Stability of EG cylindrical shells with shear stresses on a Pasternak foundation

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Abstract. This article is the result of an investigation on the influence of a Pasternak elastic foundation on the stability of exponentially graded (EG) cylindrical shells under hydrostatic pressure, based on the first-order shear deformation theory (FOSDT) considering the shear stresses. The shear stresses shape function is distributed parabolic manner through the shell thickness. The governing equations of EG orthotropic cylindrical shells resting on the Pasternak elastic foundation on the basis of FOSDT are derived in the framework of Donnell-type shell theory. The novelty of present work is to achieve closed-form solutions for critical hydrostatic pressures of EG orthotropic cylindrical shells resting on Pasternak elastic foundation based on FOSDT. The expressions for critical hydrostatic pressures of EG orthotropic cylindrical shells with and without an elastic foundation based on CST are obtained, in special cases. Finally, the effects of Pasternak foundation, shear stresses, orthotropy and heterogeneity on critical hydrostatic pressures, based on FOSDT are investigated.

1. Introduction

Anisotropic composite cylindrical shells are widely used as a structural member in many engineering applications. In some practical applications, thin composite shells are in contact with an elastic foundation. A brief review of elastic foundation models is discussed in the studies of Hui and Hansen (1980), Gorbunov-Posadv *et al*. (1984) and Hui (1986). The influence of an elastic foundation on the stability and vibration of homogeneous isotropic and orthotropic cylindrical shells is well studied in the literature. Sofiyev and Marandi (1996) examined the dynamic stability problem of non-homogeneous isotropic cylindrical shells on elastic foundations. Ng and Lam (1999) studied the effect of elastic foundation on the dynamic stability of cylindrical shells. The

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buckling analysis of short cylindrical shells surrounded by an elastic medium was carried out by Naili and Oddou (2000) and Croll (2001) using a ring model. Paliwal and Pandey (2001) presented free vibrations of an orthotropic thin cylindrical shell on an elastic foundation. Fok (2002) studied the buckling of a long cylindrical shell embedded in an elastic medium by the Rayleigh–Ritz method. Tornabene (2011) investigated free vibrations of anisotropic doubly-curved shells and panels of revolution with a free-form meridian resting on Winkler-Pasternak elastic foundations. Akoz and Ergun (2012) presented Pasternak foundation model, which is a two parameter foundation model, is used to analyze the behavior of laterally loaded beams embedded in semi-infinite media. Shen (2013) studied postbuckling of axially-loaded laminated cylindrical shells surrounded by an elastic medium.

Non-homogeneous materials are widely used in engineering design and modern technology to enhance structural strength. Non-homogeneity of materials can be attributed to the effects of humidity, radiation, high temperature and manufacturing process, etc. Significant contributions to the theory of elasticity of non-homogeneous materials and designs have been brought in the study of Lomakin (1976). Thereafter, some studies on the behaviors of non-homogeneous structural elements have been published (Grigorenko and Vasilenko 1992, Sofiyev *et al*. 2009). Recently, a new class of composite materials known as functionally graded materials (FGMs) has drawn considerable attention. In order to take the oriented structure of FGMs, these materials are generally modeled as orthotropic with principal directions (Kar and Kanoria 2009, Wosu *et al*. 2012). Both analytical and computational methods are developed to examine different problems in heterogeneous or FG orthotropic composite structures. Pan (2003) presented an exact solution for a simply supported rectangular FG anisotropic laminated plate using the pseudo-Stroh formalism extending Pagano's solution to the FG plates. Chen *et al*. (2004a) presented thermal fracture analysis of a functionally graded orthotropic strip, where the crack is situated parallel to the free edges. Chen *et al*. (2004b) studied the three-dimensional free vibration of simply supported, fluid-filled cylindrically orthotropic FG cylindrical shells with arbitrary thickness. Batra and Jin (2005) studied natural frequencies of a FG graphite/epoxy rectangular plate based on first order shear deformation. Pelletier and Vel (2006) investigated an exact solution for the steady-state thermo-elastic response of FG orthotropic cylindrical shells using Flügge and Donnell shell theories. Ramirez *et al*. (2006) examined static analysis of FG orthotropic plates using a discrete layer approach in combination with the Ritz method. Ootao and Tanigawa (2007) examined three-dimensional solution for transient thermal stresses of an orthotropic FG rectangular plate using Laplace and finite cosine transformation methods. Baron (2011) investigated propagation of elastic waves in the anisotropic hollow cylinder with elastic properties (stiffness coefficients and mass density) functionally varying in the radial direction based on the sextic Stroh's formalism and an analytical solution, the matricant, explicitly expressed under the Peano series expansion form. Peng and Li (2012) investigated the influence of orthotropy and gradient on the elastic field in particular the hoop stress distribution in hollow annular plates rotating at constant angular speed about its axis. Overview of static and dynamic problems of isotropic and anisotropic shells with variable parameters can be found in the study of Grigorenko and Grigorenko (2013). Mantari and Soares (2014) presented sinusoidal higher order shear deformation theory for the analysis of functionally graded plates and shells. However, the research works for FG orthotropic plates and shells on elastic foundations are rare in the literature. Morimoto and Tanigawa (2007) studied the elastic stability of FG orthotropic plates on a Winkler elastic foundation under in-plane compression. Sofiyev (2011) studied the hermal buckling behavior of FGM shells resting on a two parameter elastic foundation. Bagherizadeh *et al*. (2011) presented mechanical buckling of func-

tionally graded material cylindrical shells surrounded by elastic foundation. Kumar and Lal (2012) investigated vibrations of non-homogeneous orthotropic rectangular plates with bilinear thickness variation resting on Winkler foundation. Sofiyev *et al*. (2012) presented the effect of the Winkler-Pasternak type elastic foundations on the stability and vibration behaviors of non-homogeneous orthotropic shells. Najafov *et al*. (2013) studied the torsional vibration and stability of functionally graded orthotropic cylindrical shells on Winkler-Pasternak type elastic foundations. Firouz-Abadi *et al*. (2013) presented whirling frequencies of thin spinning cylindrical shells resting on the Winkler elastic foundation. Shen and Wang (2013) presented thermal buckling and postbuckling behavior for fiber reinforced composite laminated cylindrical shells embedded in a large outer elastic medium and subjected to a uniform temperature rise. Sofiyev *et al*. (2014) investigated the stability analysis of clamped nonhomogeneous shells on Winkler elastic foundation. Shariyat and Asemi (2014) presented shear buckling analysis of rectangular orthotropic FG plates surrounded by elastic foundations using three-dimensional non-linear elasticity-based 3D cubic B-spline finite element.

In above mentioned studies, the materials of plates and cylindrical shells were assumed to be inhomogeneous orthotropic such as FG orthotropic and for derivation of the basic equations classic shell theory (CST) was used. The shear stresses (or deformation) play a significant role in the stability behavior of shells composed of traditional and new generation composites. As the effect of shear stresses is not considered, it can lead to significant errors for the buckling loads of homogeneous (H) composite cylindrical shells. Thus, the shear deformation theory (SDT) becomes more interesting than the CST. Due to the increased relevance of heterogeneous orthotropic cylindrical shells in the design of composite structures, their buckling characteristics with account taken of combined effect of non-homogeneity and shear deformation (or stresses) has vital importance. However, investigations involving the application of shear-deformable shell theories for the buckling analysis are limited in number. Shirakawa (1983) investigated effects of shear deformation and rotary inertia on the buckling and vibration of cylindrical shells. Palazotto and Linnemann (1991) studied the buckling and vibration characteristics of composite cylindrical panels incorporating effects of a higher-order shear theory. Han and Simitses (1991) investigated buckling behavior of symmetric laminates composite cylindrical shell subjected to lateral or hydrostatic pressure based on Sanders-type of first-order shear deformation theory (FOSDT). Soldatos and Timarci (1993) presented a unified formulation of laminated composite, shear deformable, five-degrees-of-freedom cylindrical shell theories. Kardomateas (1997) presented Koiter-based solution for the initial postbuckling behavior of moderately thick orthotropic and shear deformable cylindrical shells under external pressure. Eslami and Shariyat (1999) developed a higher order shear deformation theory to study the dynamic buckling and postbuckling of thick composite cylindrical shells and the solution was sought on the basis of numerical methods. Shen (2008) used the boundary layer theory for the buckling and post-buckling of an anisotropic laminated cylindrical shell with the shear deformation under the external pressure. Civalek (2008) investigated vibration analysis of conical panels using the method of discrete singular convolution. Li and Lin (2010) studied the buckling and post-buckling of shear deformable anisotropic composite cylindrical shell subjected to various external pressure loads. Ferreira *et al*. (2011) investigated buckling analysis of isotropic and laminated plates by radial basis functions according to a higher-order shear deformation. Asadi and Qatu (2012) presented static analysis of thick laminated shells with different boundary conditions, using two first order shear deformation theories (FOSDTs). Ádány (2014) examined flexural buckling of simply-supported thin-walled columns with consideration of membrane shear deformations, based on shell model. Sofiyev and

Kuruoglu (2014) studied buckling and vibration of shear deformable functionally graded orthotropic cylindrical shells under external pressures. Jung and Han (2014) studied the initial buckling response of laminated composite plates and shells under the combined in-plane loading using a finite element method, based on a modified FOSDT.

There are very few studies on the static and dynamic behaviors of shear deformable heterogeneous structural elements resting on elastic foundations. Alipour *et al.* (2010) presented a semi-analytical solution for free vibration of variable thickness two-directional-functionally graded plates on elastic foundations based on FOSDT. Atmane *et al.* (2011) investigated free vibration analysis of functionally graded plates resting on Winkler–Pasternak elastic foundations using a new shear deformation theory. Thai and Choi (2012) presented a refined shear deformation theory for free vibration of functionally graded plates on an elastic foundation. Bouderba *et al.* (2013) studied thermomechanical bending response of FGM thick plates resting on Winkler-Pasternak elastic foundations based on refined trigonometric shear deformation theory. Zenkour *et al.* (2013) examined bending of cross-ply laminated plates resting on two-parameter elastic foundations under thermo-mechanical loading using a unified shear deformation plate theory. Tornabene *et al.* (2014) studied the Winkler-Pasternak foundation effect on the static and dynamic analyses of laminated doubly-curved and degenerate shells and panels.

In this study, the stability behavior of EG orthotropic cylindrical shells including shear stresses resting on a Pasternak elastic foundation under a uniform hydrostatic pressure is investigated. The expressions for the dimensionless critical hydrostatic pressures of EG orthotropic cylindrical shells resting on a Pasternak elastic foundation, based on FOSDT and CST are obtained. The shear stresses shape function is distributed parabolic manner through the shell thickness. The effects of the Pasternak elastic foundation, shear stresses, material heterogeneity, material orthotropy and shell characteristics on the values of critical hydrostatic pressures are examined independently.

2. Formulation of the problem

Fig. 1 shows the nomenclature of a circular cylindrical shell resting on a Pasternak elastic
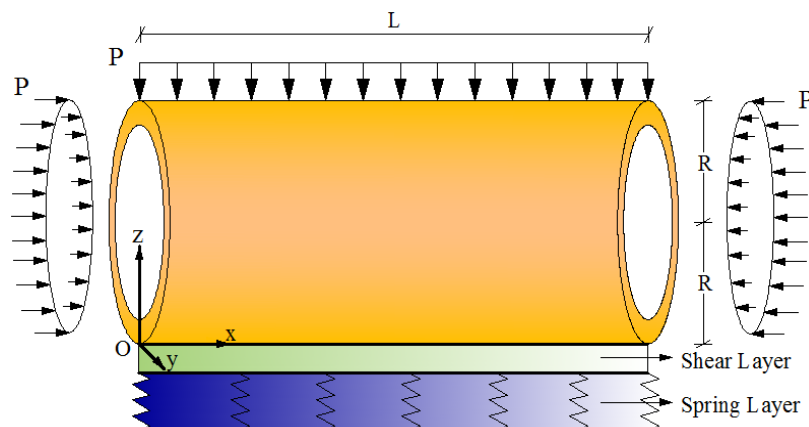

Fig. 1 Nomenclature and coordinate system of a cylindrical shell resting on a Pasternak elastic foundation and subjected to a uniform hydrostatic pressure

foundation with radius R, axial length L and thickness h. The cylindrical shell subjected to the uniform hydrostatic pressure, P. The origin of the coordinate system ($Oxyz$) is located at the end of the cylindrical shell on the reference surface. The parameters x, y and z denote length in the axial, circumferential and normal to the reference surface direction, respectively. The load-displacement relationship of the foundation is assumed to be $p_0 = K_w w - K_p (w_{,xx} + w_{,yy})$, where p_0 is the force per unit area, K_w (N/m^3) is the Winkler foundation stiffness, K_p (N/m) is shearing layer stiffness of the foundation, w is the displacement and a comma denotes partial differentiation with respect to the corresponding coordinates (Shen 2013). Let $\Phi(x, y)$ be the stress function for the stress resultants defined by $T_x = h\Phi_{,yy}$ $T_{xy} = -h\Phi_{,xy}$ and $T_y = h\Phi_{,xx}$. Assume that the Young's moduli and shear moduli of the orthotropic shell are exponential function of the coordinate in the thickness direction (Pan 2003, Ootao and Tanigawa 2007).

3. Governing relations and equations

The equations relating the stresses to strains for an EG orthotropic cylindrical shell, in term of structural axes coordinates are given by the following matrix equation (Ootao and Tanigawa 2007)

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} B_{11}(Z) & B_{12}(Z) & 0 & 0 & 0 \\ B_{21}(Z) & B_{22}(Z) & 0 & 0 & 0 \\ 0 & 0 & B_{44}(Z) & 0 & 0 \\ 0 & 0 & 0 & B_{55}(Z) & 0 \\ 0 & 0 & 0 & 0 & B_{66}(Z) \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} \tag{1}$$

where σ_x, σ_y, σ_{yz}, σ_{xz}, σ_{xy} are the stresses, ε_x, ε_y, γ_{yz}, γ_{xz}, γ_{xy} are the strains of the cylindrical shell and the quantities $B_{ij}(Z)$, ($i, j = 1, 2, \ldots, 6$) are

$$B_{11}(Z) = \frac{E_{01}e^{\mu(Z-0.5)}}{1-v_{12}v_{21}}, \quad B_{12}(Z) = \frac{v_{21}E_{01}e^{\mu(Z-0.5)}}{1-v_{12}v_{21}} = \frac{v_{12}E_{02}e^{\mu(Z-0.5)}}{1-v_{12}v_{21}} = B_{21}(Z),$$

$$B_{22}(Z) = \frac{E_{02}e^{\mu(Z-0.5)}}{1-v_{12}v_{21}}, \quad B_{44}(Z) = G_{023}e^{\mu(Z-0.5)}, \quad B_{55}(Z) = G_{013}e^{\mu(Z-0.5)}, \tag{2}$$

$$B_{66}(Z) = G_{012}e^{\mu(Z-0.5)}$$

where E_{01} and E_{02} are Young's moduli of the homogeneous orthotropic material along x and y directions, respectively; G_{012}, G_{013}, G_{023} are shear moduli which characterize angular chances between principal directions x and y, x and z, y and z, respectively; where v_{12} and v_{21} are Poisson ratios of the orthotropic cylindrical shell, which are constant and μ is the variation coefficient of Young's moduli and shear moduli satisfying $0 \leq \mu \leq 1$.

The shear stresses of cylindrical shells varies depending on the thickness coordinate as follows (Ambartsumian 1964)

$$\sigma_z = 0, \quad \sigma_{xz} = f_1(Z)\varphi(x,y), \quad \sigma_{yz} = f_2(Z)\psi(x,y) \tag{3}$$

where $\varphi(x, y)$ and $\psi(x, y)$ are arbitrary functions of the coordinates x and y which are to be

determined; f_i (Z), $(i = 1, 2)$ are the functions which characterized the variation of shear stresses σ_{xz} and σ_{yz} with respect to the shell thickness.

Substituting relations (3) into third and fourth equations of the system (1), we obtain

$$\gamma_{xz} = f_1(Z)a_{55}(Z)\varphi(x,y), \quad \gamma_{yz} = f_2(Z)a_{44}(Z)\psi(x,y) \tag{4}$$

where the following definitions apply

$$a_{55}(Z) = \frac{1}{B_{55}(Z)}, \quad a_{44}(Z) = \frac{1}{B_{44}(Z)} \tag{5}$$

Due to assumptions of the shear deformation theory, we obtain (Soldatos and Timarci 1993)

$$u_{x,z} = -w_{,x} + \gamma_{xz}, \quad u_{y,z} = -w_{,y} + \gamma_{yz} \tag{6}$$

Integration of Eq. (6) with respect to z from zero to z with the condition that for $z = 0$, $u_x = u(x, y)$ and $u_y = v(x, y)$, the following expressions for the in-plane displacements of any point in the cylindrical shell are obtained

$$u_z = w, \quad u_x = u - zw_{,x} + I_{01}\varphi, \quad u_y = v - zw_{,y} + I_{02}\psi \tag{7}$$

where $u_x = u(x, y)$ and $u_y = v(x, y)$ are displacements along coordinates x and y, respectively, and the following definitions apply

$$I_{01} = \int_0^z a_{55}(Z)f_1(Z)dz, \quad I_{02} = \int_0^z a_{44}(Z)f_2(Z)dz \tag{8}$$

The strain components ε_x, ε_y, γ_{yz}, γ_{xz}, γ_{xy} are related to the displacements u_x, u_y, u_z by the equations

$$\varepsilon_x = u_{x,x}, \quad \varepsilon_y = u_{y,y} + w/R, \quad v \quad \gamma_{xz} = u_{z,x} + u_{x,z}, \quad \gamma_{xy} = u_{x,y} + u_{y,x} \tag{9}$$

Substituting u_x and u_y from Eq. (7) into Eq. (9) we obtain expressions for the corresponding deformation components

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \varepsilon_x^0 - zw_{,xx} + I_{01}\varphi_{,x} \\ \varepsilon_y^0 - zw_{,yy} + I_{02}\psi_{,y} \\ \gamma_{xy}^0 - 2zw_{,xy} + I_{01}\varphi_{,y} + I_{02}\psi_{,x} \end{pmatrix} \tag{10}$$

where $\varepsilon_x^0, \varepsilon_y^0, \varepsilon_{xy}^0$ are the strains on the reference surface.

The force and moment resultants are defined according to (Reddy 2004)

$$\left(T_x, T_y, T_{xy}, Q_x, Q_y\right) = \int_{-h/2}^{h/2} \left(\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}\right) dz,$$

$$\left(M_x, M_y, M_{xy}\right) = \int_{-h/2}^{h/2} \left(\sigma_x, \sigma_y, \sigma_{xy}\right) z dz \tag{11}$$

where T_x and T_y are normal forces, T_{xy} is the tangential force, Q_x and Q_y are shear forces, M_x and M_y are bending moments and M_{xy} is the torque moment.

The governing equations of cylindrical shells resting on a Pasternak elastic foundation and subjected to a uniform hydrostatic pressure are given as (Shirakawa 1983, Morimoto and Tanigawa 2007)

$$
\begin{aligned}
& M_{x,x} + M_{xy,y} - Q_x = 0, \\
& M_{xy,x} + M_{y,y} - Q_y = 0, \\
& \varepsilon^0_{x,yy} + \varepsilon^0_{y,xx} - \gamma^0_{xy,xy} + w_{,xx}/R = 0, \\
& Q_{x,x} + Q_{y,y} + T_y/R - 0.5PRw_{,xx} - PRw_{,yy} - K_w w + K_p(w_{,xx} + w_{,yy}) = 0
\end{aligned}
\tag{12}
$$

The governing Eq. (12) can be expressed in terms of Φ, w, φ, ψ by using Eqs. (1), (3), (4), (10), (11) and the relation for the Airy stress function as

$$
\begin{bmatrix}
L_{11} & L_{12} & L_{13} & L_{14} \\
L_{21} & L_{22} & L_{23} & L_{24} \\
L_{31} & L_{32} & L_{33} & L_{34} \\
L_{41} & L_{42} & L_{43} & L_{44}
\end{bmatrix}
\begin{bmatrix}
\Phi \\
w \\
\varphi \\
\psi
\end{bmatrix}
= 0
\tag{13}
$$

where L_{ij} ($i, j = 1, 2, 3, 4$) are differential operators and given in Appendix A.

Eq. (13) is governing equations for the stability of EG orthotropic cylindrical shells under a uniform hydrostatic pressure and resting on a Pasternak elastic foundation, based on FOSDT.

4. Solution of governing equations

The case of an EG orthotropic cylindrical shell under the simply supported boundary conditions (Shen 2008)

$$
w = 0, \quad M_x = 0, \quad \psi = 0, \quad \Phi_{,yy} = 0 \quad \text{when} \quad x = 0,\, x = L
\tag{14}
$$

can now be considered. For the solution of equations system (13), the set of displacement, stress and rotary functions satisfying these boundary conditions can be written as (Soldatos and Timarci 1993)

$$
\begin{aligned}
& \Phi = \phi_{mn} \sin(\lambda x)\sin(\eta y), \quad w = f_{mn}\sin(\lambda x)\sin(\eta y), \\
& \varphi = \varphi_{mn}\cos(\lambda x)\sin(\eta y), \quad \psi = \psi_{mn}\sin(\lambda x)\cos(\eta y)
\end{aligned}
\tag{15}
$$

where $\phi_{mn}, f_{mn}, \varphi_{mn}, \psi_{mn}$ are unknown amplitudes, $\lambda = \dfrac{m\pi}{L}$, $\eta = \dfrac{n}{R}$, in which, m is the half wave number in axial direction and n is the circumferential wave number.

Introduction of (15) into the system of Eq. (13), yields a set of algebraic equations for $\phi_{mn}, f_{mn}, \varphi_{mn}, \psi_{mn}$

$$\begin{bmatrix} Q_{11} & -Q_{12} & Q_{13} & Q_{14} \\ Q_{21} & -Q_{22} & Q_{23} & Q_{24} \\ Q_{31} & -Q_{32} & Q_{33} & Q_{34} \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} \end{bmatrix} \begin{bmatrix} \phi_{mn} \\ f_{mn} \\ \varphi_{mn} \\ \psi_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{16}$$

where the following definitions apply

$$Q_{11} = h[(c_{11} - c_{31})\lambda^2\eta^2 + c_{12}\lambda^4], \quad Q_{12} = (c_{14} + c_{32})\lambda^2\eta^2 + c_{13}\lambda^4$$

$$Q_{13} = c_{15}\lambda^3 + c_{35}\lambda\eta^2 + I_5\lambda, Q_{14} = (c_{18} + c_{38})\eta\lambda^2$$

$$Q_{21} = h[c_{21}\eta^4 + (c_{22} - c_{31})\lambda^2\eta^2], \quad Q_{22} = (c_{32} + c_{23})\lambda^2\eta^2 + c_{24}\eta^4$$

$$Q_{23} = (c_{25} + c_{35})\lambda\eta^2, Q_{24} = c_{28}\eta^3 + c_{38}\lambda^2\eta + I_6\eta,$$

$$Q_{31} = h[b_{22}\lambda^4 + (b_{12} + b_{21} + b_{31})\lambda^2\eta^2 + b_{11}\eta^4], \tag{17}$$

$$Q_{32} = b_{23}\lambda^4 + (b_{24} + b_{13} - b_{32})\lambda^2\eta^2 + b_{14}\eta^4 + \lambda^2/R,$$

$$Q_{33} = b_{25}\lambda^3 + (b_{15} + b_{35})\lambda\eta^2,$$

$$Q_{34} = (b_{28} + b_{38})\lambda^2\eta + b_{18}\eta^3, \quad Q_{41} = \lambda^2 h/R,$$

$$Q_{42} = -PR(0.5\lambda^2 + \eta^2) + K_w + K_p(\lambda^2 + \eta^2), Q_{43} = I_5\lambda, \ Q_{44} = I_6\eta.$$

For the non-trivial solution of system of Eq. (16), the determinant of this set of equations must be zero

$$\det[Q] = 0 \tag{18}$$

Solving the set of Eq. (18), we obtain an expression for the critical hydrostatic pressure of an EG orthotropic cylindrical shell resting on a Pasternak elastic foundation on the basis of FOSDT

$$P_{FOSDT}^{crwp} = \frac{Q_{41}\Gamma_{41} + Q_{43}\Gamma_{43} + Q_{44}\Gamma_{44} + [K_w + K_p(\lambda^2 + \eta^2)]\Gamma_{42}}{\Gamma_{42}(0.5\lambda^2 + \eta^2)R} \tag{19}$$

where the following definition apply

$$\Gamma_{41} = Q_{12}Q_{23}Q_{34} - Q_{12}Q_{24}Q_{33} + Q_{22}Q_{33}Q_{14} - Q_{22}Q_{13}Q_{34} + Q_{32}Q_{13}Q_{24} - Q_{32}Q_{23}Q_{14}$$

$$\Gamma_{42} = Q_{11}Q_{23}Q_{34} - Q_{11}Q_{24}Q_{33} + Q_{21}Q_{33}Q_{14} - Q_{21}Q_{13}Q_{34} + Q_{31}Q_{13}Q_{24} - Q_{31}Q_{23}Q_{14}$$

$$\Gamma_{43} = Q_{11}Q_{22}Q_{34} - Q_{11}Q_{24}Q_{32} + Q_{21}Q_{32}Q_{14} - Q_{21}Q_{12}Q_{34} + Q_{31}Q_{12}Q_{24} - Q_{31}Q_{22}Q_{14} \tag{20}$$

$$\Gamma_{44} = -Q_{11}Q_{22}Q_{33} + Q_{11}Q_{23}Q_{32} - Q_{21}Q_{32}Q_{13} + Q_{21}Q_{12}Q_{33} - Q_{31}Q_{12}Q_{23} + Q_{31}Q_{22}Q_{13}$$

The dimensionless critical hydrostatic pressure for an EG orthotropic cylindrical shell on a Pasternak elastic foundation, based on the FOSDT expressed as follow

$$P_{1FOSDT}^{crwp} = P_{FOSDT}^{crwp}/E_{02} \tag{21}$$

The strain compatibility and stability equations of EG orthotropic cylindrical shells under a

hydrostatic pressure and resting on a Pasternak elastic foundation based on CST can be expressed as follows

$$\begin{bmatrix} \widetilde{L}_{11} \widetilde{L}_{12} \\ \widetilde{L}_{21} \widetilde{L}_{22} \end{bmatrix} \begin{bmatrix} \Phi \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (22)$$

where the following definitions apply

$$\widetilde{L}_{11} = hc_{12}()_{xxxx} + h(c_{11} - 2c_{31} + c_{22})()_{xxyy} + hc_{21}()_{yyyy} + h()_{xx}/R$$

$$\widetilde{L}_{12} = -c_{13}()_{xxxx} - (c_{14} + 2c_{32} + c_{13})()_{xxyy} - c_{14}()_{yyyy}$$
$$\qquad - 0.5PR()_{xx} - PR()_{yy} - K_w + K_p[()_{xx} + ()_{yy}] \qquad (23)$$

$$\widetilde{L}_{21} = hb_{22}()_{xxxx} + h(b_{12} + b_{21} + b_{31})()_{xxyy} + hb_{11}()_{yyyy}$$

$$\widetilde{L}_{22} = -(b_{13} - b_{32} + b_{24})()_{xxyy} - b_{14}()_{yyyy} - b_{23}()_{xxxx} + ()_{xx}/R$$

Substituting Eq. (15) into Eq. (22), after mathematical operations, for the critical hydrostatic pressure of EG orthotropic cylindrical shells resting on a Pasternak elastic foundation, based on the CST, the following expression is obtained

$$P_{CST}^{crwp} = \frac{1}{(0.5\lambda^2 + \eta^2)R} \left\{ \left[c_{13}\lambda^4 + (c_{14} + 2c_{32} + c_{23})\lambda^2\eta^2 + c_{24}\eta^4 + K_w + K_p(\lambda^2 + \eta^2) \right] \right.$$
$$+ \left[\lambda^2/R - c_{12}\lambda^4 - (c_{11} - 2c_{31} + c_{22})\lambda^2\eta^2 - c_{21}\eta^4 \right] \qquad (24)$$
$$\left. \times \frac{\left[\lambda^2/R + b_{23}\lambda^4 + (b_{13} - b_{32} + b_{24})\lambda^2\eta^2 + b_{14}\eta^4 \right]}{b_{22}\lambda^4 + (b_{12} + b_{31} + b_{21})\lambda^2\eta^2 + b_{11}\eta^4} \right\}$$

The dimensionless critical hydrostatic pressure of an EG orthotropic cylindrical shell on a Pasternak elastic foundation based on the CST is expressed as follows

$$P_{CST}^{crwp} = P_{CST}^{crwp} / E_{02} \qquad (25)$$

In a special case, the expressions for critical hydrostatic pressures of EG orthotropic cylindrical shells without an elastic foundation based on CST and FOSDT can be obtained by letting $K_w = K_p = 0$ in Eqs. (19), (22), (24) and (25).

The expressions for critical hydrostatic pressures of homogeneous orthotropic cylindrical shells on a Pasternak elastic foundation based on CST and FOSDT can be obtained by letting $\mu = 0$ in Eqs. (19), (22), (24) and (25).

The minimum values of dimensional and dimensionless critical hydrostatic pressures based on CST and FOSDT obtained by minimizing Eqs. (19) and (22), and (24) and (25), respectively, with respect to (m, n).

5. Numerical analysis

The accuracy of the present study, the values of the critical hydrostatic pressure (in kPa) for

shear deformable homogeneous orthotropic cylindrical shells without an elastic foundation for different *L/R* ratio shown in Table 1 and are compared with those presented by Han and Simitses (1991), and Li and Lin (2010). To this end, μ should be assumed zero and the homogeneous orthotropic material properties of Han and Simitses (1991), and Li and Lin (2010) are adopted. Two orthotropic material properties are taken to be (Material 1): $E_{01} = 149.66$ GPa, $E_{02} = 9.93$ GPa, $G_{012} = G_{013} = G_{023} = 4.48$ GPa, v_{12} 0.28; (Material 2): $E_{01} = 9.93$ GPa, $E_{02} = 149.66$ GPa, $G_{012} = G_{013} = G_{023} = 4.48$ GPa, v_{21} 0.28, respectively. The shell characteristics are taken to be $R/h = 30$, $L/R = 2$ and 5. The circumferential wave number (n_{cr}) in parentheses corresponds to the critical hydrostatic pressure. The results show that the present results in very well agreement with the results of Li and Lin (2010), but lower than those of Han and Simitses (1991).

In addition, the critical hydrostatic pressures for homogeneous isotropic cylindrical shells without an elastic foundation, based on CST are compared with the finite element results of Kasagi and Sridharan (1993), and boundary layer theory solution of Shen and Noda (2007) and presented in Table 2. Here $Z_b = \dfrac{L^2 \sqrt{1-v_0^2}}{Rh}$ is the Batdorf shell parameter. The data are taken to be $E_0 = 10 \times 10^6$ psi, $v_0 = 0.33$, $R/h = 200$ (Kasagi and Sridharan 1993). It can be seen that, the present results agree very well with the results of Kasagi and Sridharan (1993), and Shen and Noda (2007).

Numerical results for stability of H and EG orthotropic cylindrical shells with and without a Pasternak elastic foundation and subjected to a uniform hydrostatic pressure, based on FOSDT and CST are presented in Tables 3-4, and Fig. 2. The homogeneous material properties adopted as in Reddy (2004), are: $E_{01} = 25E_{02}$, $G_{012} = G_{013} = 0.5E_{0.2}$, $G_{023} = 0.2E_{02}$, and $v_{12} = 0.25$. For these examples the cylindrical shell characteristics are $R/h = 30$ to 50 and $L/R = 0.25$ to 1.0. The stiff-

Table 1 Comparison the present results with the results of Han and Simitses (1991), and Li and Lin (2010)

	L/R = 2			L/R = 5		
Orthotropic Materials	Han and Simitses (1991) HOST	Li and Lin (2010) HOST	Present study (FOSDT)	Han and Simitses (1991) HOST	Li and Lin (2010) HOST	Present study (FOSDT)
	P_{SDT}^{cr} (kPa), (n_{cr})					
(Mat 1)	1517(5)	1425.7(5)	1425.7(5)	683(4)	620.6(4)	620.6(4)
(Mat 2)	6798(3)	5243.2(3)	5243.2(3)	3847(3)	2403.8(3)	2403.8(2)

Table 2 Comparison of P_{CST}^{cr} (in psi) for homogeneous isotropic cylindrical shells with different length

	P_{CST}^{cr} (psi), (n_{cr})		
Z_b	Kazagi and Sridharan (1993)	Shen and Noda (2007)	Present study
50	35.09(13)	35.167(13)	35.205(13)
100	24.26(11)	24.305(11)	24.322(11)
500	10.42(8)	10.436(8)	10.440(8)
1000	7.388(7)	7.398(7)	7.400(7)
5000	3.412(5)	3.416(5)	3.416(5)
10000	2.312(4)	2.315(4)	2.315(4)

ness is characterized by (K_w, K_p) for a Pasternak elastic foundation model, by $(K_w, 0)$ for a Winkler elastic foundation model, and by $(K_w, K_p) = (0, 0)$ for an unconstrained shell. The shear stresses shape function is distributed parabolic manner through the shell thickness, i.e., $f_1(Z) = f_2(Z) = 1 - 4Z^2$. The EG compositional profile is taken to be $e^{\mu(Z-0.5)}$ and exponential factor is $\mu = 1$. As $\mu = 0$, it correspond to the homogeneous case. The circumferential wave number (n_{cr}) in brackets corresponds to dimensionless critical hydrostatic pressures and the longitudinally wave number is taken to be $m = 1$.

The values of dimensionless critical hydrostatic pressures for H and EG orthotropic cylindrical shells with and without an elastic foundation based on FOSDT and CST are presented in Table 3. The Pasternak foundation stiffness is taken to be $(K_w, K_p) = (2 \times 10^7 \, \text{N/m}^3; 3 \times 10^3 \, \text{N/m})$. The values of dimensionless critical hydrostatic pressures for H and EG orthotropic cylindrical shells with and without Pasternak elastic foundation based on FOSDT and CST decrease as R/h and L/R increase. The circumferential wave numbers corresponding to critical hydrostatic pressures decrease as L/R increases, whereas, changes irregularly as R/h increases depending on the ratio L/R. Considering the effect of a Pasternak elastic foundation, increase the values of the dimensionless critical hydrostatic pressures for H and EG orthotropic cylindrical shells. The influence of the Pasternak elastic foundation on the values of P_{1SDT}^{crwp} for H and EG orthotropic cylindrical shells increases, as the ratios L/R and R/h increase. For example, the influence of a Pasternak elastic foundation on the values of P_{1SDT}^{crwp} for EG (or H) shells increases from 1.12% to 3.92% (or from 0.68% to 2.37%)

Table 3 Variation of dimensionless critical hydrostatic pressures and corresponding circumferential wave numbers for H and EG orthotropic cylindrical shells with and without a Pasternak elastic foundation versus R/h and L/R

	$P_1^{cr} \times 10^3$, (n_{cr})					
	FOSDT	CST	FOSDT	CST	FOSDT	CST
	$R/h = 30$		$R/h = 40$		$R/h = 50$	
L/R	EG shells without an elastic foundation $(K_w, K_p) = 0$					
0.25	1.883(28)	3.324(26)	0.977(27)	1.403(26)	0.561(27)	0.719(26)
0.5	0.704(14)	0.841(13)	0.323(14)	0.358(14)	0.173(14)	0.185(14)
1.0	0.228(8)	0.239(8)	0.104(8)	0.107(8)	0.058(9)	0.059(9)
	H shells without an elastic foundation $(K_w, K_p) = 0$					
0.25	3.106(28)	5.524(26)	1.615(27)	2.332(26)	0.929(27)	1.195(26)
0.5	1.166(14)	1.397(13)	0.535(14)	0.595(14)	0.287(14)	0.307(14)
1.0	0.376(8)	0.395(8)	0.172(8)	0.177(8)	0.095(9)	0.097(9)
	EG shells on an elastic foundation $(K_w = 2 \times 10^7 \, \text{N/m}^3; K_p = 3 \times 10^3 \, \text{N/m})$					
0.25	1.904(28)	3.346(26)	0.999(27)	1.426(26)	0.583(27)	0.741(27)
0.5	0.739(14)	0.878(14)	0.357(14)	0.393(14)	0.207(15)	0.220(15)
1.0	0.298(9)	0.311(8)	0.166(10)	0.170(10)	0.112(11)	0.114(11)
	H shells on an elastic foundation $(K_w = 2 \times 10^7 \, \text{N/m}^3; K_p = 3 \times 10^3 \, \text{N/m})$					
0.25	3.127(28)	5.546(26)	1.637(27)	2.354(26)	0.951(27)	1.217(27)
0.5	1.201(14)	1.435(14)	0.569(14)	0.630(14)	0.322(14)	0.342(14)
1.0	0.448(8)	0.467(8)	0.237(9)	0.243(9)	0.153(10)	0.155(10)

and from 30.7% to 93.1% (or from 19.15% to 61.05%), respectively, as R/h increases from 30 to 50 with $L/R = 0.25$ and 1.0, respectively. It is observed that the influence of an elastic foundation on the dimensionless critical hydrostatic pressures for H and EG orthotropic cylindrical shells is slight in short shells, i.e., for $L/R = 0.25$. The influence of shear stresses on the values of dimensionless critical hydrostatic pressures for EG (or H) shells resting on the Pasternak elastic foundation decreases from 43.1% to 4.18% (or from 43.62% to 4.07%) and 21.32% to 1.75% (or from 21.86% to 1.29%), respectively, as L/R increases from 0.25 to 1 for $R/h = 30$ and 50, respectively. The influence of heterogeneity on the values of dimensionless critical hydrostatic pressures for the unconstrained orthotropic cylindrical shell on the basis of CST and FOSDT almost remain constant and approximately around 39%, as R/h increases from 30 to 50 for fixed L/R, whereas, this influence on the dimensionless critical hydrostatic pressures of orthotropic cylindrical shell resting on a Pasternak elastic foundation decreases from 39% to 34% and from 39% to 27%, as L/R increases from 0.25 to 1 for $R/h = 30$ and 50, respectively.

The variation of the values of dimensionless critical hydrostatic pressures and corresponding circumferential wave numbers for H and EG orthotropic cylindrical shells versus the foundations moduli K_w and K_p for $R/h = 30$ and different L/R are tabulated in Table 4. The values of dimensionless critical hydrostatic pressures and corresponding circumferential wave numbers for H and EG orthotropic cylindrical shells on the basis of CST and FOSDT increase with increasing of the foundations moduli K_w and K_p. The effect of heterogeneity on the dimensionless critical

Table 4 Variation of the values of dimensionless critical hydrostatic pressures and corresponding circumferential wave numbers for H and EG orthotropic cylindrical shells versus the foundations moduli K_w and K_p with different L/R

|		$P_1^{cr} \times 10^3$, (n_{cr})					
		FOSDT	CST	FOSDT	CST	FOSDT	CST
		$L/R = 0.25$		$L/R = 0.5$		$L/R = 1.0$	
K_w (N/m^3)	K_p (N/m)	EG cylindrical shells					
0	0	1.883(28)	3.324(26)	0.704(14)	0.841(13)	0.228(8)	0.239(8)
	0	1.885(28)	3.327(26)	0.713(14)	0.851(13)	0.255(8)	0.266(8)
10^7	2×10^3	1.897(28)	3.338(26)	0.724(14)	0.863(13)	0.266(8)	0.278(8)
	4×10^3	1.908(28)	3.350(26)	0.736(14)	0.874(14)	0.278(8)	0.289(8)
	0	1.887(28)	3.329(26)	0.722(14)	0.860(14)	0.282(9)	0.294(8)
2×10^7	2×10^3	1.899(28)	3.341(26)	0.733(14)	0.872(14)	0.293(9)	0.305(8)
	4×10^3	1.910(28)	3.352(26)	0.745(14)	0.883(14)	0.304(9)	0.316(8)
K_w (N/m^3)	K_p (N/m)	H cylindrical shells					
0	0	3.106(28)	5.524(26)	1.166(14)	1.397(13)	0.376(8)	0.395(8)
	0	3.108(28)	5.526(26)	1.175(14)	1.407(13)	0.404(8)	0.423(8)
10^7	2×10^3	3.119(28)	5.538(26)	1.186(14)	1.419(13)	0.415(8)	0.434(8)
	4×10^3	3.131(28)	5.550(26)	1.198(14)	1.430(13)	0.426(8)	0.445(8)
	0	3.110(28)	5.529(26)	1.184(14)	1.417(13)	0.432(8)	0.450(8)
2×10^7	2×10^3	3.122(28)	5.540(26)	1.195(14)	1.429(14)	0.443(8)	0.462(8)
	4×10^3	3.133(28)	5.552(26)	1.207(14)	1.440(14)	0.454(8)	0.473(8)
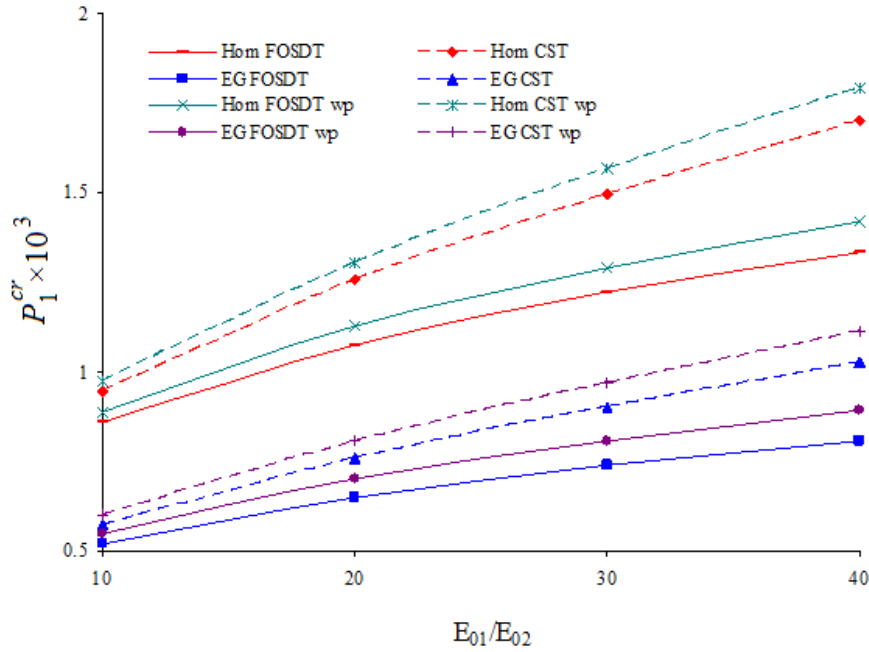
Fig. 2 Variation of the values of dimensionless critical hydrostatic pressures for H and EG orthotropic cylindrical shells with and without an elastic foundation versus E_{01}/E_{02}

hydrostatic pressure decreases, while the effect of shear stresses almost remains constant with increasing of the foundation stiffness. The effect of a Pasternak elastic foundation on the values of P_{FOSDT}^{crwp} slightly higher from its effect on the values of P_{CST}^{crwp}, as a percentage. Furthermore, the influence of shear stresses on the dimensionless critical hydrostatic pressures is considerable for short shells, while the influence of an elastic foundation is significant for medium length shells. The influence of a Pasternak elastic foundation on the dimensionless critical hydrostatic pressures is higher than the Winkler elastic foundation.

The variation of dimensionless critical hydrostatic pressures for H and EG orthotropic cylindrical shells with and without a Pasternak elastic foundation versus E_{01}/E_{02} are plotted in Fig 2. To explain the effect of the degree of anisotropy of the shell material on the stability process, the ratios of the Young's moduli were assumed to be E_{01}/E_{02} = 10; 20; 30; 40. The material properties and shell characteristics are taken to be $E_{01} = 2 \times 10^{11}$ Pa, v_{12} 0.2, $L/R = 0.5$ and $R/h = 30$. The Pasternak foundation stiffness is taken to be $(K_w, K_p) = (2 \times 10^8 \text{ N/m}^3; 5 \times 10^4 \text{ N/m})$. The values of dimensionless critical hydrostatic pressures for H and EG orthotropic shells with and without an elastic foundation on the basis of CST and FOSDT increase with increasing the ratio, E_{01}/E_{02}. The effect of heterogeneity on the dimensionless critical hydrostatic pressure for unconstrained orthotropic cylindrical shell with the CST remains constant, while this effect decreases for orthotropic cylindrical shell resting on a Pasternak elastic foundation, as the ratio, E_{01}/E_{02}, increases, based on CST and FOSDT. The effect of shear tresses on the values of dimensionless critical hydrostatic pressures for H and EG orthotropic shells with and without an elastic foundation increases, as E_{01}/E_{02} increases from 10 to 40 by steps 10.

6. Conclusions

In this study, the effect of a Pasternak elastic foundation on the stability of EG orthotropic cylindrical shells including shear stresses subjected to a uniform hydrostatic pressure is investigated. The shear stresses shape functions are distributed parabolic manner through the shell thickness. The governing equations of EG orthotropic cylindrical shells on the basis of FOSDT are derived in the framework of Donnell-type shell theory. The boundary condition is considered to be simply-supported. The novelty of the present work is to achieve the closed-form solutions for the critical hydrostatic pressures of EG orthotropic cylindrical shells resting on a Pasternak elastic foundation based on FOSDT. Finally, the effects of a Pasternak elastic foundation, shear stresses, heterogeneity, material orthotropy and shell characteristics on the values of dimensionless critical hydrostatic pressures are investigated.

References

Ádány, S. (2014), "Flexural buckling of simply-supported thin-walled columns with consideration of membrane shear deformations: Analytical solutions based on shell model", *Thin-Wall. Struct.*, **74**, 36-48.

Akoz, A.Y. and Ergun, H. (2012), "Analysis of partially embedded beams in two-parameter foundation", *Struct. Eng. Mech., Int. J.*, **42**(1), 1-12.

Alipour, M.M., Shariyat, M. and Shaban, M. (2010), "A semi-analytical solution for free vibration of variable thickness two-directional-functionally graded plates on elastic foundations", *Int. J. Mech. Mater. Des.*, **6**(4), 293-304.

Ambartsumian, S.A. (1964), *Theory of Anisotropic Plates*; *Strength, Stability, Vibration*, Technomic, Stamford, USA.

Asadi, E. and Qatu, M.S. (2012), "Static analysis of thick laminated shells with different boundary conditions using GDQ", *Thin-Wall. Struct.*, **51**, 76-81.

Atmane, H.A., Tounsi, A., Mechab, I. and Bedia, E.A.A. (2010), "Free vibration analysis of functionally graded plates resting on Winkler–Pasternak elastic foundations using a new shear deformation theory", *Int. J. Mech. Mater. Des.*, **6**(2), 113-121.

Bagherizadeh, E., Kiani, Y. and Eslami, M.R. (2011), "Mechanical buckling of functionally graded material cylindrical shells surrounded by Pasternak elastic foundation", *Compos. Struct.*, **93**(11), 3063-3071.

Baron, C. (2011), "Propagation of elastic waves in an anisotropic functionally graded hollow cylinder in vacuum", *Ultrasonics*, **51**(2), 123-130.

Batra, R.C. and Jin, J. (2005), "Natural frequencies of a functionally graded anisotropic rectangular plate", *J. Sound Vib.*, **282**(1), 509-516.

Bouderba, B., Houari, M.S.A. and Tounsi, A. (2013), "Thermomechanical bending response of FGM thick plates resting on Winkler-Pasternak elastic foundations", *Steel Compos. Struct., Int. J.*, **14**(1), 85-104.

Chen, J., Soh, A.K., Liu, J. and Liu, Z. (2004a), "Thermal fracture analysis of a functionally graded orthotropic strip with a crack", *Int. J. Mech. Mater. Des.*, **1**(2), 131-141.

Chen, W.Q., Bian, Z.G. and Ding, H.J. (2004b), "Three-dimensional vibration analysis of fluid-filled orthotropic FGM cylindrical shells", *Int. J. Mech. Sci.*, **46**(1), 159-171.

Civalek, Ö. (2008), "Vibration analysis of conical panels using the method of discrete singular convolution", *Comm. Numer. Meth. Eng.*, **24**(3), 169-181.

Croll, J.G.A. (2001), "Buckling of cylindrical tunnel liners", *J. Eng. Mech.*, **127**(4), 333-341.

Eslami, M.R. and Shariyat, M. (1999), "A higher-order theory for dynamic buckling and postbuckling analysis of laminated cylindrical shells", *J. Press. Vess. T. – ASME*, **121**(1), 94-102.

Ferreira, A.J.M., Roque, C.M.C., Neves, A.M.A., Jorge, R.M.N., Soares, C.M.M. and Reddy, J.N. (2011), "Buckling analysis of isotropic and laminated plates by radial basis functions according to a higher-order shear deformation theory", *Thin-Wall. Struct.*, **49**(7), 804-811.

Firouz-Abadi, R.D., Torkaman-Asadi, M.A. and Rahmanian, M. (2013), "Whirling frequencies of thin spinning cylindrical shells surrounded by an elastic foundation", *Acta Mech.*, **224**(4), 881-892.

Fok, S.L. (2002), "Analysis of the buckling of long cylindrical shells embedded in an elastic medium using the energy method", *J. Strain Anal. Eng. Design*, **37**(5), 375-383.

Gorbunov-Possadov, M.I., Malikova, T.A. and Solomin, V.I. (1984), *Design of Structures on Elastic Foundation*, Strojizdat, Moscow, Russia.

Grigorenko, Y.M. and Grigorenko, A.Y. (2013), "Static and dynamic problems for anisotropic inhomogeneous shells with variable parameters and their numerical solution (review)", *Int. Appl. Mech.*, **49**(2), 123-193.

Grigorenko, Y.M. and Vasilenko, A.T. (1992), *Static Problems for Anisotropic Inhomogeneous Shells*, Nauka, Moscow, Russia.

Han, B. and Simitses, G.J. (1991), "Analysis of anisotropic laminated cylindrical shells subjected to destabilizing loads. Part II: Numerical results", *Compos. Struct.*, **19**(2), 183-205.

Hui, D. (1986), "Imperfection-sensitivity of elastically supported beams and its relation to the double-cusp instability model", *Proc. Roy. Soc. Lond. Math. Phys. Sci.*, **405**(1828), 143-158.

Hui, D. and Hansen, J.S. (1980), "Two-mode buckling of an elastically supported plate and its relation to catastrophe-theory", *J. Appl. Mech.*, **47**(3), 607-612.

Jung, W.Y. and Han, S.C. (2014) "Shear buckling responses of laminated composite shells using a modified 8-node ANS shell element", *Compos. Struct.*, **109**, 119-129.

Kar, A. and Kanoria, M. (2009), "Generalized thermoelastic functionally graded orthotropic hollow sphere under thermal shock with three-phase-lag effect", *Europe J. Mech. A-Solids*, **28**(4), 757-767.

Kardomateas, G.A. (1997), "Koiter-based solution for the initial postbuckling behavior of moderately thick orthotropic and shear deformable cylindrical shells under external pressure", *J. Appl. Mech.*, **64**(4), 885-896.

Kasagi, A. and Sridharan, S. (1993), "Buckling and post-buckling analysis of thick composite cylindrical shells under hydrostatic pressure" *Compos. Eng.*, **3**(5), 467-487.

Kumar, Y. and Lal, R. (2012), "Vibrations of non-homogeneous orthotropic rectangular plates with bilinear thickness variation resting on Winkler foundation", *Meccanica*, **47**(4), 893-915.

Li, Z.M. and Lin, Z.Q. (2010), "Non-linear buckling and postbuckling of shear deformable anisotropic laminated cylindrical shell subjected to varying external pressure loads", *Compos. Struct.* **92**(2), 553-567.

Lomakin, V.A. (1976), *The Elasticity Theory of Non-homogeneous Materials*, Nauka, Moscow, Russia.

Mantari, J.L. and Soares, C.G. (2014), "Optimized sinusoidal higher order shear deformation theory for the analysis of functionally graded plates and shells", *Compos. B Eng.*, **56**(1), 126-136.

Morimoto, T. and Tanigawa, Y. (2007), "Elastic stability of inhomogeneous thin plates on an elastic foundation", *Arch. Appl. Mech.*, **77**(9), 653-674.

Naili, S. and Oddou, C. (2000), "Buckling of short cylindrical shell surrounded by an elastic medium", *J. Appl. Mech.*, **67**(1), 212-214.

Najafov, A.M., Sofiyev, A.H. and Kuruoglu, N. (2013), "Torsional vibration and stability of functionally graded orthotropic cylindrical shells on elastic foundations", *Meccanica*, **48**(4), 829-840.

Ng, T.Y. and Lam, K.Y. (1999), "Effects of elastic foundation on the dynamic stability of cylindrical shells", *Struct. Eng. Mech., Int. J.*, **8**(2), 193-205.

Ootao, Y. and Tanigawa, Y. (2007), "Three-dimensional solution for transient thermal stresses of an orthotropic functionally graded rectangular plate", *Compos. Struct.*, **80**(1), 10-20.

Palazotto, A.N. and Linnemann, P.E. (1991), "Vibration and buckling characteristics of composite cylindrical panels incorporating the effects of a higher-order shear theory", *Int. J. Solid. Struct.*, **28**(3), 341-361.

Paliwal, D.N. and Pandey, R.K. (2001), "Free vibrations of an orthotropic thin cylindrical shell on a Pasternak foundation", *AIAA J.*, **39**(11), 2188-2191.

Pan, E. (2003), "Exact solution for functionally graded anisotropic elastic composite laminates", *J. Compos. Mater.*, **37**(21), 1903-1920.

Pelletier, J.L. and Vel, S.S. (2006), "An exact solution for the steady-state thermoelastic response of

functionally graded orthotropic cylindrical shells", *Int. J. Solid. Struct.*, **43**(5), 1131-1158.

Peng, X.L. and Li, X.F. (2012), "Elastic analysis of rotating functionally graded polar orthotropic disks", *Int. J. Mech. Sci.*, **60**(1), 84-91.

Ramirez, F., Heyliger, P.R. and Pan, E. (2006), "Static analysis of functionally graded elastic anisotropic plates using a discrete layer approach", *Compos. B Eng.*, **37**(1), 10-20.

Reddy, J.N. (2004), *Mechanics of Laminated Composite Plates and Shells*: *Theory and Analysis*, CRC Press, New York, USA.

Shariyat, M. and Asemi, K. (2014), "Three-dimensional non-linear elasticity-based 3D cubic B-spline finite element shear buckling analysis of rectangular orthotropic FGM plates surrounded by elastic foundations", *Compos. B Eng.*, **56**, 934-947.

Shen, H.S. (2008), "Boundary layer theory for the buckling and postbuckling of an anisotropic laminated cylindrical shell. Part II: Prediction under external pressure", *Compos. Struct.*, **82**(3), 362-370.

Shen, H.S. (2013), "Postbuckling of axially-loaded laminated cylindrical shells surrounded by an elastic medium", *Mech. Adv. Mater. Struct.*, **20**(2), 130-150.

Shen, H.S. and Noda, N. (2007), "Post-buckling of pressure-loaded FGM hybrid cylindrical shells in thermal environments", *Compos. Struct.*, **77**(4), 546-560.

Shen, H.S. and Wang, H. (2013), "Thermal postbuckling of functionally graded fiber reinforced composite cylindrical shells surrounded by an elastic medium", *Compos. Struct.*, **102**, 250-260.

Shirakawa, K. (1983), "Effects of shear deformation and rotatory inertia on vibration and buckling of cylindrical shells", *J. Sound Vib.*, **91**(3), 425-437.

Sofiyev, A.H. (2011), "Thermal buckling of FGM shells resting on a two parameter elastic foundation", *Thin-Wall. Struct.*, **49**(10), 1304-1311.

Sofiyev, A.H. and Kuruoglu, N. (2014), "Buckling and vibration of shear deformable functionally graded orthotropic cylindrical shells under external pressures", *Thin-Wall. Struct.*, **78**, 121-130.

Sofiyev, A.H. and Marandi, B. (1996), "Dynamic stability problem of non-homogeneous cylindrical shells on elastic foundations", *Proc. Inst. Math. Mech. Academy Sci. Azerbaijan*, **5**(8), 128-131. [In Russian]

Sofiyev, A.H., Omurtag, M.H. and Schnack, E. (2009), "The vibration and stability of orthotropic conical shells with non-homogeneous material properties under a hydrostatic pressure", *J. Sound Vib.*, **319**(3-5), 963-983.

Sofiyev, A.H., Schnack, E., Haciyev, V.C. and Kuruoglu, N. (2012), "Effect of the two-parameter elastic foundation on the critical parameters of non-homogeneous orthotropic shells", *Int. J. Struct. Stabil. Dynam.*, **12**(5), 24 p.

Sofiyev, A.H., Deniz, A., Ozyigit, P. and Pinarlik, M. (2014), "Stability analysis of clamped nonhomogeneous shells on the elastic foundation", *Acta Phys. Pol.*, **125**(2), 459-461.

Soldatos, K.P. and Timarci, T. (1993), "A unified formulation of laminated composite, shear deformable, five-degrees-of-freedom cylindrical shell theories", *Compos. Struct.*, **25**(1-4), 165-171.

Thai, H.T. and Choi, D.H. (2012), "A refined shear deformation theory for free vibration of functionally graded plates on elastic foundation", *Compos. B Eng.*, **43**(5), 2335-2347.

Tornabene, F. (2011), "Free vibrations of anisotropic doubly-curved shells and panels of revolution with a free-form meridian resting on Winkler-Pasternak elastic foundations", *Compos. Struct.*, **94**(1), 186-206.

Tornabene, F., Fantuzzi, N., Viola, E. and Reddy, J.N. (2014), "Winkler-Pasternak foundation effect on the static and dynamic analyses of laminated doubly-curved and degenerate shells and panels", *Compos. B Eng.*, **57**, 269-296.

Wosu, S.N., Hui, D. and Daniel, L. (2012), "Hygrothermal effects on the dynamic compressive properties of graphite/epoxy composite material", *Compos. B Eng.*, **43**(3), 841-855.

Zenkour, A.M., Alam, M.N.M. and Radwan, A.F. (2013), "Bending of cross-ply laminated plates resting on elastic foundations under thermo-mechanical loading", *Int. J. Mech. Mater. Des.*, **9**(3), 239-251.

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Appendix

The differential operators L_{ij} $(i, j = 1, 2, 3, 4)$ are

$$
\begin{aligned}
&L_{11} = h(c_{11} - c_{31})(),_{xxyy} + hc_{12}(),_{xxxx}, \quad L_{12} = -c_{13}(),_{xxxx} - (c_{14} + c_{32})(),_{xxyy}, \\
&L_{13} = c_{15}(),_{xxx} + c_{35}(),_{xyy} - I_5(),_x, \quad L_{14} = (c_{18} + c_{38})(),_{xxy}, \\
&L_{21} = hc_{21}(),_{yyyy} + (c_{22} - c_{31})(),_{xxyy}, \quad L_{22} = -(c_{32} + c_{23})(),_{xxyy} - c_{24}(),_{yyyy}, \\
&L_{23} = (c_{25} + c_{35})(),_{xyy}, \quad L_{24} = c_{28}(),_{yyy} + c_{38}(),_{xxy} - I_6(),_y, \\
&L_{31} = hb_{22}(),_{xxxx} + h(b_{12} + b_{21} + b_{31})(),_{xxyy} + hb_{11}(),_{yyyy}, \\
&L_{32} = -b_{23}(),_{xxxx} - (b_{24} + b_{13} - b_{32})(),_{xxyy} - b_{14}(),_{yyyy} + (),_{xx}/R, \\
&L_{33} = b_{25}(),_{xxx} + (b_{15} + b_{35})(),_{xyy}, \quad L_{34} = (b_{28} + b_{38})(),_{xxy} + b_{18}(),_{yyy}, \\
&L_{41} = h(),_{xx}/R, \quad L_{42} = -0.5PR(),_{xx} - PR(),_{yy} - K_w + K_p \left[(),_{xx} + (),_{yy}\right], \\
&L_{43} = I_5(),_x, \quad L_{44} = I_6(),_y
\end{aligned}
\tag{A-1}
$$

where the following definitions apply

$$
\begin{aligned}
&c_{11} = \Lambda_{11}^1 b_{11} + \Lambda_{12}^1 b_{21}, \; c_{12} = \Lambda_{11}^1 b_{12} + \Lambda_{12}^1 b_{12}, \; c_{13} = \Lambda_{11}^1 b_{13} + \Lambda_{12}^1 b_{23} + \Lambda_{11}^2 \\
&c_{14} = \Lambda_{11}^1 b_{14} + \Lambda_{12}^1 b_{24} + \Lambda_{12}^2, c_{15} = \Lambda_{11}^1 b_{15} + \Lambda_{12}^1 b_{25} + \Lambda_{15}^1, c_{18} = \Lambda_{11}^1 b_{18} + \Lambda_{12}^1 b_{28} + \Lambda_{18}^1 \\
&c_{21} = \Lambda_{21}^1 b_{11} + \Lambda_{22}^1 b_{21}, \; c_{22} = \Lambda_{21}^1 b_{12} + \Lambda_{22}^1 b_{22}, \; c_{23} = \Lambda_{21}^1 b_{13} + \Lambda_{22}^1 b_{23} + \Lambda_{21}^2, \\
&c_{24} = \Lambda_{21}^1 b_{14} + \Lambda_{22}^1 b_{24} + \Lambda_{22}^2, c_{25} = \Lambda_{21}^1 b_{15} + \Lambda_{22}^1 b_{25} + \Lambda_{25}^1, c_{28} = \Lambda_{21}^1 b_{18} + \Lambda_{22}^1 b_{28} + \Lambda_{28}^1 \\
&c_{31} = \Lambda_{66}^1 b_{31}, \; c_{32} = \Lambda_{66}^1 b_{32} + 2\Lambda_{66}^2, \; c_{35} = \Lambda_{35}^1 - \Lambda_{66}^1 b_{35}, \; c_{38} = \Lambda_{38}^1 - \Lambda_{66}^1 b_{38} \\
&b_{11} = \frac{\Lambda_{22}^0}{\Delta}, \; b_{12} = -\frac{\Lambda_{12}^0}{\Delta}, \; b_{13} = \frac{\Lambda_{12}^0 \Lambda_{21}^1 - \Lambda_{11}^1 \Lambda_{22}^0}{\Delta}, \; b_{14} = \frac{\Lambda_{12}^0 \Lambda_{22}^1 - \Lambda_{12}^1 \Lambda_{22}^0}{\Delta}, \\
&b_{15} = \frac{\Lambda_{25}^0 \Lambda_{12}^0 - \Lambda_{15}^0 \Lambda_{22}^0}{\Delta}, \; b_{18} = \frac{\Lambda_{28}^0 \Lambda_{12}^0 - \Lambda_{18}^0 \Lambda_{22}^0}{\Delta}, \; b_{21} = -\frac{\Lambda_{21}^0}{\Delta}, \; b_{22} = \frac{\Lambda_{11}^0}{\Delta}, \\
&b_{23} = \frac{\Lambda_{11}^1 \Lambda_{21}^0 - \Lambda_{21}^1 \Lambda_{11}^0}{\Delta}, \; b_{24} = \frac{\Lambda_{12}^1 \Lambda_{21}^0 - \Lambda_{22}^1 \Lambda_{11}^0}{\Delta}, \; b_{25} = \frac{\Lambda_{15}^0 \Lambda_{21}^0 - \Lambda_{25}^0 \Lambda_{11}^0}{\Delta}, \\
&b_{28} = \frac{\Lambda_{18}^0 \Lambda_{21}^0 - \Lambda_{28}^0 \Lambda_{11}^0}{\Delta}, \; \Delta = \Lambda_{11}^0 \Lambda_{22}^0 - \Lambda_{12}^0 \Lambda_{21}^0, \; b_{31} = \frac{1}{\Lambda_{66}^0}, \; b_{32} = -\frac{2\Lambda_{66}^1}{\Lambda_{66}^0}, \\
&b_{35} = \frac{\Lambda_{35}^0}{\Lambda_{66}^0}, \; b_{38} = \frac{\Lambda_{38}^0}{\Lambda_{66}^0}, \; I_5 = \int_{-h/2}^{h/2} f_1(Z)\mathrm{d}z, \; I_6 = \int_{-h/2}^{h/2} f_2(Z)\mathrm{d}z.
\end{aligned}
\tag{A-2}
$$

in which

$$\Lambda_{11}^{k_1} = \frac{E_{01}\Theta(z)}{1-\nu_{12}\nu_{21}}, \ \Lambda_{12}^{k_1} = \frac{\nu_{21}E_{01}\Theta(z)}{1-\nu_{12}\nu_{21}}, \ \Lambda_{22}^{k_1} = \frac{E_{02}\Theta(z)}{1-\nu_{12}\nu_{21}}, \ \Lambda_{66}^{k_1} = G_{012}\Theta(z),$$

$$\Lambda_{15}^{k_2} = \int_{-h/2}^{h/2} z^{k_2} I_{01} Q_{11}(Z) \mathrm{d}z, \Lambda_{18}^{k_2} = \int_{-h/2}^{h/2} z^{k_2} I_{02} Q_{12}(Z) \mathrm{d}z, \ \Lambda_{25}^{k_2} = \int_{-h/2}^{h/2} z^{k_2} I_{01} Q_{21}(Z) \mathrm{d}z,$$

$$\Lambda_{28}^{k_2} = \int_{-h/2}^{h/2} z^{k_2} I_{02} Q_{22}(Z) \mathrm{d}z, \ \Lambda_{35}^{k_2} = \int_{-h/2}^{h/2} z^{k_2} I_{01} Q_{66}(Z) \mathrm{d}z, \Lambda_{38}^{k_2} = \int_{-h/2}^{h/2} z^{k_2} I_{02} Q_{66}(Z) \mathrm{d}z,$$

$$\Theta(z) = \int_{-h/2}^{h/2} e^{\mu(Z-0.5)} z^{k_1} dz, \quad k_1 = 0,1,2; \ k_2 = 0,1.$$

(A-3)