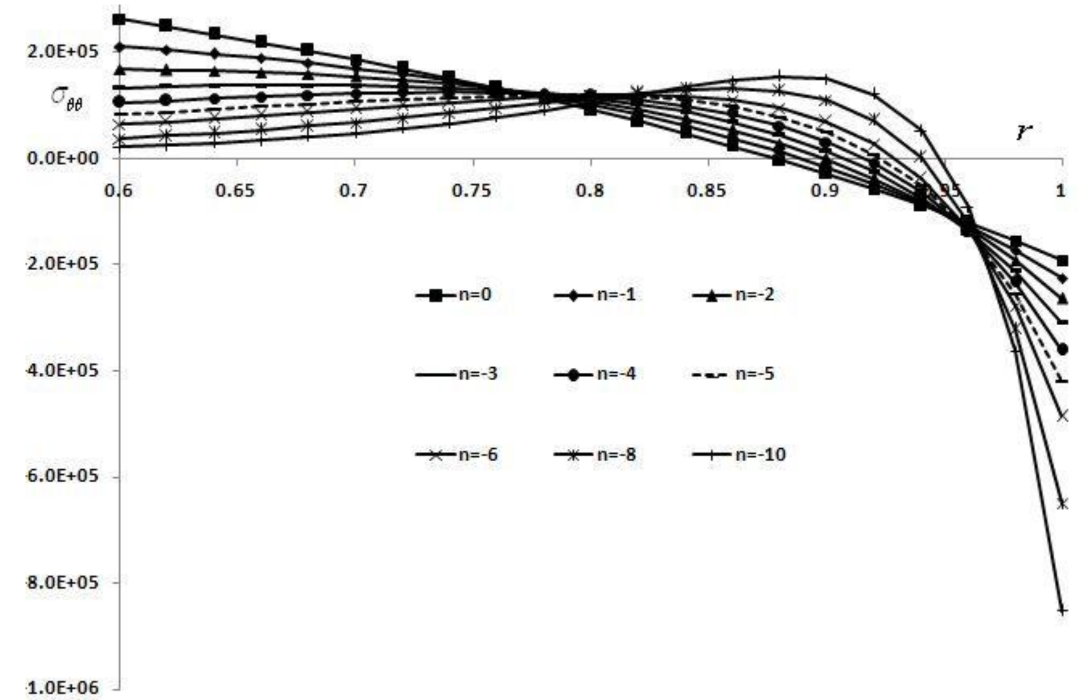


Fig. 5 The radial distribution of circumferential stress along the thickness direction of a triangular FG beam for negative values of nonhomogeneity index

Two identical values of circumferential stress for both positive and negative values of nonhomogeneity index can be observed in Figs. 4 and 5 as presented in Figs. 2 and 3.

3.1.3 Circular section

The radial distribution of circumferential stress can be investigated for a curved beam with circular section in terms of different positive and negative values of nonhomogeneity index. Shown in Fig. 6 is the radial distribution of circumferential stress along the thickness direction of a circular FG beam for positive values of nonhomogeneity index. The same observed trend in Fig. 2 can be observed in this figure. The obtained results indicate that with increasing the nonhomogeneity index, the maximum circumferential stress at inner surface decreases monotonically. Conversely, the maximum negative value of circumferential stress at outer surface increases considerably with increasing the nonhomogeneity index.

Fig. 7 shows the radial distribution of circumferential stress along the thickness direction of a circular FG beam for negative values of nonhomogeneity index. Increasing the maximum values of circumferential stress at inner surface with increasing the nonhomogeneity index is as results of this figure.

3.1.3.1 Comparison between present and previous results

The results of this research can be compared with related reference. For this goal, radial distribution of circumferential stress is selected. The comparison between present and previous results can be shown in Fig. 8. Dimensionless parameters have been considered compatible with

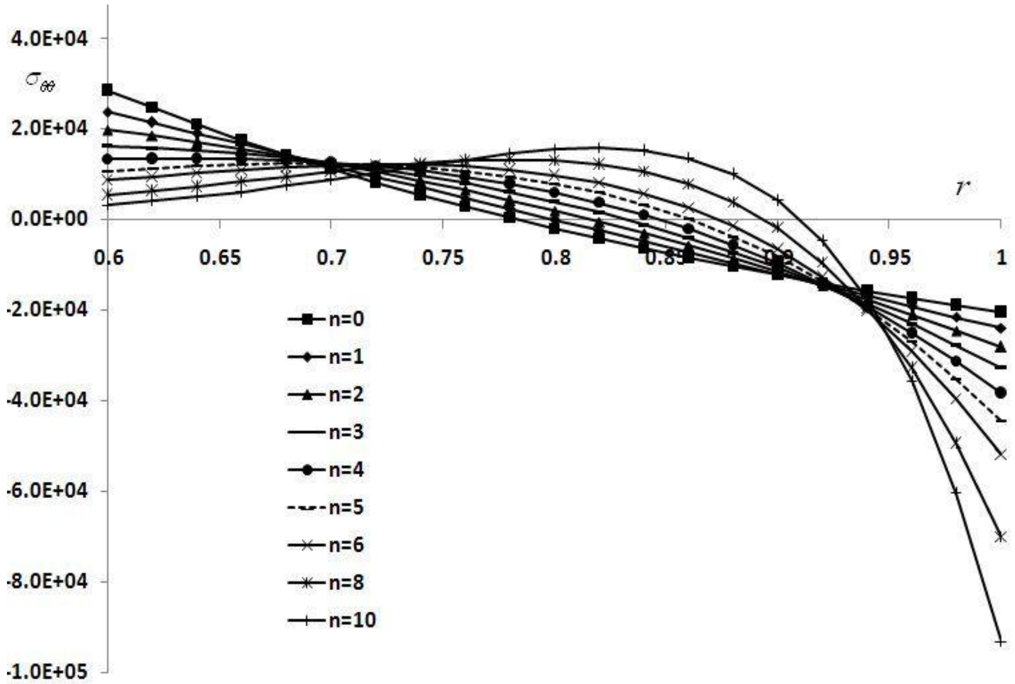


Fig. 6 The radial distribution of circumferential stress along the thickness direction of a circular FG beam for positive values of nonhomogeneity index

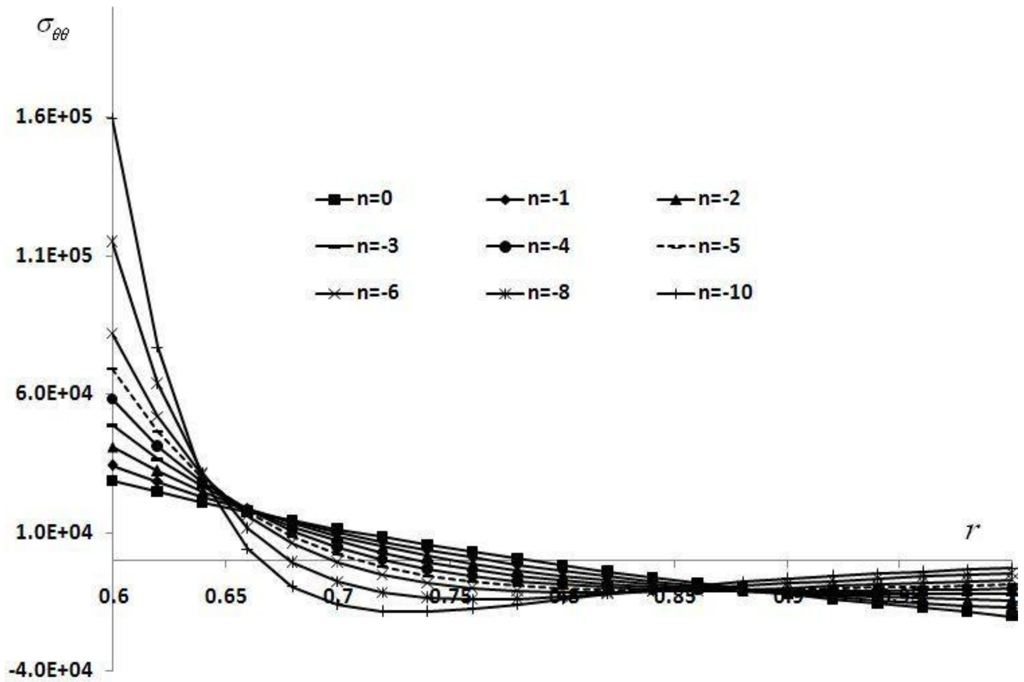


Fig. 7 The radial distribution of circumferential stress along the thickness direction of a circular FG beam for negative values of nonhomogeneity index

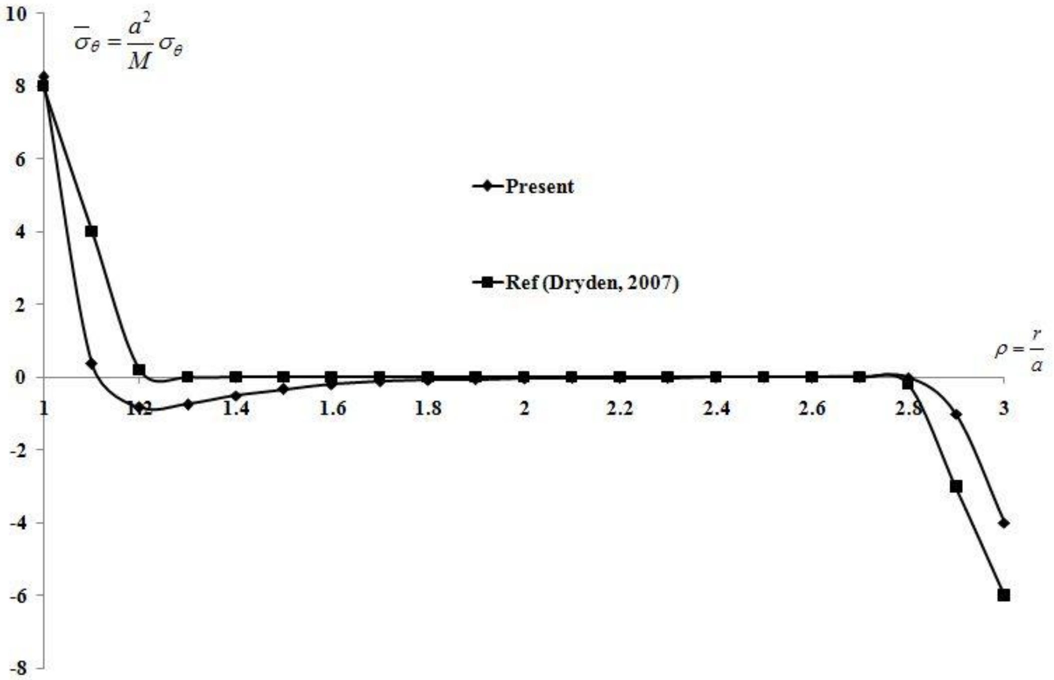


Fig. 8 Comparison between present and previous distribution of circumferential stress along the thickness direction of a circular FG beam

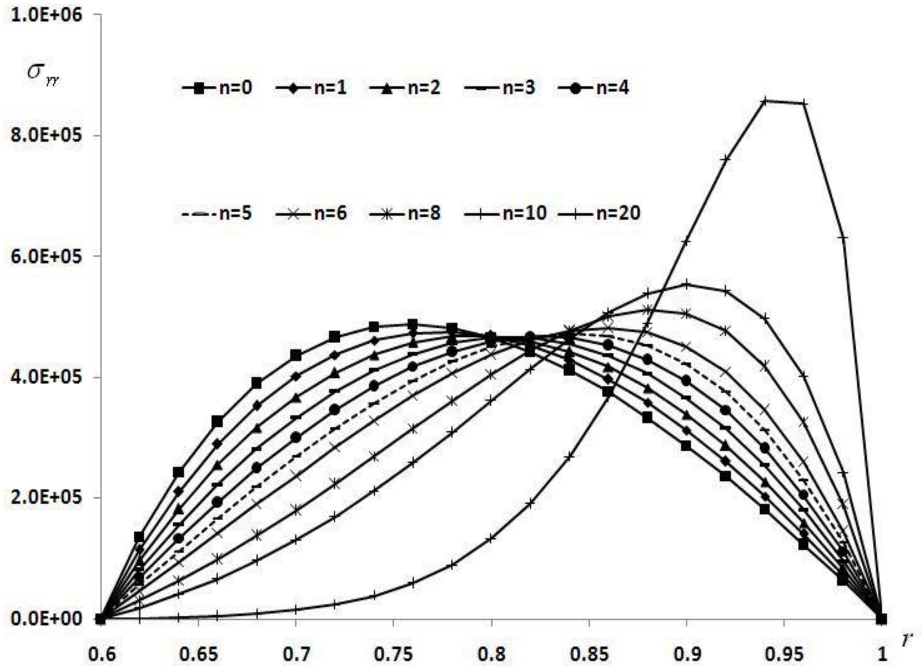


Fig. 9 The radial distribution of radial stress along the thickness direction of a rectangular FG beam for positive values of nonhomogeneity index

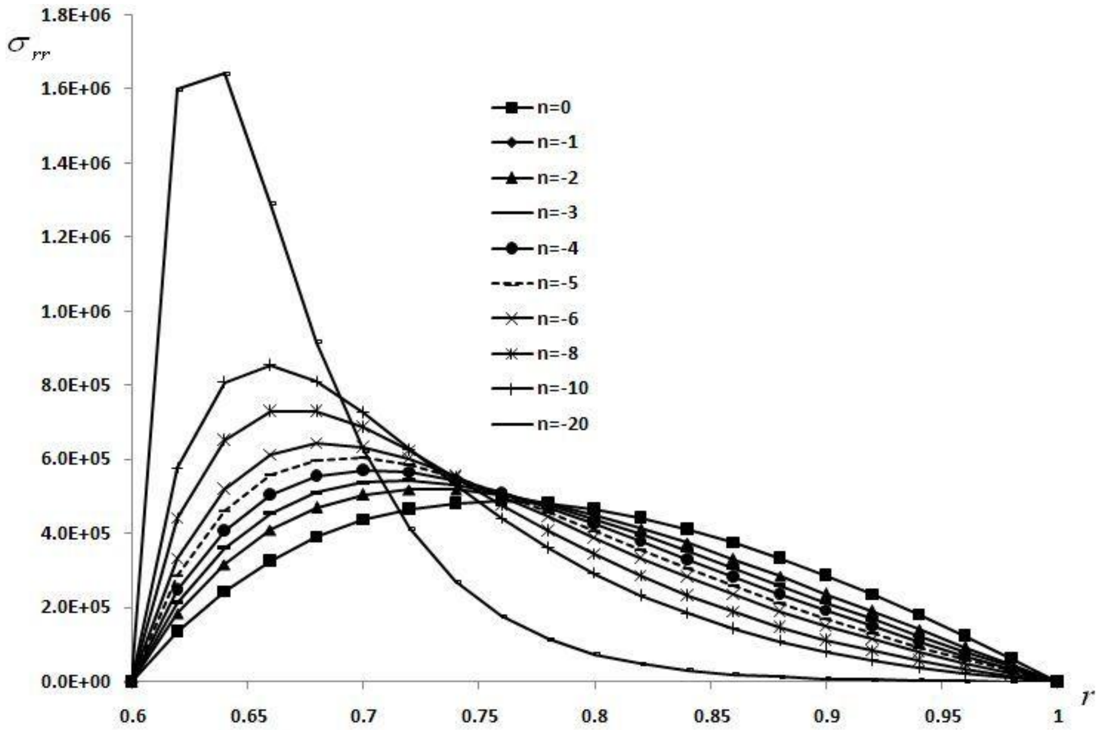


Fig. 10 The radial distribution of radial stress along the thickness direction of a rectangular FG beam for negative values of nonhomogeneity index

literature (Dryden 2007). These dimensionless parameters are $\bar{\sigma}_\theta = \dfrac{a^2}{M}\sigma_\theta$ and $\rho = \dfrac{r}{a}$.

Using derived equation for radial distribution of radial stress (Eq. (10)), this distribution can be presented. Figs. 9 and 10 show the radial distribution of radial stress for a FG curved beam with rectangular cross section in terms of positive and negative nonhomogeneity index.

Investigation on the Fig. 9 indicates that with increasing the positive nonhomogeneity index, the location of maximum radial stress moves to outer surface. For better understanding this moving, the results of radial stress have been presented for two large values of nonhomogeneity index (n = 10, 20). These results show that for very large values of nonhomogeneity index, the maximum radial stress is located near the outer surface. The location of maximum radial stress is depending on the values non homogeneous index. Large positive values of non homogeneous index guide this location to outer surface while large negative values guide to inner surface.

Conversely, investigation on the Fig. 10 indicates that with increasing the negative nonhomogeneity index, the location of maximum radial stress moves to inner surface.

4. Conclusions

Elastic solution of a curved beam made of functionally graded material was developed in this paper. Functionality has been considered along the radial direction based on a power function.

Stress distribution along the radial direction was evaluated analytically in integral form in terms of distribution of stiffness of material. The results were evaluated for various types of cross sections and different values of non-homogeneity index. The obtained numerical results indicate that the distribution of stress can be controlled by selection of the various nonhomogeneity indexes. Some important conclusions can be stated as follows:

(1) Investigation on the radial distribution of circumferential stress for various types of section indicates that two identical values of circumferential stress for both positive and negative values of nonhomogeneity index can be observed.
(2) Investigation on the varying the nonhomogeneous index indicates that the maximum value of stress is located at inner surface of the curved beam. This maximum value increases with increasing the nonhomogeneity index.
(3) Investigation on the radial stress indicates that with increasing the values of nonhomogeneity index from negative to positive values, the location of maximum radial stress moves from inner surface to outer surface.

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