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A new simple shear and normal deformations theory for functionally graded beams

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Abstract. In the present work, a simple and refined trigonometric higher-order beam theory is developed for bending and vibration of functionally graded beams. The beauty of this theory is that, in addition to modeling the displacement field with only 3 unknowns as in Timoshenko beam theory, the thickness stretching effect ($\varepsilon_z \neq 0$) is also included in the present theory. Thus, the present refined beam theory has fewer number of unknowns and equations of motion than the other shear and normal deformations theories, and it considers also the transverse shear deformation effects without requiring shear correction factors. The neutral surface position for such beams in which the material properties vary in the thickness direction is determined. Based on the present refined trigonometric higher-order beam theory and the neutral surface concept, the equations of motion are derived from Hamilton's principle. Numerical results of the present theory are compared with other theories to show the effect of the inclusion of transverse normal strain on the deflections and stresses.

Keywords: functionally graded beam; a simple third-unknown theory; normal strain; neutral surface position

1. Introduction

Functionally graded materials (FGMs) are a class of composites that have continuous variation in material properties from one surface to another and thus eliminate the stress concentration at the interface of the layers found in laminated composites. The most familiar FGM is compositionally graded from a refractory ceramic to a metal. Currently FGMs are widely used in aerospace, nuclear reactor, energy sources, biomechanical, optical, civil, automotive, electronic, chemical, mechanical, and shipbuilding industries.

Due to increasing of FGM applications in engineering structures, many beam theories have been developed to predict the response of functionally graded (FG) beams. It should be noted that

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the classical beam theory (CBT), which is based on the Euler–Bernoulli hypothesis, is suitable for slender FG beams only. For moderately deep FG beams, the CBT underestimates deflection and overestimates natural frequency due to ignoring the transverse shear deformation effect (Yang and Chen 2008, Bouremana *et al*. 2013). The first-order shear deformation beam theory (FBT) known as Timoshenko beam theory is simple to implement and applied for moderately deep FG beams and gives acceptable results but depends on a shear correction factor which is hard to find as it depends on many parameters (Bouremana *et al*. 2013, Chakraborty *et al*. 2003, Li 2008, Sina *et al*. 2009, Wei *et al*. 2012). However, there is no need of shear correction factors when using higher-order shear deformation theories (HPT). Among these theories we can cite the third-order theory of Reddy (Reddy 1984, Yesilce 2010, Yesilce and Catal 2009, 2011), the sinusoidal theories (Touratier 1991, Tounsi *et al*. 2013a, Bouderba *et al*. 2013, Bachir Bouiadjra *et al*. 2013, Ait Amar Meziane *et al*. 2014, Draiche *et al*. 2014), the hyperbolic theories (Soldatos 1992, El Meiche *et al*. 2011, Fekrar *et al*. 2012, Kettaf *et al*. 2013), the inverse hyperbolic theories (Sahoo and Singh 2013, Grover *et al*. 2013), the exponential theory of Karama *et al*. (2003), and the unified formulation of Carrera (Carrera 2003, Carrera *et al*. 2011a). Based on the assumption of a higher-order variation of axial displacement through the depth of the beam various higher- order shear deformation theories are also developed (Benatta *et al*. 2008, Kadoli *et al*. 2008, Sallai *et al*. 2009, Li *et al*. 2010, Mahi *et al*. 2011, Wattanasakulpong *et al*. 2011, Ould Larbi *et al*. 2013, Hadji *et al*. 2014).

Since, the material properties of functionally graded beam vary through the thickness direction, the neutral surface of such beam may not coincide with its geometric middle surface. Therefore, stretching and bending deformations of FG beam are coupled. Some researchers (Bousahla *et al*. 2014, Ould Larbi *et al*. 2013, Yahoobi and Feraidoon 2010, Morimoto *et al*. 2006) have shown that there is no stretching-bending coupling in constitutive equations if the reference surface is properly selected.

The purpose of this study is to develop a new shear and normal deformation beam theory for FG beams based on the concept of neutral surface position. Just third unknown displacement functions are used in the present theory against four or more unknown displacement functions used in the corresponding ones. The effects due to transverse shear and normal deformations are both included. The theory accounts for adequate distribution of the transverse shear stresses through the beam thickness and tangential stress-free boundary conditions on the beam boundary surface, thus a shear correction factor is not required. The effectiveness of the present theory is demonstrated and results are compared with the corresponding FGM solution.

2. Theoretical formulations

2.1 Physical neutral surface

A beam made of functionally graded materials with a uniform cross-section of area A, height h, and length L is considered here. The Cartesian coordinate system, (x, y, z), with the origin at the left end of the beam is used in this analysis. The xoy plane is taken to be the undeformed mid-plane of the beam, the x axis coincides with the centroidal axis of the beam, and the z axis is perpendicular to the $x-y$ plane. Due to asymmetry of material properties of FG beams with respect to middle plane, the stretching and bending equations are coupled. But, if the origin of the coordinate system is suitably selected in the thickness direction of the FG beam so as to be the neutral surface, the properties of the FG beam being symmetric with respect to it. To specify the

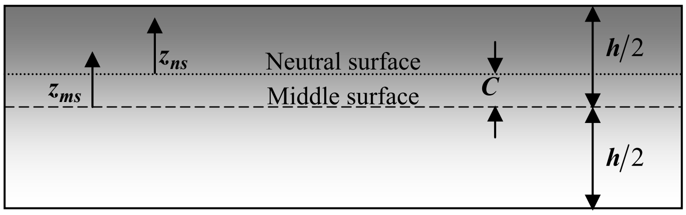

Fig. 1 The position of middle surface and neutral surface for a functionally graded beam

position of neutral surface of FG beams, two different planes are considered for the measurement of z, namely, z_{ms} and z_{ns} measured from the middle surface and the neutral surface of the beam, respectively, as depicted in Fig. 1.

The volume-fraction of ceramic V_C is expressed based on z_{ms} and z_{ns} coordinates as

$$V_C = \left(\frac{z_{ms}}{h} + \frac{1}{2}\right)^k = \left(\frac{z_{ns} + C}{h} + \frac{1}{2}\right)^k \tag{1}$$

where k is the power law index which takes the value greater or equal to zero and C is the distance of neutral surface from the mid-surface. Material non-homogeneous properties of a functionally graded beam may be obtained by means of the Voigt rule of mixture (Suresh and Mortensen 1998, Benachour *et al*. 2011, Bourada *et al*. 2012, Bachir Bouiadjra *et al*. 2012, Zidi *et al*. 2014, Klouche Djedid *et al*. 2014). Thus, using Eq. (1), the material non-homogeneous properties of FG beam P, as a function of thickness coordinate, become

$$P(z) = P_M + P_{CM}\left(\frac{z_{ns} + C}{h} + \frac{1}{2}\right)^k, \qquad P_{CM} = P_C - P_M \tag{2}$$

where P_M and P_C are the corresponding properties of the metal and ceramic, respectively. In the present work, we assume that the elasticity modules E and the mass density ρ are described by Eq. (2). The position of the neutral surface of the FG beam is determined to satisfy the first moment with respect to Young's modulus being zero as follows (Bouremana *et al*. 2013, Ould Larbi *et al*. 2013, Yahoobi and Feraidoon 2010)

$$\int_{-h/2}^{h/2} E(z_{ms})(z_{ms} - C)dz_{ms} = 0 \tag{3}$$

Consequently, the position of neutral surface can be obtained as

$$C = \frac{\int_{-h/2}^{h/2} E(z_{ms})z_{ms}dz_{ms}}{\int_{-h/2}^{h/2} E(z_{ms})dz_{ms}} \tag{4}$$

It is clear that the parameter C is zero for homogeneous isotropic beams, as expected.

2.2 Kinematics and constitutive equations

The displacement field of the present theory is chosen based on the following assumptions: (1) the origin of the Cartesian coordinate system is taken at the neutral surface of the FG beam; (2) the transverse displacement is partitioned into bending and thickness stretching components; (3) the axial displacement is partitioned into extension, bending and shear components; (4) the bending part of the axial displacement is similar to those given by CBT; and (5) the shear part of the axial displacement gives rise to the hyperbolic variations of shear strains and hence to shear stresses through the thickness of the beam in such a way that the shear stresses vanish on the top and bottom surfaces of the beam. Based on these assumptions, the following displacement field relations can be obtained

$$\begin{aligned} u(x,z_{ns},t) &= u_0(x,t) - z_{ns}\frac{\partial w_0}{\partial x} + f(z_{ns})\frac{\partial \varphi}{\partial x} \\ v(x,z_{ns},t) &= 0 \\ w(x,z_{ns},t) &= w_0(x,,t) + g(z_{ns})\,\varphi(x,t) \end{aligned} \tag{5}$$

In Eq. (5), u_0 and w_0 are the displacements of the neutral surface along the axes x, and z, respectively; and the additional displacement φ accounts for the effect of normal stress (thickness stretching effect). In this study, the shape functions $f(z_{ns})$ and $g(z_{ns})$ are chosen based on both the hyperbolic function proposed by Zenkour (2013) and the neutral surface concept as

$$f(z_{ns}) = h\sinh\left(\frac{z_{ns}+c}{h}\right) - \left(\frac{4(z_{ns}+c)^3}{3h^2}\right)\cosh\left(\frac{1}{2}\right), \quad \text{and} \quad g(z_{ns}) = \frac{1}{12}f'(z_{ns}) \tag{6}$$

The strains associated with the displacements in Eq. (5), there must be

$$\varepsilon_x = \varepsilon_x^0 + z_{ns}\,k_x + f(z_{ns})\,\eta_x \tag{7a}$$

$$\gamma_{xz} = \left[f'(z_{ns}) + g(z_{ns})\right]\gamma_{xz}^0 \tag{7b}$$

$$\varepsilon_z = g'(z_{ns})\,\varepsilon_z^0 \tag{7c}$$

where the prime denotes differentiation with respect to z_{ns} and, ε_x^0, k_x, η_x, γ_{xz}^0, ε_z^0 must be defined

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x}, \quad k_x = -\frac{\partial^2 w}{\partial x^2}, \quad \eta_x = \frac{\partial^2 \varphi}{\partial x^2}, \quad \gamma_{xz}^0 = \frac{\partial \varphi}{\partial x}, \quad \varepsilon_z^0 = \varphi \tag{8}$$

By assuming that the material of FG beam obeys Hooke’s law, the stresses in the beam become

$$\sigma_x = Q_{11}(z_{ns})\varepsilon_x + Q_{13}(z_{ns})\varepsilon_z, \quad \tau_{xz} = Q_{55}(z_{ns})\gamma_{xz}, \quad \text{and} \quad \sigma_z = Q_{13}(z_{ns})\varepsilon_x + Q_{33}(z_{ns})\varepsilon_z \tag{9a}$$

where

$$Q_{11}(z_{ns}) = Q_{33}(z_{ns}) = \frac{E(z_{ns})}{(1-\nu^2)}, \quad Q_{13}(z_{ns}) = \nu\, Q_{11}(z_{ns}), \quad \text{and} \quad Q_{55}(z_{ns}) = \frac{E(z_{ns})}{2(1+\nu)} \tag{9b}$$

2.3 Equations of motion

In order to derive the equations of motion, Hamilton's principle is used

$$\int_{t_1}^{t_2} (\delta U + \delta V - \delta K)dt = 0 \tag{10}$$

where U, K and V denote the strain energy, kinetic energy and the work done by external forces, respectively.

The variation of the strain energy can be stated as

$$\begin{aligned} \delta U &= \int_0^L \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} (\sigma_x \delta\, \varepsilon_x + \sigma_z \delta\, \varepsilon_z + \tau_{xz}\delta\, \gamma_{xz}) dz_{ns} dx \\ &= \int_0^L \left(N \frac{d\delta\, u_0}{dx} - M \frac{d^2\delta\, w_0}{dx^2} - P \frac{d^2\delta\, \varphi}{dx^2} + N_z \delta\, \varphi + Q \frac{d\delta\, \varphi}{dx} \right) dx \end{aligned} \tag{11}$$

where N, M, P, N_z and Q are the stress resultants defined as

$$\begin{aligned} (N, M, P) &= \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} (1, z_{ns}, f(z_{ns}))\, \sigma_x dz_{ns}, \quad N_z = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} \sigma_z g'(z_{ns}) dz_{ns}, \\ \text{and} \quad Q &= \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} \tau_{xz} \left[f'(z_{ns}) + g(z_{ns}) \right] dz_{ns} \end{aligned} \tag{12}$$

The variation of work done by externally transverse loads q can be expressed as

$$\delta V = -\int_0^L q \delta\, w dx \tag{13}$$

The variation of the kinetic energy is obtained as

$$\begin{aligned}
\delta K &= \int_0^L \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} \rho(z_{ns})\left[\dot{u}\delta\,\dot{u} + \dot{w}\delta\,\dot{w}\right]dz_{ns}dx \\
&= \int_0^L \{ I_0\left[\dot{u}_0\delta\dot{u}_0 + \dot{w}_0\delta\dot{w}_0\right] + J_0\left[\dot{w}_0\,\delta\,\dot{\varphi} + \dot{\varphi}\,\delta\,\dot{w}_0\right] \\
&\quad - I_1\left(\dot{u}_0\frac{d\delta\dot{w}_0}{dx} + \frac{d\dot{w}_0}{dx}\delta\,\dot{u}_0\right) + I_2\left(\frac{d\dot{w}_0}{dx}\frac{d\delta\,\dot{w}_0}{dx}\right) \\
&\quad - J_1\left(\dot{u}_0\frac{d\delta\dot{\varphi}}{dx} + \frac{d\dot{\varphi}}{dx}\delta\,\dot{u}_0\right) + J_2\left(\frac{d\dot{w}_0}{dx}\frac{d\delta\,\dot{\varphi}}{dx} + \frac{d\dot{\varphi}}{dx}\frac{d\delta\,\dot{w}_0}{dx}\right) + K_0\dot{\varphi}\,\delta\,\dot{\varphi}\}dx
\end{aligned} \tag{14}$$

Where dot-superscript convention indicates the differentiation with respect to the time variable t; and (I_i, J_i, K_i) are mass inertias defined as

$$\left(I_0, I_1, I_2\right) = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} \left(1, z_{ns}, z_{ns}^2\right)\rho(z_{ns})dz_{ns} \tag{15a}$$

$$\left(J_0, J_1, J_2\right) = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} \left(g, f, z_{ns}f\right)\rho(z_{ns})dz_{ns} \tag{15b}$$

$$\left(K_0, K_2\right) = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} \left(g^2, f^2\right)\rho(z_{ns})dz_{ns} \tag{15c}$$

Substituting the expressions for δU, δV, and δK from Eqs. (11), (13), and (14) into Eq. (10) and integrating by parts, and collecting the coefficients of δu_0, δw_0 and $\delta\varphi$, the following equations of motion of the FG beam are obtained

$$\delta\,u_0 : \ \frac{dN}{dx} = I_0\ddot{u}_0 - I_1\frac{d\ddot{w}_0}{dx} + J_1\frac{d\ddot{\varphi}}{dx} \tag{16a}$$

$$\delta\,w_0 : \ \frac{d^2M}{dx^2} + q = I_0\ddot{w}_0 + J_0\ddot{\varphi} + I_1\frac{d\ddot{u}_0}{dx} - I_2\frac{d^2\ddot{w}_0}{dx^2} + J_2\frac{d^2\ddot{\varphi}}{dx^2} \tag{16b}$$

$$\delta\,\varphi : \ \frac{d^2P}{dx^2} - \frac{dQ}{dx} + N_z = J_1\frac{d\ddot{u}_0}{dx} - J_0\ddot{w}_0 - J_2\frac{d^2\ddot{w}_0}{dx^2} - K_0\ddot{\varphi} + K_2\frac{d^2\ddot{\varphi}}{dx^2} \tag{16c}$$

Equations (16) can be expressed in terms of displacements (u_0, w_0 and φ) by using Eqs. (5), (7), (9) and (12) as follows

$$A_{11}\frac{\partial^2 u_0}{\partial x^2}+B_{11}^s\frac{\partial^3\varphi}{\partial x^3}+L\frac{\partial\varphi}{\partial x}=I_0\ddot{u}_0-I_1\frac{d\ddot{w}_0}{dx}+J_1\frac{d\ddot{\varphi}}{dx} \tag{17a}$$

$$-D_{11}\frac{\partial^4 w_0}{\partial x^4}+D_{11}^s\frac{\partial^4\varphi}{\partial x^4}+L^a\frac{\partial^2\varphi}{\partial x^2}+q=I_0\ddot{w}_0+J_0\ddot{\varphi}+I_1\frac{d\ddot{u}_0}{dx}-I_2\frac{d^2\ddot{w}_0}{dx^2}+J_2\frac{d^2\ddot{\varphi}}{dx^2} \tag{17b}$$

$$\begin{aligned}&L\frac{\partial u_0}{\partial x}-L^a\frac{\partial^2 w_0}{\partial x^2}+2R\frac{\partial^2\varphi}{\partial x^2}+R^a\varphi+B_{11}^s\frac{\partial^3 u_0}{\partial x^3}-D_{11}^s\frac{\partial^4 w_0}{\partial x^4}+H_{11}^s\frac{\partial^4\varphi}{\partial x^4}-A_{55}^s\frac{\partial^2\varphi}{\partial x^2}\\&=J_1\frac{d\ddot{u}_0}{dx}-J_0\ddot{w}_0-J_2\frac{d^2\ddot{w}_0}{dx^2}-K_0\ddot{\varphi}+K_2\frac{d^2\ddot{\varphi}}{dx^2}\end{aligned} \tag{17c}$$

Where A_{11}, D_{11}, etc., are the beam stiffness, defined by

$$\left(A_{11},D_{11},B_{11}^s,D_{11}^s,H_{11}^s\right)=\int_{-\frac{h}{2}-C}^{\frac{h}{2}-C}Q_{11}\left(1,z^2,f(z_{ns}),z_{ns}\,f(z_{ns}),f^2(z_{ns})\right)dz_{ns} \tag{18}$$

and

$$A_{44}^s=A_{55}^s=\int_{-\frac{h}{2}-C}^{\frac{h}{2}-C}Q_{55}\left[g(z_{ns})\right]^2dz_{ns},\quad \left[L,L^a,R,R^a\right]=Q_{13}\int_{-\frac{h}{2}-C}^{\frac{h}{2}-C}\left[1,z_{ns},f(z_{ns}),g'(z_{ns})\right]g'(z_{ns})dz_{ns} \tag{19}$$

3. Analytical solution

The equations of motion admit the Navier solutions for simply supported beams. The variables u_0,
w_0 and φ can be written by assuming the following variations

$$\begin{Bmatrix}u_0\\w\\\varphi\end{Bmatrix}=\sum_{m=1}^{\infty}\begin{Bmatrix}U_m\cos(\lambda\,x)e^{i\omega t}\\W_m\sin(\lambda\,x)\,e^{i\omega t}\\\Phi_m\sin(\lambda\,x)e^{i\omega t}\end{Bmatrix} \tag{20}$$

where U_m, W_m and Φ_m are arbitrary parameters to be determined, ω is the eigenfrequency associated with mth eigenmode, and $\lambda = m\pi/L$. The transverse load q is also expanded in Fourier series as

$$q(x)=\sum_{m=1}^{\infty}Q_m\sin(\lambda\,x) \tag{21}$$

where Q_m is the load amplitude calculated from

$$Q_m = \frac{2}{L}\int_0^L q(x)\sin(\lambda\, x)dx \tag{22}$$

For the case of uniform distributed load, we have

$$Q_m = \frac{4q_0}{m\pi}, \quad (m = 1, 3, 5....) \tag{23}$$

Substituting the expansions of u_0, w, φ, and q from Eqs. (20) and (21) into the equations of motion Eq. (17) the analytical solutions can be obtained from the following equations

$$\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix}\right)\begin{Bmatrix} U_m \\ W_m \\ \Phi_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ Q_m \\ 0 \end{Bmatrix} \tag{24}$$

where

$$\begin{aligned} &a_{11} = A_{11}\lambda^2, \quad a_{12} = -B_{11}\lambda^3, \quad a_{13} = \lambda\left(B_{11}^s\lambda^2 - L\right), \\ &a_{22} = D_{11}\lambda^4, \quad a_{23} = \lambda^2(L^a - \lambda^2 D_{11}^S), \quad a_{33} = R^a + \lambda^2(A_{55}^s - 2R) + H_{11}^s\lambda^4 \end{aligned} \tag{25a}$$

$$\begin{aligned} &m_{11} = I_0, \quad m_{12} = -I_1\lambda, \quad m_{13} = J_1\lambda, \\ &m_{22} = I_0 + I_2\lambda^2, \quad m_{23} = -J_2\lambda^2 + J_0, \quad m_{33} = K_0 + K_2\lambda^2 \end{aligned} \tag{25b}$$

4. Numerical results

The bending and free vibration analysis of simply supported FG beams is illustrated in what follows. The results are determined from the new simple and accurate trigonometric higher-order beam theory. The theory is formulated in such way that the thickness stretching effect is taken into account, i.e., the Koiter's recommendation regarding stretching effect of the structure (Koiter 1959) is obeyed.

The FG beam is taken to be made of aluminum and alumina with the following material properties:

Ceramic (P_C: Alumina, Al_2O_3): E_c = 380 GPa; ν = 0.3; ρ_c 3960 kg/m^3.

Metal (P_M: Aluminium, Al): E_m = 70 GPa; ν = 0.3; ρ_m = 2707 kg/m^3.

For convenience, the following dimensionless forms are used

$$\begin{aligned} &\bar{w} = 100\frac{E_m h^3}{q_0 L^4} w\left(\frac{L}{2}\right), \quad \bar{u} = 100\frac{E_m h^3}{q_0 L^4} u\left(0, -\frac{h}{2} - C\right), \quad \bar{\sigma}_x = \frac{h}{q_0 L}\sigma_x\left(\frac{L}{2}, \frac{h}{2} - C\right), \\ &\bar{\tau}_{xz} = \frac{h}{q_0 L}\tau_{xz}\left(0, -C\right), \quad \bar{\omega} = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}} \end{aligned}$$

4.1 Bending analysis

The static results in this section are compared with: (a) the analytical solutions given by Li *et al*. (2010) and the efficient shear deformation beam theory proposed by Ould Larbi *et al*. (2013); and (b) a recently developed first shear deformation beam theory (Bouremana *et al*. 2013). For this aim, Table 1 contains nondimensional displacements and stresses of FG beams under uniform load $q0$ for different values of power law index k and span-to-depth ratio L / h. Results are in good agreements with the published results. The small difference observed between the results is due to the effect of thickness stretching which is omitted in other theories (Bouremana *et al*. 2013, Li *et al*. 2010, Ould Larbi *et al*. 2013). The importance of thickness stretching effect is studied by Carrera *et al*. (2011b), Houari *et al*. (2013), Bessaim *et al*. (2013), Saidi *et al*. (2013), Hebali *et al*.

Table 1 Nondimensional deflections and stresses of FG beams under uniform load

k	Method	$L/h=5$				$L/h=20$			
		$\overline{w}$	$\overline{u}$	$\overline{\sigma}_x$	$\overline{\tau}_{xz}$	$\overline{w}$	$\overline{u}$	$\overline{\sigma}_x$	$\overline{\tau}_{xz}$
0	Li *et al*. (2010) $\varepsilon_z = 0$	3.1657	0.9402	3.8020	0.7500	2.8962	0.2306	15.0130	0.7500
	Ould Larbi *et al*. (2013) $\varepsilon_z = 0$	3.1651	0.9406	3.8043	0.7489	2.8962	0.2305	15.0136	0.7625
	Bouremana *et al*. (2013) $\varepsilon_z = 0$	3.1657	0.9209	3.7500	0.5990	2.8963	0.2303	15.0000	0.5993
	Present $\varepsilon_z \neq 0$	3.1357	0.9261	3.8614	0.7438	2.8906	0.2300	15.2708	0.7656
0.5	Li *et al*. (2010) $\varepsilon_z = 0$	4.8292	1.6603	4.9925	0.7676	4.4645	0.4087	19.7005	0.7676
	Ould Larbi *et al*. (2013) $\varepsilon_z = 0$	4.8282	1.6608	4.9956	0.7660	4.4644	0.4087	19.7013	0.7795
	Bouremana *et al*. (2013) $\varepsilon_z = 0$	4.8347	1.6331	4.9206	0.6270	4.4648	0.4083	19.6825	0.6266
	Present $\varepsilon_z \neq 0$	4.7584	1.6124	5.0789	0.7604	4.4292	0.4010	20.0787	0.7824
1	Li *et al*. (2010) $\varepsilon_z = 0$	6.2599	2.3045	5.8837	0.7500	5.8049	0.5686	23.2054	0.7500
	Ould Larbi *et al*. (2013) $\varepsilon_z = 0$	6.2590	2.3052	5.8875	0.7489	5.8049	0.5685	23.2063	0.7625
	Bouremana *et al*. (2013) $\varepsilon_z = 0$	6.2600	2.2722	5.7959	0.5988	5.8050	0.5681	23.1834	0.5995
	Present $\varepsilon_z \neq 0$	6.1271	2.2162	5.9841	0.7438	5.7131	0.5515	23.6405	0.7656
2	Li *et al*. (2010) $\varepsilon_z = 0$	8.0602	3.1134	6.8812	0.6787	7.4415	0.7691	27.0989	0.6787
	Ould Larbi *et al*. (2013) $\varepsilon_z = 0$	8.0683	3.1146	6.8878	0.6870	7.4421	0.7691	27.1005	0.7005
	Bouremana *et al*. (2013) $\varepsilon_z = 0$	8.0307	3.0741	6.7678	0.5101	7.4400	0.7686	27.0704	0.5102
	Present $\varepsilon_z \neq 0$	7.8501	2.9703	6.9957	0.6838	7.2688	0.7390	27.5763	0.7044
5	Li *et al*. (2010) $\varepsilon_z = 0$	9.7802	3.7089	8.1030	0.5790	8.8151	0.9133	31.8112	0.5790
	Ould Larbi *et al*. (2013) $\varepsilon_z = 0$	9.8345	3.7128	8.1187	0.6084	8.8186	0.9134	31.8151	0.6218
	Bouremana *et al*. (2013) $\varepsilon_z = 0$	9.6484	3.6496	7.9427	0.3926	8.8068	0.9120	31.7710	0.3927
	Present $\varepsilon_z \neq 0$	9.6028	3.5488	8.2440	0.6079	8.6396	0.8798	32.3457	0.6271
10	Li *et al*. (2010) $\varepsilon_z = 0$	10.8979	3.8860	9.7063	0.6436	9.6879	0.9536	38.1372	0.6436
	Ould Larbi *et al*. (2013) $\varepsilon_z = 0$	10.9413	3.8898	9.7203	0.6640	9.6907	0.9537	38.1408	0.6788
	Bouremana *et al*. (2013) $\varepsilon_z = 0$	10.7194	3.8098	9.5231	0.4288	9.6770	0.9524	38.0915	0.4292
	Present $\varepsilon_z \neq 0$	10.7561	3.7501	9.8597	0.6625	9.5715	0.9278	38.7327	0.6835

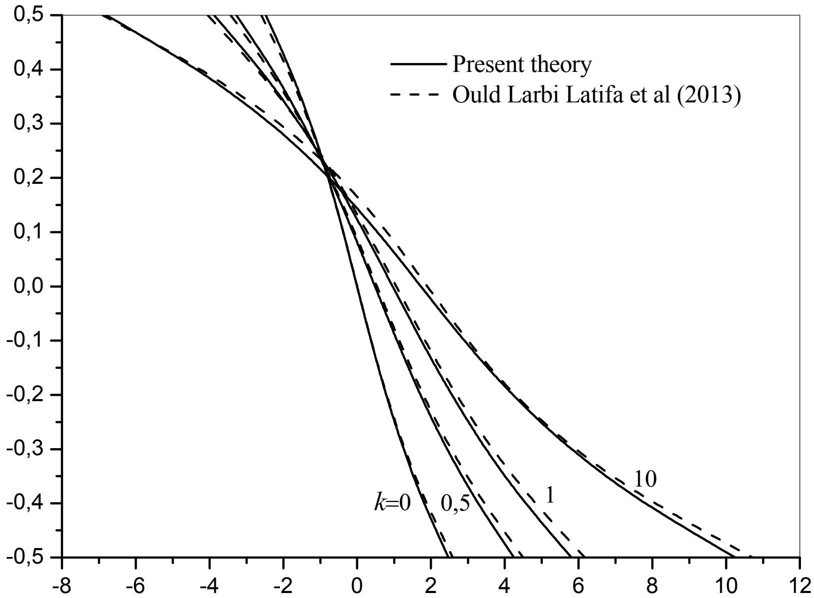

Fig. 2 The variation of the axial displacement $\bar{u}$ through-the-thickness of a FG beam ($L = 2h$)

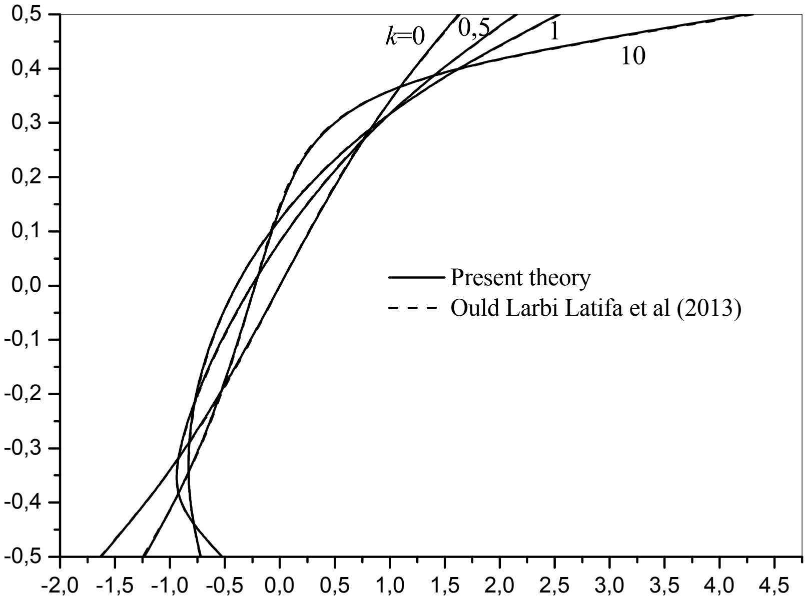

Fig. 3 The variation of the axial stress $\bar{\sigma}_x$ through-the-thickness of a FG beam ($L = 2h$)

(2014), Fekrar *et al*. (2014), Belabed *et al*. (2014). It should be taken into consideration in the modeling of thick and moderately thick plates.

The through thickness variations of the axial displacement ($\bar{u}_0$) and stresses ($\bar{\sigma}_x$ and $\bar{\tau}_{xz}$) are illustrated in Figs. 2-4 for thick beams with $L/h = 2$ and under uniform load. Again, the results

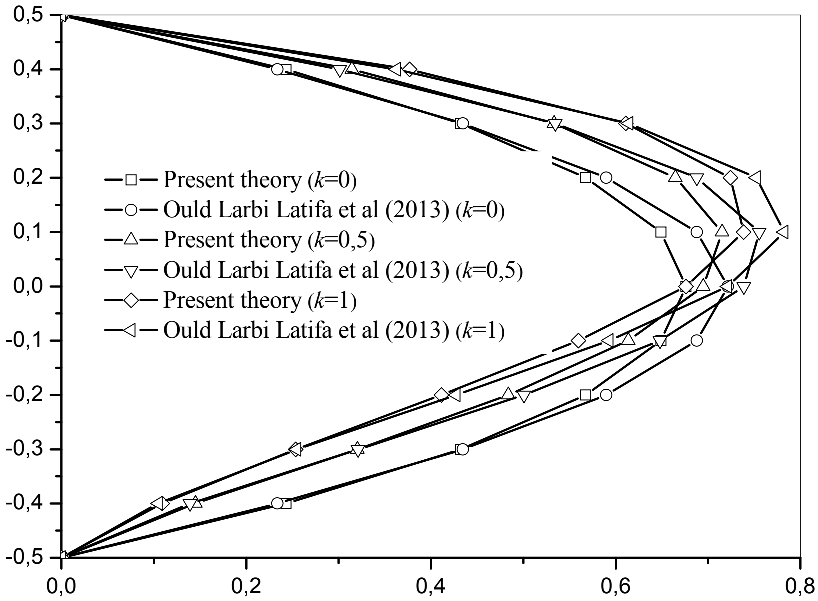

Fig. 4 The variation of the transverse shear stress $\bar{\tau}_{xz}$ through-the-thickness of a FG beam (L = 2h)

Table 2 First three nondimensional frequencies $\bar{\omega}$ of FG beams

L/h	Mode	Theory	k					
			0	0.5	1	2	5	10
5	1	CBT	5.3953	4.5931	4.1484	3.7793	3.5949	3.4921
		Ould Larbi *et al*. (2013) $\varepsilon_z = 0$	5.1529	4.4108	3.9905	3.6263	3.4001	3.2812
		Present $\varepsilon_z \neq 0$	5.1665	4.4347	4.0271	3.6723	3.4374	3.3048
	2	CBT	20.6187	17.5415	15.7982	14.3260	13.5876	13.2376
		Ould Larbi *et al*. (2013) $\varepsilon_z = 0$	17.8844	15.4613	14.0121	12.6404	11.5349	11.0216
		Present $\varepsilon_z \neq 0$	17.9979	15.5965	14.1780	12.8232	11.6761	11.1231
	3	CBT	43.3483	36.8308	33.0278	29.7458	28.0850	27.4752
		Ould Larbi *et al*. (2013) $\varepsilon_z = 0$	34.2248	29.8496	27.1085	24.3196	21.6987	20.5555
		P Present $\varepsilon_z \neq 0$	34.5558	30.2017	27.4946	24.7075	21.9842	20.7758
20	1	CBT	5.4777	4.6641	4.2163	3.8472	3.6628	3.5547
		Ould Larbi *et al*. (2013) $\varepsilon_z = 0$	5.4603	4.6511	4.2051	3.8361	3.6484	3.5389
		Present $\varepsilon_z \neq 0$	5.4650	4.6690	4.2383	3.8813	3.6858	3.5606
	2	CBT	21.8438	18.5987	16.8100	15.3334	14.5959	14.1676
		Ould Larbi *et al*. (2013) $\varepsilon_z = 0$	21.5734	18.3964	16.6345	15.1617	14.3732	13.9257
		Present $\varepsilon_z \neq 0$	21.6003	18.4736	16.7706	15.3434	14.5227	14.0144
	3	CBT	48.8999	41.6328	37.6173	34.2954	32.6357	31.6883
		Ould Larbi *et al*. (2013) $\varepsilon_z = 0$	47.5940	40.6534	36.7686	33.4681	31.5719	30.5342
		Present $\varepsilon_z \neq 0$	47.6822	40.8457	37.0853	33.8792	31.9080	30.7403

obtained by the new simple trigonometric higher-order beam theory are in good agreement with those predicted by the efficient shear deformation beam theory proposed by Ould Larbi *et al*. (2013). As is noted above, the beam theory proposed by Ould Larbi *et al*. (2013) slightly overestimates the axial displacement and stresses due to ignoring the thickness stretching effect.

4.2 Free vibration analysis

In this section, the vibration results obtained by the present theory are compared with those of Ould Larbi *et al*. (2013) and CBT. Table 2 presents the first three nondimensional frequencies of FG beams for different values of power law index k and span-to-depth ratio L/h. Results are in good agreements with the published result of Ould Larbi *et al*. (2013). As is indicated in the above section, the small difference observed between the results obtained by the present theory and Ould Larbi *et al*. (2013) is due to the effect of thickness stretching which is omitted this latter (Ould Larbi *et al*. (2013). It can be observed also that there is a remarkable difference between the frequencies of CBT and those of shear deformable beam theories for thicker FG beam.

5. Conclusions

A simple and refined trigonometric higher-order beam theory with thickness stretching effect for the bending and vibration analysis of FG beams is presented. The equations of motion are derived by employing the Hamilton's principle. Results show that the present theory is able to include the thickness stretching effect and providing very accurate results compared with the other existing higher-order beam theories. The formulation lends itself particularly well in analysing nanostructures (Heireche *et al*. 2008, Tounsi *et al*. 2008, Benzair *et al*. 2008, Amara *et al*. 2010, Tounsi *et al*. 2010, 2013b, c, Berrabah *et al*. 2013) which will be considered in the near future.

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