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Bending and buckling analyses of functionally graded material (FGM) size-dependent nanoscale beams including the thickness stretching effect

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Abstract. This paper addresses theoretically the bending and buckling behaviors of size-dependent nanobeams made of functionally graded materials (FGMs) including the thickness stretching effect. The size-dependent FGM nanobeam is investigated on the basis of the nonlocal continuum model. The nonlocal elastic behavior is described by the differential constitutive model of Eringen, which enables the present model to become effective in the analysis and design of nanostructures. The present model incorporates the length scale parameter (nonlocal parameter) which can capture the small scale effect, and furthermore accounts for both shear deformation and thickness stretching effects by virtue of a sinusoidal variation of all displacements through the thickness without using shear correction factor. The material properties of FGM nanobeams are assumed to vary through the thickness according to a power law. The governing equations and the related boundary conditions are derived using the principal of minimum total potential energy. A Navier-type solution is developed for simply-supported boundary conditions, and exact expressions are proposed for the deflections and the buckling load. The effects of nonlocal parameter, aspect ratio and various material compositions on the static and stability responses of the FGM nanobeam are discussed in detail. The study is relevant to nanotechnology deployment in for example aircraft structures.

Keywords: nanobeam; nonlocal elasticity theory; bending; buckling; stretching effect; functionally graded materials; navier solution; aspect ratio

1. Introduction

Structural beams fabricated from nanomaterials (Harik and Salas 2003) and of nanometer dimensions are referred to as *nanobeams* (nanowires, nanotubes, nanorods). These nano-structural

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elements are extensively utilized as nanostructure components for nanoelectromechanical (NEMS) and microelectromechanical systems (MEMS) (Mukherjee and Aluru 2006), which arise frequently in the aerospace industry. Hence, the understanding of mechanical behavior of nanobeams is critical for optimizing the performance of such structures. In such applications, the *size effect* plays major role which should be addressed to properly quantify the behavior of such small scale structures. It has now been generally established that classical continuum mechanics fails to predict the size-dependent response of the structures at micro-and nano-scales since the classical theory framework does not feature intrinsic length scales. In order to overcome this problem, many higher order continuum (nonlocal) theories have been proposed. These formulations contain additional material constants, such as the modified couple stress theory (Yang *et al*. 2002), the strain gradient theory (Aifantis 1999), the micropolar theory (Eringen 1967), the nonlocal elasticity theory (Eringen 1972), and the surface elasticity model (Gurtin *et al*. 1998). Such theories aim to robustly characterize the size effect in micro, nano-scale structures by introducing an *intrinsic length scale* in the constitutive relations. Among these theories, the nonlocal elasticity theory has emerged as a very promising and accurate approach. Introduced by Eringen (1983), nonlocal elasticity can successfully account for the scale effect in elasticity and has been shown to effectively simulate many complex phenomena in multi-scale mechanics including lattice dispersion of elastic waves, wave propagation in composites, dislocation mechanics, fracture mechanics and surface tension effects in fluids. Peddieson *et al*. (2003) first applied the nonlocal Eringen elasticity theory (Eringen 1983) to nanotechnology and derived expressions for the static deformations of beam structures based on a simplified nonlocal beam model. Subsequently, based on the nonlocal constitutive relation of Eringen, numerous studies have appeared which have developed nonlocal beam models for predicting the responses of nanostructures. These investigations include static analysis (Wang and Liew 2007, Pijaudier-Cabot and Bažant 1987, Lim and Wang 2007, Reddy and Pang 2008), buckling calculations (Zhang *et al*. 2004, 2006, Wang *et al*. 2006, Murmu and Pradhan 2009a, Amara *et al*. 2010, Narendar and Gopalakrishnan 2011a, Tounsi *et al*. 2013a, b, Semmah *et al*. 2014, Zidour *et al*. 2014), vibration modelling (Yoon *et al*. 2003, Zhang *et al*. 2005a, b, Benzair *et al*. 2008, Murmu and Pradhan 2009b, Hemmatnezhad and Ansari 2013, Boumia *et al*. 2014, Baghdadi *et al*. 2014), wave propagation simulations (Lu *et al*. 2007, Tounsi *et al*. 2008, Heireche *et al*. 2008a, b, c, Song *et al*. 2008, Narendar and Gopalakrishnan 2011b, Besseghier *et al*. 2011, Naceri *et al*. 2011, Gafour *et al*. 2013) and thermo-mechanical (Mustapha and Zhong 2010a, Maachou *et al*. 2011, Zidour *et al*. 2012) computations of nanostructures. Recently, Mustapha and Zhong (2010b) investigated the free vibration of an axially-loaded non-prismatic single-walled carbon nanotube embedded in a two-parameter elastic medium with a Bubnov-Galerkin method. Roque *et al*. (2011) used the nonlocal elasticity theory of Eringen to study bending, buckling and free vibration of Timoshenko nanobeams with a meshless numerical method. Reddy (2007) implemented a range of different beam theories including those of Euler-Bernoulli, Timoshenko, Levinson (1981) and Reddy (1984) to simulate bending, buckling and vibration of nonlocal beams. Benguediab *et al*. (2014) proposed a comprehensive nonlocal shear deformation beam theory for bending, buckling and vibration analysis of homogeneous nanobeams founded on Eringen's nonlocal elasticity theory.

Developments in the field of materials engineering have stimulated a new class of high-performance materials with smooth and continuous variation of the material properties, which can be strategically manipulated for specific applications such as aerospace structures, solar power collectors, bridges, machine components etc. These materials are designated as functionally graded materials (FGMs). Yaghoobi and his co-workers (Yaghoobi and Torabi 2013a, b, Yaghoobi and

Yaghoobi 2013, Yaghoobi and Fereidoon 2014) studied the buckling behavior of FGM structures. Tounsi and his co-workers (Klouche Djedid *et al*. 2014, Ait Amar Meziane *et al*. 2014, Tounsi *et al*. 2013c, Zidi *et al*. 2014, Bouderba *et al*. 2013, Bachir Bouiadjra *et al*. 2012, 2013, Bouremana *et al*. 2013, Bourada *et al*. 2012, Fekrar *et al*. 2012, El Meiche *et al*. 2011, Benachour *et al*. 2011, Mahi *et al*. 2010, Sallai *et al*. 2009, Benatta *et al*. 2008) investigated the mechanical response of FGM structures. They used a through-the-thickness variation of the material properties according to a power law. Recently, the application of FGMs has expanded to the realm of nano-structures and typical examples in this regard are nano-electromechanical systems (NEMS), thin films in the form of shape memory alloys, atomic force microscopes (AFMs), nano-implants in medical engineering, nanotubes in aircraft wings, nanobeams for spacecraft chassis structures etc. All these applications achieve high sensitivity and enhanced performance. However, thusfar, relatively sparse research has been communicated on FGM nanobeam structural mechanics, based on the nonlocal elasticity theory. Janghorban and Zare (2011) investigated nonlocal free vibration of axially FGM nanobeams by using the differential quadrature computational method. Eltaher *et al*. (2012) studied free vibration of FGM nanobeam based on the nonlocal Euler-Bernoulli beam theory. The static bending and buckling of FGM nanobeam has also been examined based on the nonlocal Timoshenko and Euler-Bernoulli beam theory by Şimşek and Yurtçu (2013). A recent review of applications of nonlocal Eringen elasticity in a range of nanobeam problems is provided by Murmu and Adhikari (2012).

In this paper, a nonlocal beam theory is proposed for bending and buckling of FGM nanobeams. Contrary to the other theories elaborated in (Roque *et al*. 2011, Reddy 2007, Benguediab *et al*. 2014, Şimşek and Yurtçu 2013, Berrabah *et al*. 2013), *where the stretching effect is neglected*, in the current investigation this so-called ‘‘stretching effect’’ is taken into consideration. The displacement field of the proposed theory is chosen based on the following assumptions (Bousahla *et al*. 2014, Fekrar *et al*. 2014, Belabed *et al*. 2014, Hebali *et al*. 2014, Houari *et al*. 2013, Bessaim *et al*. 2013, Saidi *et al*. 2013): (1) The transverse displacement is partitioned into bending, shear and stretching components; (2) the axial displacement consists of extension, bending and shear components; (3) the bending component of axial displacement is similar to that given by the Euler-Bernoulli beam theory; and (4) the shear component of axial displacement gives rise to the sinusoidal variation of shear strain and hence to shear stress through the thickness of the beam in such a way that shear stress vanishes on the top and bottom surfaces. The material properties of the FGM nanobeam are assumed to vary in the thickness direction. Based on the nonlocal constitutive relations of Eringen (1983), the governing equations are derived using the principal of minimum total potential energy. To illustrate the accuracy of the present theory, the obtained results are compared with those predicted by the Euler-Bernoulli beam theory and Timoshenko beam theory. Finally, the influences of nonlocal parameter, power law index, and aspect ratio on the bending, buckling and vibration responses of FGM nanobeam are discussed.

2. Theoretical formulations

2.1 Material properties

A functionally graded material, simply-supported nanobeam, of length L, width b, and thickness h, is shown in Fig. 1. It is assumed that material properties of the FGM nanobeam, such as Young’s modulus (E), Poisson’s ratio (v), and the shear modulus (G), vary continuously

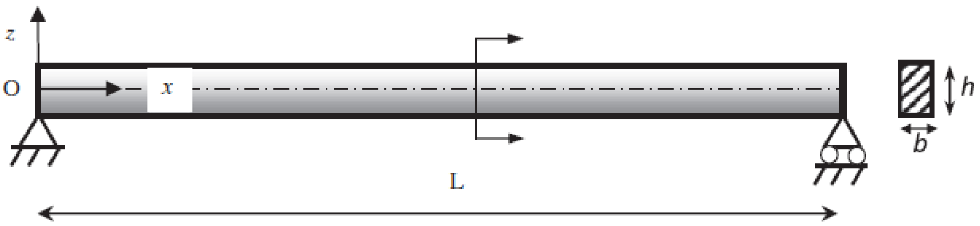


Fig. 1 Gradation of material properties through the thickness of the FG beam

through the nanobeam thickness according to a power-law form (Eltaher *et al*. 2012, Şimşek and Yurtçu 2013), which can be described by

$$P(z) = (P_t - P_b)\left(\frac{z}{h} + \frac{1}{2}\right)^k + P_b \tag{1}$$

where P_t and P_b are the corresponding material property at the top and bottom surfaces of the nanobeam, k is a non-negative number that dictates the material variation profile through the thickness of the nanobeam.

2.2 Kinematics

Based on the assumptions made above, the displacement field of the present theory can be obtained as

$$u(x,z,t) = u_0(x,t) - z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x} \tag{2a}$$

$$w(x,z,t) = w_b(x,t) + w_s(x,t) + w_{st}(x,z,t) \tag{2b}$$

where u_0 is the axial displacement along the midplane of the nanobeam; w_b, w_s and w_{st} are the bending, shear and stretching components of transverse displacement along the midplane of the beam. Furthermore

$$f(z) = \left(z - \frac{h}{\pi}\sin\frac{\pi z}{h}\right) \tag{2c}$$

The component due to the stretching effect w_{st} can be given as

$$w_{st}(x,z,t) = g(z)\,\varphi(x,t) \tag{2d}$$

The additional displacement φ accounts for the effect of normal stress is included and $g(z)$ is given as follows

$$g(z) = 1 - f'(z) \tag{2e}$$

The nonzero strains of the proposed beam theory are

$$\varepsilon_x = \varepsilon_x^0 + z\, k_x^b + f(z)\, k_x^s, \quad \gamma_{xz} = g(z)\,\gamma_{xz}^0, \quad \text{and} \quad \varepsilon_z = g'(z)\,\varepsilon_z^0 \tag{3}$$

where

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x}, \quad k_x^b = -\frac{\partial^2 w_b}{\partial x^2}, \quad k_x^s = -\frac{\partial^2 w_s}{\partial x^2}, \quad \gamma_{xz}^0 = \frac{\partial w_s}{\partial x} + \frac{\partial \varphi}{\partial x}, \quad \varepsilon_z^0 = \varphi \tag{4}$$

2.3 Nonlocal theory and constitutive relations

Response of materials at the nanoscale is different from those of their bulk counterparts. In the theory of nonlocal elasticity Eringen (1983), the stress at a reference point x is considered to be a functional of the strain field at every point in the body. For example, in the non - local elasticity, the uniaxial constitutive law is expressed as (Eringen 1983)

$$\sigma_x - \mu \frac{d^2\sigma_x}{dx^2} = Q_{11}\varepsilon_x + Q_{13}\varepsilon_z \tag{5a}$$

$$\tau_{xz} - \mu \frac{d^2\tau_{xz}}{dx^2} = Q_{55}\gamma_{xz} \tag{5b}$$

$$\sigma_z - \mu \frac{d^2\sigma_z}{dx^2} = Q_{13}\varepsilon_x + Q_{33}\varepsilon_z \tag{5c}$$

The Q_{ij} expressions in terms of engineering constants are

$$Q_{11}(z) = Q_{33}(z) = \frac{E(z)}{1-\upsilon^2}, \quad Q_{13}(z) = \nu\, Q_{11}(z), \quad Q_{55}(z) = \frac{E(z)}{2(1+\upsilon)} \tag{5d}$$

And $\mu = (e_0 a)^2$ is a nonlocal parameter revealing the nanoscale effect on the response of nanobeams, e_0 is a constant appropriate to each material and a is an internal characteristic length.

2.4 The governing equations based on the nonlocal elasticity theory

The governing equations will be derived by using principal of the minimum total potential energy as follows

$$\delta\Pi = \delta(U_{\text{int}} - W_{ext}) = 0 \tag{6}$$

where Π is the total potential energy. δU_{int} is the virtual variation of the strain energy; and δW_{ext} is the variation of work done by external forces. The first variation of the strain energy is given as

$$\delta U_{\text{int}} = \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_x \delta\varepsilon_x + \sigma_z \delta\varepsilon_z + \tau_{xz}\delta\gamma_{xz})\, dzdx \tag{7}$$

↓

↑

$$= \int_0^L \left(N \frac{d\delta u_0}{dx} + N_z \delta \varphi - M_b \frac{d^2 \delta w_b}{dx^2} - M_s \frac{d^2 \delta w_s}{dx^2} + Q\left[\frac{d\delta w_s}{dx} + \frac{d\delta \varphi}{dx} \right] \right) dx \tag{7}$$

where N, M_b, M_s, N_z and Q are the stress resultants defined as

$$(N, M_b, M_s) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1, z, f(z))\, \sigma_x dz, \quad N_z = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_z g'(z) dz, \quad \text{and} \quad Q = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} g(z) dz \tag{8}$$

The first variation of the work done by the axial compressive force is given by

$$\delta V = \int_0^L q \delta\, w dx + \int_0^L N_0 \frac{d\, w}{dx} \frac{d\delta\, w}{dx} dx \tag{9}$$

where q and N_0 are the transverse and axial loads, respectively.

Substituting the expressions for δU_{int}, and v from Eqs. (7) and (9) into Eq. (6) and integrating by parts, and collecting the coefficients of δu_0, δw_b, δw_s and $\delta\varphi$, the following governing equations of the FGM nanobeam are obtained

$$\delta\, u_0 : \quad \frac{dN}{dx} = 0 \tag{10a}$$

$$\delta\, w_b : \quad \frac{d^2 M_b}{dx^2} + q - N_0 \frac{d^2 w}{dx^2} = 0 \tag{10b}$$

$$\delta\, w_s : \quad \frac{d^2 M_s}{dx^2} + \frac{dQ}{\partial x} + q - N_0 \frac{d^2 w}{dx^2} = 0 \tag{10c}$$

$$\delta\varphi : \quad \frac{dQ}{dx} - N_z - N_0 \frac{d^2 w}{dx^2} = 0 \tag{10d}$$

By virtue of Eqs. (3), (5), and (8), the *force-strain* and the *moment-strain relations* of the present nonlocal beam theory can be obtained as follows

$$N - \mu \frac{d^2 N}{dx^2} = A_{11} \frac{du_0}{dx} - B_{11} \frac{d^2 w_b}{dx^2} - B_{11}^s \frac{d^2 w_s}{dx^2} + X_{13}\, \varphi \tag{11a}$$

$$M_b - \mu \frac{d^2 M_b}{dx^2} = B_{11} \frac{du_0}{dx} - D_{11} \frac{d^2 w_b}{dx^2} - D_{11}^s \frac{d^2 w_s}{dx^2} + Y_{13} \varphi \tag{11b}$$

$$M_s - \mu \frac{d^2 M_s}{dx^2} = B_{11}^s \frac{du_0}{dx} - D_{11}^s \frac{d^2 w_b}{dx^2} - H_{11}^s \frac{d^2 w_s}{dx^2} + Y_{13}^s \varphi \tag{11c}$$

$$Q-\mu\frac{d^2Q}{dx^2}=A_{55}^s\left(\frac{dw_s}{dx}+\frac{d\,\varphi}{dx}\right) \tag{11d}$$

$$N_z-\mu\frac{d^2N_z}{dx^2}=X_{13}\frac{du_0}{dx}-Y_{13}\frac{d^2w_b}{dx^2}-Y_{13}^s\frac{d^2w_s}{dx^2}+Z_{33}\,\varphi \tag{11e}$$

where

$$\left(A_{11},B_{11},D_{11},B_{11}^s,D_{11}^s,H_{11}^s\right)=\int_{-h/2}^{h/2}Q_{11}\left(1,z,z^2,f(z),z\,f(z),f^2(z)\right)dz \tag{12a}$$

and

$$A_{55}^s=\int_{h/2}^{h/2}Q_{55}[g(z)]^2dz,\quad \left[X_{13},Y_{13},Y_{13}^s\right]=\int_{h/2}^{h/2}Q_{13}[1,z,f(z)]g'(z)dz,\quad Z_{33}=\int_{h/2}^{h/2}Q_{33}[g'(z)]^2dz, \tag{12b}$$

By substituting Eq. (11) into Eq. (10), the nonlocal governing equations can be expressed in terms of displacements (u_0, w_b, w_s, φ) as

$$A_{11}\frac{d^2u_0}{dx^2}-B_{11}\frac{d^3w_b}{dx^3}-B_{11}^s\frac{d^3w_s}{dx^3}+X_{13}\frac{d\varphi}{dx}=0 \tag{13a}$$

$$B_{11}\frac{d^3u_0}{dx^3}-D_{11}\frac{d^4w_b}{dx^4}-D_{11}^s\frac{d^4w_s}{dx^4}+Y_{13}\frac{d^2\varphi}{dx^2}+\left(1-\mu\frac{d^2}{dx^2}\right)\left(q-N_0\frac{d^2w}{dx^2}\right)=0 \tag{13b}$$

$$B_{11}^s\frac{d^3u_0}{dx^3}-D_{11}^s\frac{d^4w_b}{dx^4}-H_{11}^s\frac{d^4w_s}{dx^4}+A_{55}^s\frac{d^2w_s}{dx^2}+\left(A_{55}^s+Y_{13}^s\right)\frac{d^2\,\varphi}{dx^2}+\left(1-\mu\frac{d^2}{dx^2}\right)\left(q-N_0\frac{d^2w}{dx^2}\right)=0 \tag{13c}$$

$$-X_{13}\frac{du_0}{dx}+Y_{13}\frac{d^2w_b}{dx^2}+\left(A_{55}^s+Y_{13}^s\right)\frac{\partial^2w_s}{\partial x^2}+A_{55}^s\frac{\partial^2\varphi}{\partial x^2}-Z_{33}\varphi-\left(1-\mu\frac{d^2}{dx^2}\right)\left(N_0\frac{d^2w}{dx^2}\right)=0 \tag{13d}$$

The equations of motion of local beam theory can be retrieved from Eq. (13) by setting the nonlocal parameter μ equal to zero.

2.5 Analytical solution

The equations of motion admit Navier analytical solutions for *simply supported* beams. The variables u_0, w_b, w_s, and φ can be written by assuming the following variations

$$\begin{Bmatrix}u_0\\w_b\\w_s\\\varphi\end{Bmatrix}==\sum_{m=1}^{\infty}\begin{Bmatrix}U_m\cos(\lambda\,x)\,e^{i\,\omega\,t}\\W_{bm}\sin(\lambda\,x)\,e^{i\,\omega\,t}\\W_{sm}\sin(\lambda\,x)\,e^{i\,\omega\,t}\\\Phi_{stm}\sin(\lambda\,x)\,e^{i\,\omega\,t}\end{Bmatrix} \tag{14}$$

where U_m, W_{bm}, W_{sm} and Φ_{stm} are arbitrary parameters to be determined, ω is the eigenfrequency associated with mth eigenmode, and $\lambda = m\pi / L$. The transverse load q is also expanded in the Fourier sine series as

$$q(x) = \sum_{n=1}^{\infty} Q_n \sin \alpha\, x, \quad Q_n = \frac{2}{L} \int_0^L q(x) \sin(\alpha\, x)\, dx \tag{15}$$

The Fourier coefficients Q_n associated with some typical loads are given

$$Q_n = q_0, \quad n = 1 \quad \text{for sinusoidal load,} \tag{16a}$$

$$Q_n = \frac{4q_0}{n\pi}, \quad n = 1,3,5\ldots\ldots \quad \text{for uniform load,} \tag{16b}$$

$$Q_n = \frac{2q_0}{L} \sin\frac{n\pi}{2}, \quad n = 1,2,3\ldots\ldots \quad \text{for point load } Q_0 \text{ at the midspan,} \tag{16c}$$

Substituting Eqs. (14) and (15) into Eq. (13), the analytical solutions can be obtained by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} - \zeta & a_{23} - \zeta & a_{24} - \zeta \\ a_{13} & a_{23} - \zeta & a_{33} - \zeta & a_{34} - \zeta \\ a_{14} & a_{24} - \zeta & a_{34} - \zeta & a_{44} - \zeta \end{bmatrix} \begin{Bmatrix} U_m \\ W_{bm} \\ W_{sm} \\ \Phi_{stm} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \alpha\, Q_n \\ \alpha\, Q_n \\ 0 \end{Bmatrix} \tag{17}$$

where

$$\begin{aligned} &a_{11} = A_{11}\lambda^2, && a_{12} = -B_{11}\lambda^3, && a_{13} = -B_{11}^s\lambda^3, && a_{14} = -X_{13}\lambda, \\ &a_{22} = D_{11}\lambda^4, && a_{23} = D_{11}^s\lambda^4, && a_{24} = Y_{13}\lambda^2, && a_{33} = A_{55}\lambda^2 + H_{11}\lambda^4, \\ &a_{34} = \lambda^2 Y_{13}^s + A_{55}\lambda^2, && a_{44} = \lambda^2 A_{55}^s + Z_{33}, && \zeta = \alpha N_0 \lambda^2, && \alpha = 1 + \mu\lambda^2 \end{aligned} \tag{18}$$

3. Numerical results and interpretation

In this section, analytical solutions obtained in the previous sections are utilized for numerical examples. The obtained results are compared with those reported by Şimşek and Yurtçu (2013) based on nonlocal Timoshenko beam theory (TBT) for a wide range of nonlocal parameter (e_0a), the material distribution parameter (k) and thickness ratio (L / h). The FGM nanobeam has the following prescribed material properties: E_t = 0.25 TPa, E_b = 1 TPa, $\nu_t = \nu_b = 0.3$. A conservative estimate of the *nonlocal* parameter $0 \le e_0a \le 2$ nm for single walled carbon nanotubes (SWCNTs) has been provided by Wang (2005). Therefore, in this study, the nonlocal parameter is taken as e_0a = 0, 0.5, 1, 1.5, 2 nm to investigate nonlocal effects on the responses of FGM nanobeam. For convenience, the following dimensionless quantities are defined

$$\overline{w} = 100w\frac{E_b I}{q_0 L^4} \quad \text{for uniform load} \tag{19a}$$

$$\overline{N} = N_{cr} \frac{L^2}{E_b I} \quad \text{i.e., critical buckling load parameter} \tag{19b}$$

Table 1 shows the non-dimensional *maximum* deflections $\overline{w}$ of a simply supported FGM nanobeam subjected to uniform load. The calculated values are obtained using 100 terms in series in Eqs. (14) and (15) with Maple software. It should be noted that the case, $e_0a = 0$ corresponds to *local* beam theory. The obtained results (Models 1 and 2) are compared with those predicted by TBT (Şimşek and Yurtçu 2013). Since the effect of thickness stretching is neglected in Model 1 ($\varepsilon_z = 0$), it leads to the solutions close to TBT (Şimşek and Yurtçu 2013) for all values of thickness ratio, L / h, material distribution parameter k and nonlocal parameter e_0a. The slight difference between the results obtained by Model 1 ($\varepsilon_z = 0$) and TBT is due to the use of a *constant shear correction factor* for any values of the material distribution parameter k (Menaa *et al*. 2012). In addition, the results of Model 2 ($\varepsilon_z \neq 0$) are also provided to show the importance of including the thickness stretching effect. Indeed, it is evident from inspection of **Table 1** that the inclusion of the thickness stretching effect leads to a reduction in the magnitudes of deflection of FGM nanobeams. In other words, with the thickness stretching effect incorporated, FGM nanobeams exhibit *greater stiffness*, and this characteristic is particularly important in applications.

Table 2 documents the values for the computed *non-dimensional critical buckling loads*. The present computations are benchmarked with the earlier results of Şimşek and Yurtçu (2013) and good correlation is observed with Model 1 ($\varepsilon_z = 0$). The results obtained using Model 2 ($\varepsilon_z \neq 0$) show that the inclusion of the thickness stretching effect manifests in an enhancement in the critical buckling loads. According to this table, buckling loads *decrease* with increasing nonlocal parameter (e_0a). However, the increase of power law index k leads to an increase of critical buckling loads.

Fig. 2 shows the variation of the non-dimensional deflection and the buckling load of the FGM nanobeam with geometrical aspect ratio. The local and nonlocal results are given for $e_0a = 0$ and $e_0a = 1$ nm, respectively. The material distribution parameter is assumed to be constant i.e., $k = 1$. In this example, the aspect ratio varies from $L / h = 10$ to $L / h = 50$. It is apparent that deflections predicted by the nonlocal theory exceed in magnitude those computed with the local (classical) continuum theory. On the other hand, the *nonlocal* solution of the buckling load is lower in magnitude than the local buckling load due to the small scale effects. Also, it can be observed that the inclusion of the thickness stretching effect leads to a marked reduction in nanobeam deflection and an increase in buckling load values for FGM nanobeams. These results effectively demonstrate that the inclusion of small scale parameter softens the nanobeam (reduces stiffness), whereas the inclusion of thickness stretching effect makes it stiffer. As such both small scale and thickness stretching effects exert a significant influence on nanobeam structural performance.

Fig. 3 shows the effect of the nonlocal parameter on dimensionless deflections and critical buckling loads. The results in this figure are obtained by using the present *nonlocal* shear deformation beam theory including the thickness stretching effect (Model 2). The material distribution parameter is assumed to be constant (i.e., $k = 1$). These figures show that the responses vary in a *nonlinear* fashion with the nonlocal parameter. It can be seen that the effect of nonlocal parameter e_0a on deflections and critical buckling loads of FGM nanobeams is significant, especially at relatively higher aspect ratios. Therefore, it can be concluded that FGM nanobeam responses are *aspect ratio-dependent* based on nonlocal elasticity.

Table 1 Dimensionless transverse deflections ($\overline{w}$) of the FG nanobeam for uniform load

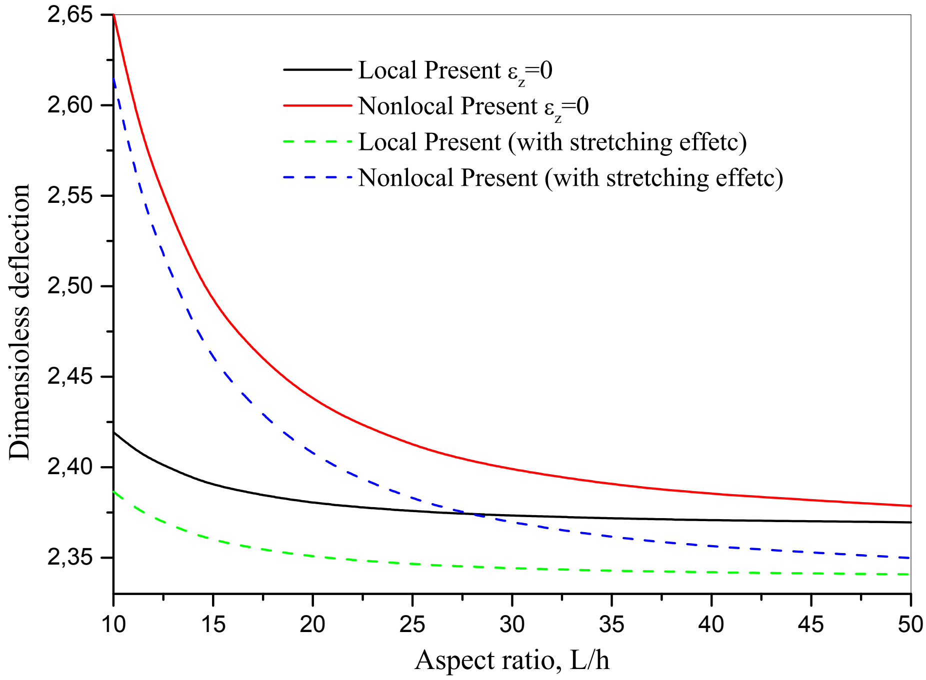
L/h	k	Nonlocal parameter, e_0a (nm)														
		0			0.5			1			1.5			2		
		TBT[a]	Model 1	Model 2	TBT[a]	Model 1	Model 2	TBT[a]	Model 1	Model 2	TBT[a]	Model 1	Model 2	TBT[a]	Model 1	Model 2
	0	5.3383	5.3381	5.3197	5.4659	5.4659	5.4469	5.8487	5.8485	5.8284	6.4867	6.4865	6.4643	7.3798	7.3797	7.3548
	0.3	3.2169	3.2178	3.1831	3.2938	3.2946	3.2591	3.5245	3.5254	3.4874	3.9090	3.9102	3.8681	4.4472	4.4482	4.4007
10	1	2.4194	2.4193	2.3864	2.4772	2.4772	2.4434	2.6508	2.6508	2.6147	2.9401	2.9401	2.9005	3.3451	3.3449	3.2997
	3	1.9249	1.9234	1.9067	1.9710	1.9693	1.9522	2.1091	2.1074	2.0892	2.3393	2.3373	2.3171	2.6615	2.6596	2.6367
	10	1.5799	1.5790	1.5721	1.6176	1.6169	1.6099	1.7310	1.7301	1.7227	1.9190	1.9190	1.9107	2.1843	2.1831	2.1739
	0	5.2227	5.2228	5.2141	5.2366	5.2366	5.2279	5.2784	5.2786	5.2698	5.3480	5.3480	5.3391	5.4455	5.4456	5.4365
	0.3	3.1486	3.1473	3.1179	3.1570	3.1557	3.1265	3.1822	3.1809	3.1514	3.2241	3.2230	3.1930	3.2829	3.2815	3.2510
30	1	2.3732	2.3731	2.3442	2.3795	2.3795	2.3504	2.3985	2.3984	2.3692	2.4301	2.4301	2.4004	2.4744	2.4744	2.4441
	3	1.8894	1.8892	1.8759	1.8944	1.8943	1.8810	1.9095	1.9094	1.8960	1.9347	1.9344	1.9209	1.9700	1.9698	1.9560
	10	1.5489	1.5488	1.5449	1.5530	1.5530	1.5490	1.5654	1.5653	1.5614	1.5860	1.5861	1.5821	1.6149	1.6149	1.6108
	0	5.2096	5.2097	5.2021	5.2108	5.2110	5.2034	5.2146	5.2146	5.2071	5.2208	5.2210	5.2134	5.2296	5.2296	5.2220
	0.3	3.1408	3.1394	3.1103	3.1416	3.1404	3.1116	3.1438	3.1426	3.1137	3.1476	3.1465	3.1175	3.1529	3.1517	3.1228
100	1	2.3679	2.3680	2.3393	2.3685	2.3686	2.3399	2.3702	2.3702	2.3416	2.3730	2.3731	2.3444	2.3770	2.3771	2.3484
	3	1.8853	1.8853	1.8725	1.8858	1.8858	1.8729	1.8871	1.8871	1.8743	1.8894	1.8893	1.8765	1.8926	1.8926	1.8797
	10	1.5453	1.5453	1.5418	1.5457	1.5457	1.5422	1.5468	1.5468	1.5433	1.5487	1.5487	1.5451	1.5513	1.5513	1.5477

[a] Şimşek and Yurtçu (2013)
Model 1: Present theory with $\varepsilon_z = 0$
Model 2: Present theory with $\varepsilon_z \neq 0$

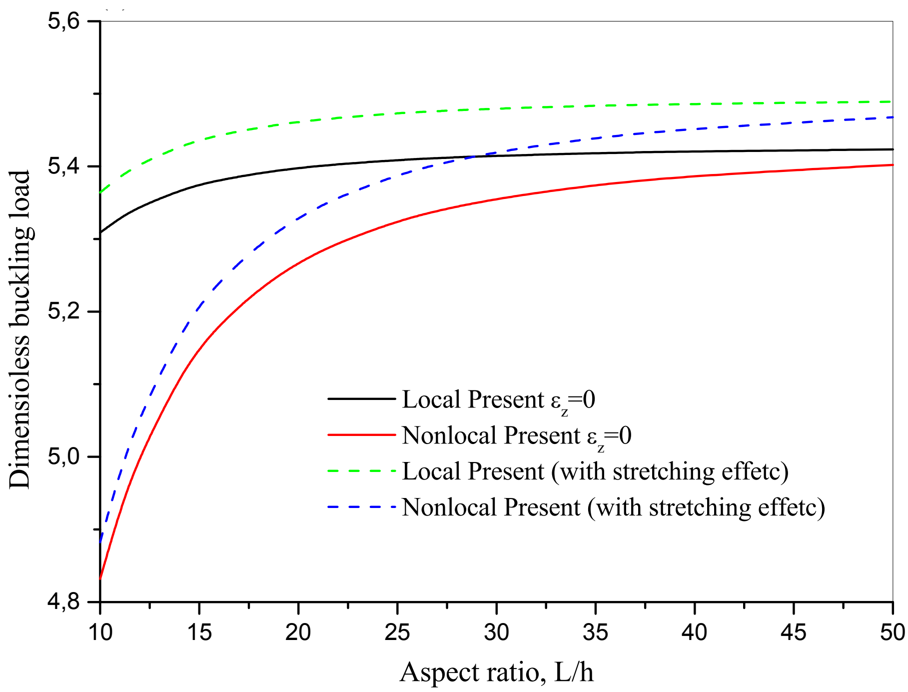
Table 2 Dimensionless critical buckling load $(\overline{N})$ of the FG nanobeam

		Nonlocal parameter, e_0a (nm)														
L/h	k	0			0.5			1			1.5			2		
		TBT[a]	Model 1	Model 2	TBT[a]	Model 1	Model 2	TBT[a]	Model 1	Model 2	TBT[a]	Model 1	Model 2	TBT[a]	Model 1	Model 2
10	0	2.4056	2.4052	2.4058	2.3477	2.3473	2.3478	2.1895	2.1892	2.1897	1.9685	1.9682	1.9686	1.7247	1.7244	1.7248
	0.3	3.9921	3.9906	4.0204	3.8959	3.8945	3.9236	3.6335	3.6322	3.6592	3.2667	3.2655	3.2898	2.8621	2.8611	2.8824
	1	5.3084	5.3086	5.3642	5.1805	5.1808	5.2350	4.8315	4.8317	4.8823	4.3437	4.3440	4.3894	3.8059	3.8060	3.8459
	3	6.6720	6.6780	6.7131	6.5113	6.5172	6.5515	6.0727	6.0781	6.1101	5.4596	5.4645	5.4932	4.7835	4.7878	4.8130
	10	8.1289	8.1338	8.1397	7.9332	7.9379	7.9437	7.3987	7.4031	7.4085	6.6518	6.6558	6.6606	5.8281	5.8316	5.8358
30	0	2.4603	2.4604	2.4634	2.4536	2.4537	2.4567	2.4336	2.4337	2.4367	2.4011	2.4011	2.4041	2.3570	2.3570	2.3599
	0.3	4.0811	4.0826	4.1195	4.0699	4.0714	4.1083	4.0368	4.0383	4.0749	3.9828	3.9843	4.0203	3.9096	3.9110	3.9464
	1	5.4146	5.4147	5.4797	5.3998	5.3999	5.4647	5.3559	5.3560	5.4203	5.2843	5.2843	5.3478	5.1871	5.1872	5.2495
	3	6.8011	6.8018	6.8473	6.7825	6.7832	6.8285	6.7273	6.7280	6.7730	6.6373	6.6380	6.6824	6.5153	6.5160	6.5595
	10	8.2962	8.2968	8.3143	8.2735	8.2741	8.2916	8.2062	8.2068	8.2241	8.0964	8.0970	8.1141	7.9476	7.9481	7.9649
100	0	2.4667	2.4668	2.4703	2.4661	2.4662	2.4696	2.4643	2.4643	2.4678	2.4613	2.4613	2.4648	2.4570	2.4571	2.4605
	0.3	4.0915	4.0933	4.1311	4.0905	4.0923	4.1301	4.0874	4.0893	4.1271	4.0824	4.0842	4.1220	4.0754	4.0772	4.1149
	1	5.4270	5.4271	5.4932	5.4257	5.4257	5.4918	5.4217	5.4217	5.4878	5.4150	5.4150	5.4810	5.4057	5.4057	5.4716
	3	6.8161	6.8162	6.8629	6.8144	6.8145	6.8612	6.8094	6.8095	6.8561	6.8010	6.8011	6.8476	6.7893	6.7894	6.8359
	10	8.3157	8.3158	8.3346	8.3136	8.3137	8.3326	8.3075	8.3076	8.3264	8.2972	8.2973	8.3162	8.2830	8.2831	8.3018

[a] Taken from Şimşek and Yurtçu (2013)
Model 1: Present theory with $\varepsilon_z = 0$
Model 2: Present theory with $\varepsilon_z \neq 0$



(a)



(b)

Fig. 2 Effect of the aspect ratio on (a) dimensionless deflection for uniform load; and (b) dimensionless buckling load for $k = 1$, $e_0a = 1$ nm

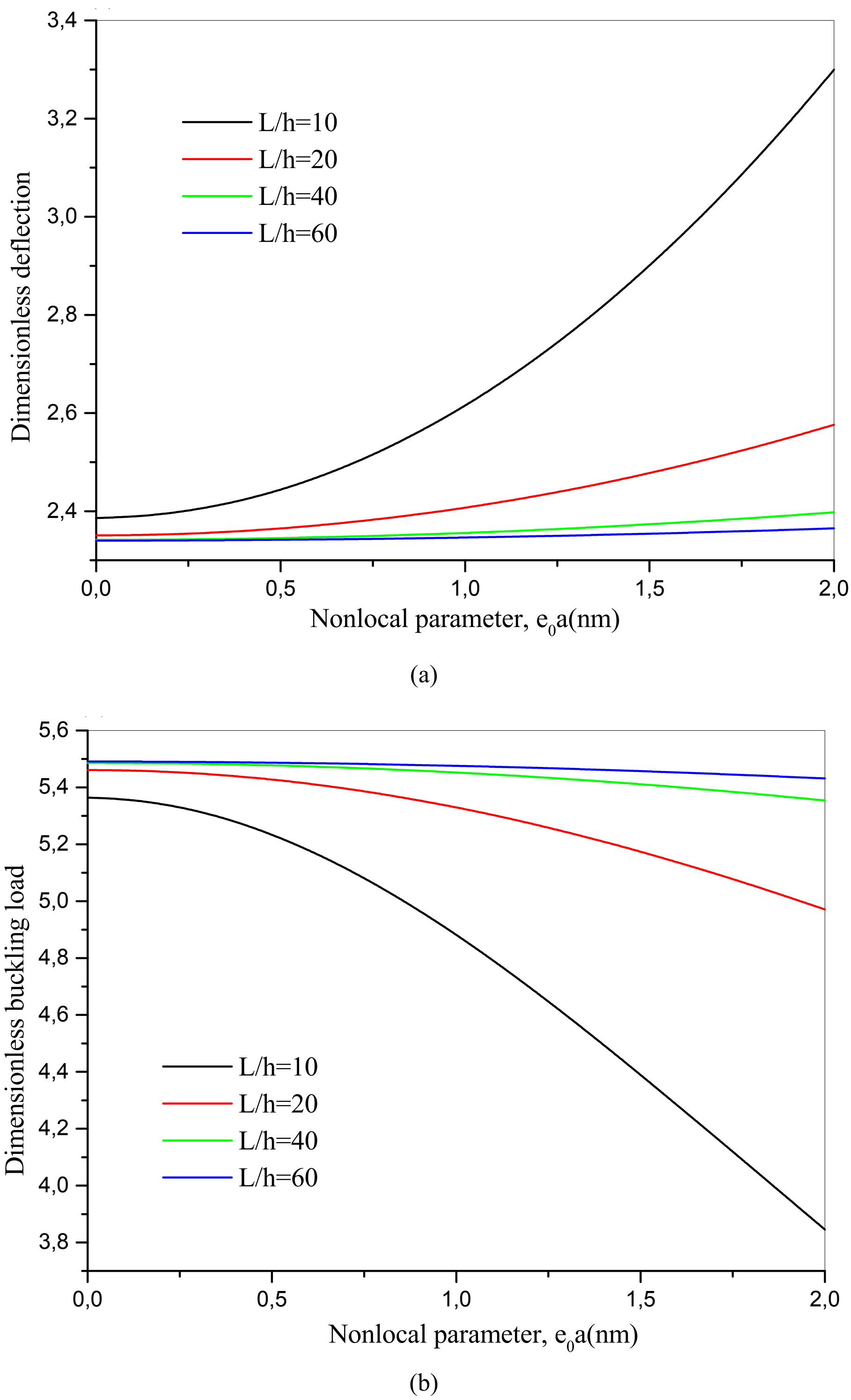


Fig. 3 Effect of nonlocal parameter on (a) dimensionless deflection for uniform load; and (b) dimensionless buckling load for $k = 1$

4. Conclusions

The bending and buckling analyses of FGM size-dependent nanoscale beams has been investigated on the basis of a nonlocal thickness-stretching sinusoidal shear deformation beam theory. The present model is capable of capturing small scale, shear deformation and thickness stretching effects of nanobeams, and additionally satisfies the zero traction boundary conditions on the top and bottom surfaces of the nanobeam without requiring a shear correction factor. Based on the nonlocal differential constitutive relation of Eringen, the nonlocal governing equations are derived using the principal of minimum total potential energy. The computations demonstrate that the inclusion of both small scale and thickness stretching effects elevates nanobeam stiffness, and hence, leads to a *reduction* of deflections and a corresponding increase of buckling loads. Therefore, the small scale and thickness stretching effects should be considered in the analysis of mechanical behavior of nanostructures. Further, it is found that, the material-distribution profile may be manipulated to select a specific design deflection and buckling load. The present computations also provide a solid benchmark for verification of finite element and other numerical simulations of FGM nanobeam mechanics.

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