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A novel first-order shear deformation theory for laminated composite plates

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Abstract. In the present study, a new simple first-order shear deformation theory is presented for laminated composite plates. Moreover, the number of unknowns of this theory is the least one comparing with the traditional first-order and the other higher-order shear deformation theories. Equations of motion and boundary conditions are derived from Hamilton's principle. Analytical solutions of simply supported antisymmetric cross-ply and angle-ply laminates are obtained and the results are compared with the exact three-dimensional (3D) solutions and those predicted by existing theories. It can be concluded that the proposed theory is accurate and simple in solving the static bending and free vibration behaviors of laminated composite plates.

Keywords: laminated composite plate; plate theory; bending; vibration

1. Introduction

Composites are widely applicable in aerospace, civil, mechanical and other fields of modern technology due to their favorable characteristics of high stiffness and strength to weight ratio. Composite plates are one of the most important structural elements that were studied by many researchers in the last 6 or 7 decades and a variety of laminated theories has been introduced. The classical plate theory (CPT), which neglects the transverse normal and shear stresses, provides reasonable results for thin laminates. However, it underpredicts deflections and overpredicts frequencies as well as buckling loads with moderately thick laminates (Reddy 1997). However, in thick and moderately thick plates, the shear de formations cannot be neglected, and the theory shows inaccurate results for them. There are numerous theories that include the transverse shear strains. One of the well-known theories is the Reissner model (Reissner 1945), which is called the first-order shear-deformation theory (FSDT) and considers the displacement field as linear variations of midplane displacements. Considering FSDT, Moradi and Mansouri (2012) studied thermal buckling of thin and thick laminated plates using this method. Some other plate theories, namely, the higher-order shear-deformation theories (HSDT), which include the effect of

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transverse shear de formations, are the Levy (1977), Hencky (1947), Lo *et al*. (1977), Nelson and Lorch (1974), and Reddy (1984) theories. The latter one is a simple higher-order theory that takes into account not only the trans verse shear strains, but also their parabolic variation across the plate thickness and requires no shear correction coefficients in computing the shear stresses. Reddy (1984, 2000) developed a third-order shear deformation theory (TSDT) with cubic variations for in-plane displacements. Xiang *et al*. (2011, 2013) proposed a n-order shear deformation theory in which Reddy's theory can be considered as a specific case. Based on the mixed variational approach, Fares *et al*. (2009) proposed a HSDT with linear and parabolic variations for in-plane and transverse displacements, respectively. Bodaghi and Saidi (2011) presented a higher order shear deformation theory for thermo-elastic buckling behavior of thick rectangular plate made of functionally graded materials. Matsunaga (2000) developed a higher order theory based on a complete power series expansion of the displacement field in the thickness coordinate. Bouderba *et al*. (2013) developed a simple trigonometric shear deformation theory to investigate thermo -mechanical behavior of simply supported functionally graded plates resting on a Winkler–Pasternak elastic foundation. Bakhti *et al*. (2013) proposed an efficient and simple refined theory for nonlinear cylindrical bending behavior of functionally graded nanocomposite plates.

Although some well-known HSDTs have five unknowns as in the case of FSDT (e.g., the third-order shear deformation theory (Reddy 1984, 2000, Bodaghi and Saidi 2011)), their equations of motion are much more complicated than those of FSDT. Thus, needs exist for the development of shear deformation theory which is simple to use.

In the present work, a new a simple FSDT is developed for the bending and free vibration analysis of laminated composite plates. Unlike the conventional FSDT, the present one contains only four unknowns. Thus, the number of unknowns and governing equations for the present theory is reduced, significantly facilitating engineering analysis. Equations of motion are derived from Hamilton's principle. Closed-form solutions of simply supported antisymmetric cross-ply and angle-ply laminates are obtained. Numerical examples are presented to verify the accuracy of the present theory.

2. Theoretical formulations

Consider a rectangular composite plate of thickness h, length a, and width b, referred to the rectangular cartesian coordinates (x, y, z). The $x-y$ plane is taken to be the undeformed mid-plane of the plate, and the z axis is perpendicular to the $x-y$ plane.

2.1 Basic assumptions

The assumptions of the present theory are as follows:

(1) The displacements are small in comparison with the plate thickness and, therefore, strains involved are infinitesimal.
(2) The transverse normal stress σ_z is negligible in comparison with in-plane stresses σ_x and σ_y.
(3) This theory assumes constant transverse shear stress and it needs a shear correction factor in order to satisfy the plate boundary conditions on the lower and upper surface.

2.2 Kinematics

Based on the assumptions made in the preceding section, the displacement field can be obtained as follows

$$u(x,y,z,t) = u_0(x,y,t) - z\frac{\partial \Phi}{\partial x} \tag{1a}$$

$$v(x,y,z,t) = v_0(x,y,t) - z\frac{\partial \Phi}{\partial y} \tag{1b}$$

$$w(x,y,z,t) = w(x,y,t) \tag{1c}$$

where, u, v, w are displacements in the x, y, z directions, u_0 and v_0 are the midplane surface displacements. Φ is function of coordinates x, y and time t.

The strains associated with the displacements in Eq. (1) are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z\begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \gamma_{yz}^s \\ \gamma_{xz}^s \end{Bmatrix} \tag{2}$$

where

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial x} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix}, \quad \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 \Phi}{\partial x^2} \\ -\dfrac{\partial^2 \Phi}{\partial y^2} \\ -2\dfrac{\partial^2 \Phi}{\partial x \partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz}^s \\ \gamma_{xz}^s \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial w}{\partial y} - \dfrac{\partial \Phi}{\partial y} \\ \dfrac{\partial w}{\partial x} - \dfrac{\partial \Phi}{\partial x} \end{Bmatrix} \tag{3}$$

The linear constitutive relations of a for a layer can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{4}$$

where (σ_x, σ_y, τ_{xy}, τ_{yz}, τ_{xz}) and (ε_x, ε_y, γ_{xy}, γ_{yz}, γ_{xz}) are the stress and strain components, respectively. Q_{ij} are the material constants in the material axes of the layer given as

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \; Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}, \; Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}, \; Q_{66} = G_{12}, \; Q_{44} = G_{23}, \; Q_{55} = G_{13} \tag{5}$$

Since the laminate is made of several orthotropic layers with their material axes oriented arbitrarily with respect to the laminate coordinates, the constitutive equations of each layer must be transformed to the laminate coordinates x, y, z. The stress-strain relations in the laminate coordinates of the k -th layer are given as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \tau_{yz} \\ \tau_{zx} \end{Bmatrix}^{(k)} = \begin{bmatrix} \overline{Q}_{44} & \overline{Q}_{45} \\ \overline{Q}_{45} & \overline{Q}_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{6}$$

where $\overline{Q}_{ij}$ are the transformed material constants given in Reddy (2004).

2.3 Equations of motion

Hamilton's principle is used herein to derive the equations of motion. The principle can be stated in analytical form as (Reddy 2002, Jones 1999)

$$0 = \int_0^T (\delta U + \delta V - \delta K) dt \tag{7}$$

where δU is the variation of strain energy; δV is the variation of potential energy; and δK is the variation of kinetic energy.

The variation of strain energy of the plate is calculated by

$$\begin{aligned} \delta U &= \int_V \left(\sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + \tau_{xy} \delta \gamma_{xy} + \tau_{yz} \delta \gamma_{yz} + \tau_{xz} \delta \gamma_{xz} \right) dA dz \\ &= \int_A \left\{ N_x \frac{\partial \delta u_0}{\partial x} - M_x \frac{\partial^2 \delta \Phi}{\partial x^2} + N_y \frac{\partial \delta v_0}{\partial y} - M_y \frac{\partial^2 \delta \Phi}{\partial y^2} + N_{xy} \left(\frac{\partial \delta u_0}{\partial y} + \frac{\partial \delta v_0}{\partial x} \right) \right. \\ &\quad \left. - 2 M_{xy} \frac{\partial^2 \delta \Phi}{\partial x \partial y} + Q_{yz} \frac{\partial \delta (w - \Phi)}{\partial y} + Q_{xz} \frac{\partial \delta (w - \Phi)}{\partial x} \right\} dA \end{aligned} \tag{8}$$

where N, M, and Q are the stress resultants defined as

$$(N_i, M_i) = \int_{-h/2}^{h/2} (1, z) \sigma_i dz, \ (i = x, y, xy) \quad \text{and} \quad (Q_{xz}, Q_{yz}) = \int_{-h/2}^{h/2} (\tau_{xz}, \tau_{yz}) dz. \tag{9}$$

The variation of potential energy of the applied loads can be expressed as

$$\delta V = -\int_A q \delta w \, dA \tag{10}$$

where q is the transverse applied load.

The variation of kinetic energy of the plate can be written as

$$\downarrow \ \delta K = \int_V (\dot{u} \delta \dot{u} + \dot{v} \delta \dot{v} + \dot{w} \delta \dot{w}) \rho dA dz \tag{11}$$

$$\uparrow \quad = \int_A \Big\{ I_0 \left[\dot{u}_0 \delta \dot{u}_0 + \dot{v}_0 \delta \dot{v}_0 + \dot{w} \delta \dot{w} \right] + I_2 \left(\frac{\partial \dot{\Phi}}{\partial x} \frac{\partial \delta \dot{\Phi}}{\partial x} + \frac{\partial \dot{\Phi}}{\partial y} \frac{\partial \delta \dot{\Phi}}{\partial y} \right) \Big\} dA \tag{11}$$

where dot-superscript convention indicates the differentiation with respect to the time variable t; ρ is the mass density; and (I_0, I_2) are mass inertias defined as

$$(I_0, I_2) = \int_{-h/2}^{h/2} (1, z^2) \rho dz \tag{12}$$

Substituting the expressions for δU, δV, and δK from Eqs. (8), (10), and (11) into Eq. (7) and integrating by parts, and collecting the coefficients of δu_0, δv_0, $\delta \Phi$ and δw, the following equations of motion of the plate are obtained

$$\delta u_0 : \; \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \ddot{u}_0 \tag{13a}$$

$$\delta v_0 : \; \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = I_0 \ddot{v}_0 \tag{13b}$$

$$\delta \Phi : \; \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - \frac{\partial Q_{xz}}{\partial x} - \frac{\partial Q_{yz}}{\partial y} = -I_2 \nabla^2 \ddot{\Phi} \tag{13c}$$

$$\delta w : \; \frac{\partial Q_{xz}}{\partial x} + \frac{\partial Q_{yz}}{\partial y} + q = I_0 \ddot{w} \tag{13d}$$

By substituting Eq. (2) into Eq. (4) and the subsequent results into Eq. (8), the stress resultants are obtained as

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon \\ k \end{Bmatrix}, \quad Q = k_s A^s \gamma, \tag{14}$$

where

$$N = \left\{ N_x, N_y, N_{xy} \right\}^t, \quad M = \left\{ M_x, M_y, M_{xy} \right\}^t, \tag{15a}$$

$$\varepsilon = \left\{ \varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0 \right\}^t, \quad k = \left\{ k_x, k_y, k_{xy} \right\}^t, \tag{15b}$$

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}, \; B = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}, \; D = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}, \tag{15c}$$

$$Q = \{Q_{xz}, Q_{yz}\}^t, \ \gamma = \{\gamma_{xz}, \gamma_{yz}\}^t, \ A^s = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix}, \tag{15d}$$

where A_{ij}, B_{ij} and D_{ij}, are the plate stiffness, defined by

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} \overline{Q}_{ij} \left(1, z, z^2\right) dz \tag{16}$$

k_s is a shear correction factor which is analogous to shear correction factor proposed by Mindlin (1951).

By substituting Eq. (14) into Eq. (13), the equations of motion can be expressed in terms of displacements (u_0, v_0, Φ, w) as

$$A_{11} d_{11} u_0 + A_{66} d_{22} u_0 + \left(A_{12} + A_{66}\right) d_{12} v_0 = I_0 \ddot{u} \tag{17a}$$

$$A_{22} d_{22} v_0 + A_{66} d_{11} v_0 + \left(A_{12} + A_{66}\right) d_{12} u_0 = I_0 \ddot{u} \tag{17b}$$

$$-D_{11} d_{1111} \Phi - 2\left(D_{12} + 2D_{66}\right) d_{1122} \Phi - D_{22} d_{2222} \Phi - A_{55}^s d_{11}(w - \Phi) - A_{44}^s d_{22}(w - \Phi) = -I_2 \nabla^2 \ddot{\Phi} \tag{17c}$$

$$A_{55}^s d_{11}(w - \Phi) + A_{44}^s d_{22}(w - \Phi) + q = I_0 \ddot{w} \tag{17d}$$

where d_{ij}, and d_{ijlm} are the following differential operators

$$d_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}, \ d_{ijlm} = \frac{\partial^4}{\partial x_i \partial x_j \partial x_l \partial x_m}, \ d_i = \frac{\partial}{\partial x_i}, \ (i, j, l, m = 1, 2). \tag{18}$$

Clearly, when the effect of transverse shear deformation is neglected (w = Φ), Eq. (17) yields the equations of laminated plate.

3. Exact solution for a simply-supported antisymmetric cross-ply and angle-ply laminates

Rectangular plates are generally classified in accordance with the type of support used. We are here concerned with the exact solution of Eqs. (17a)-(17d) for a simply supported antisymmetric cross-ply and angle-ply laminates. The following boundary conditions are imposed at the side edges for the present theory:

- Simply supported edge (cross-ply laminate)

$$v_0 = w = \Phi = N_x = M_x = 0 \quad \text{at} \quad x = 0, \ a \tag{19a}$$

$$u_0 = w = \Phi = N_y = M_y = 0 \quad \text{at} \quad y = 0, \ b \tag{19b}$$

- Simply supported edge (angle-ply laminate)

$$u_0 = w = \Phi = N_{xy} = M_x = 0 \quad \text{at} \quad x = 0,\ a \tag{20a}$$

$$v_0 = w = \Phi = N_{xy} = M_y = 0 \quad \text{at} \quad y = 0,\ b \tag{20b}$$

Following the Navier solution procedure, we assume the following solution form for u_0, v_0, Φ, and w that satisfies the boundary conditions

$$\begin{Bmatrix} \Phi \\ w \end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{Bmatrix} \Psi_{mn} \sin(\lambda\, x)\sin(\mu\, y) e^{i\,\omega\, t} \\ W_{mn} \sin(\lambda\, x)\sin(\mu\, y) e^{i\,\omega\, t} \end{Bmatrix} \tag{21a}$$

$$\begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{Bmatrix} U_{mn} \cos(\lambda\, x)\sin(\mu\, y) e^{i\,\omega\, t} \\ V_{mn} \sin(\lambda\, x)\cos(\mu\, y) e^{i\,\omega\, t} \end{Bmatrix} \text{(antisymmetric cross - ply)} \tag{21b}$$

$$\begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{Bmatrix} U_{mn} \sin(\lambda\, x)\cos(\mu\, y) e^{i\,\omega\, t} \\ V_{mn} \cos(\lambda\, x)\sin(\mu\, y) e^{i\,\omega\, t} \end{Bmatrix} \text{(antisymmetric angles - ply)} \tag{21c}$$

where U_{mn}, V_{mn}, Ψ_{mn}, and W_{mn} are arbitrary parameters to be determined, ω is the eigenfrequency associated with (m, n) th eigenmode, and $\lambda = m\pi / a$, $\mu = n\pi / b$ and $i = \sqrt{-1}$.

The transverse load q is also expanded in the double-Fourier sine series as

$$q(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Q_{mn} \sin(\lambda\, x)\sin(\mu\, y) \tag{22}$$

The coefficients Q_{mn} are given below for some typical loads

$$Q_{mn} = \frac{4}{ab}\int_0^a\int_0^b q(x, y)\sin(\lambda\, x)\sin(\mu\, y)\,dxdy = \begin{cases} q_0 & \text{for sinusoidal load} \\ \dfrac{16 q_0}{mn\pi^2} & \text{for sinusoidal load} \end{cases} \tag{23}$$

Substituting Eqs. (21) and (22) into Eq. (17), the closed-form solutions can be obtained from

$$\left([C] - \omega^2 [M]\right)\{\Delta\} = \{P\}, \tag{24}$$

where $\{\Delta\} = \{U, V, \Psi, W\}^t$, and $[C]$ and $[M]$ are the symmetric matrixes given by

$$[C] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix},\ [M] = \begin{bmatrix} m_{11} & 0 & 0 & 0 \\ 0 & m_{22} & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{34} & m_{44} \end{bmatrix} \tag{25}$$

in which

$$
\begin{aligned}
&a_{11} = -\left(A_{11}\lambda^2 + A_{66}\mu^2\right)\\
&a_{12} = -\lambda\,\mu\left(A_{12} + A_{66}\right)\\
&a_{13} = \lambda^3 B_{11}\\
&a_{14} = 0\\
&a_{22} = -\left(A_{66}\lambda^2 + A_{22}\mu^2\right)\\
&a_{23} = \mu^3 B_{22}\\
&a_{24} = 0\\
&a_{33} = -\left(D_{11}\lambda^4 + 2(D_{12} + 2D_{66})\lambda^2\mu^2 + D_{22}\mu^4 + A_{55}\lambda^2 + A_{44}\mu^2\right)\\
&a_{34} = A_{55}\lambda^2 + A_{44}\mu^2\\
&a_{44} = -A_{55}\lambda^2 - A_{44}\mu^2\\
&m_{11} = m_{22} = -I_0\\
&m_{33} = -I_2\left(\lambda^2 + \mu^2\right)\\
&m_{44} = m_{34} = -I_0
\end{aligned}
\tag{26}
$$

The components of the generalized force vector $\{P\} = \{P_1, P_2, P_3, P_4\}^t$ are given by

$$
\begin{aligned}
&P_1 = 0\\
&P_2 = 0\\
&P_3 = 0\\
&P_4 = -q_0
\end{aligned}
\tag{27}
$$

4. Results and discussion

In this section, various numerical examples are presented and discussed to verify the accuracy of the present theory in predicting the bending and free vibration responses of simply supported laminated composite plates. For verification purpose, the obtained results are compared with the exact 3D solutions and those predicted by other plate models. The description of various plate models and their corresponding number of unknowns are listed in Table 1. For all calculations, the value of shear correction factor is taken as 5/6. The following lamina properties are used:

- Material 1 (Reddy 2004)

$$E_1 = 25E_2,\; G_{12} = G_{13} = 0.5E_2,\; G_{23} = 0.2E_2,\; \nu_{12} = 0.25 \tag{28a}$$

- Material 2 (Noor 1973)

$$E_1 / E_2 = \text{open},\; G_{12} = G_{13} = 0.6E_2,\; G_{23} = 0.5E_2,\; \nu_{12} = 0.25 \tag{28b}$$

- Material 3 (Noor and Burton 1990)

Table 1 Displacement models

Model	Theory	Unknowns
CPT	Classical plate theory	3
ZSDT	Zeroth-order shear deformation theory (Ray, 2003)	5
FSDT	First-order shear deformation theory	5
TSDT	Third-order shear deformation theory (Reddy, 1984)	5
HSDT	Higher-order shear deformation theory (Swaminathan and Patil, 2008)	12
Present	Simple first-order shear deformation theory	4

$$E_1 = 15E_2,\ G_{12} = G_{13} = 0.5E_2,\ G_{23} = 0.35E_2,\ \nu_{12} = 0.3 \tag{28c}$$

For convenience, the following dimensionless forms are used

$$\bar{z} = \frac{z}{h},\quad \bar{u}(z) = 100\frac{E_2h^3}{q_0a^4}u\left(0,\frac{b}{2},z\right),\quad \bar{w} = 100\frac{E_2h^3}{q_0a^4}w\left(\frac{a}{2},\frac{b}{2}\right),\quad \bar{\sigma}_{x,y}(z) = \frac{h^2}{q_0a^2}\sigma_{x,y}\left(\frac{a}{2},\frac{b}{2},z\right),$$

$$\bar{\tau}_{xy}(z) = \frac{h^2}{q_0a^2}\tau_{xy}(0,0,z),\quad \bar{\tau}_{xz}(z) = \frac{h}{q_0a}\tau_{xz}\left(0,\frac{b}{2},z\right),\quad \bar{\tau}_{yz}(z) = \frac{h}{q_0a}\tau_{yz}\left(\frac{a}{2},0,z\right),\quad \bar{\omega} = \frac{\omega a^2}{h}\sqrt{\frac{\rho}{E_2}}$$

4.1 Bending analysis

Example 1

In the first example, thin and thick two-layer antisymmetric cross-ply (0/90) square laminates made of Material 1 and subjected to sinusoidal loads are considered. The thickness ratios a / h are taken to be 2 (corresponding to very thick plates), 5 (corresponding to thick plates), 10, 20 (corresponding to moderately thick plates), and 100 (corresponding to thin plates). In Table 2, the obtained results using the new simple first-order shear deformation theory are compared with the exact 3D solutions given by Pagano (1970) and those computed using TSDT, FSDT and CPT. In general, the present theory and existing conventional FSDT give almost identical results for

Table 2 Dimensionless deflection $\bar{w}$ of two-layer antisymmetric cross-ply (0/90) square laminates under sinusoidal loads (Material 1)

Theory	a / h			
	2	5	10	100
Exact (Pagano 1970)	4.9362	1.7287	1.2318	1.0742
TSDT	4.5619	1.6670	1.2161	1.0651
FSDT	5.4059	1.7584	1.2373	1.0653
Present	5.4059	1.7584	1.2373	1.0653
CPT	1.0636	1.0636	1.0636	1.0636

various values of thickness ratio a / h. For the case of very thick laminates with $a / h = 2$, there are small errors in values predicted by the present theory, conventional FSDT and TSDT. However, these errors become negligible when the thickness ratio a / h is greater than 5. Due to ignoring shear deformation effects, the CPT provides acceptable results for the laminated composite plates with $a / h \geq 20$.

Example 2

In this example, an antisymmetric cross-ply $(0/90)_n$ square laminate under sinusoidal loads is investigated using Material 1. In Table 3, dimensionless deflections of laminates for different values of the thickness ratio and ply number are presented. The obtained results are compared with those given by Ray (2003) using the zeroth-order shear deformation theory (ZSDT) and those reported by Reddy (2004) using TSDT, FSDT and CPT. It can be seen that the present FSDT and existing conventional FSDT give solutions identical to each other, and their solutions are also in close agreement with those generated by Ray (2003) for all values of the thickness ratio and ply number.

The variations of dimensionless deflection with respect to thickness ratio a / h and material anisotropy E_1 / E_2 are showed in Figs. 1 and 2, respectively. These figures illustrate also the accuracy of present theory for wide range of thickness ratio a / h and material anisotropy E_1. The obtained results obtained by the present theory with only four unknowns are compared with those predicted by CPT and the conventional FSDT with five unknowns. Again, the present FSDT and the conventional FSDT give identical results, whereas CPT underestimates deflections of thick laminates with $a / h < 20$ due to ignoring shear deformation effects (see Fig. 1). The through thickness variations and corresponding values of the in-plane displacement $\bar{u}$, normal stresses ($\bar{\sigma}_x, \bar{\sigma}_y$), and shear stresses ($\bar{\tau}_{xy}, \bar{\tau}_{xz}, \bar{\tau}_{yz}$) are also given in Fig. 3, for a moderately thick laminate with $a / h = 10$. It can be seen again from Fig. 3 that the results predicted by the present theory with only four unknowns are identical with those obtained using the conventional FSDT with five unknowns.

Table 3 Dimensionless deflection $\bar{w}$ of antisymmetric cross-ply $(0/90)_n$ square laminates under sinusoidal loads (Material 1)

n	Theory	a / h			
		4	10	20	100
	ZSDT (Ray 2003)	2.0010	1.2160	1.1020	1.0650
	TSDT	1.9985	1.2161	1.1018	1.0651
1	FSDT	2.1492	1.2373	1.1070	1.0653
	CPT	1.0636	1.0636	1.0636	1.0636
	Present	2.1492	1.2373	1.1070	1.0653
	ZSDT (Ray 2003)	1.5410	0.6380	0.5060	0.4630
	TSDT	1.5411	0.6382	0.5060	0.4635
3	FSDT	1.5473	0.6354	0.5053	0.4635
	CPT	0.4617	0.4617	0.4617	0.4617
	Present	1.5473	0.6354	0.5053	0.4635

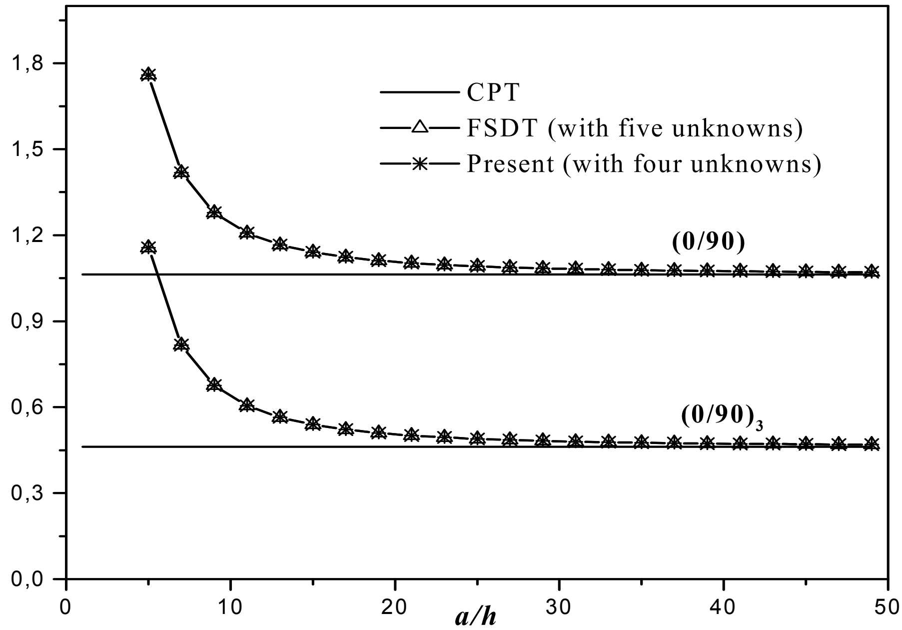

Fig. 1 Variation of dimensionless deflection of antisymmetric cross-ply $(0/90)_n$ square laminates under sinusoidal loads versus thickness ratio (Material 1)

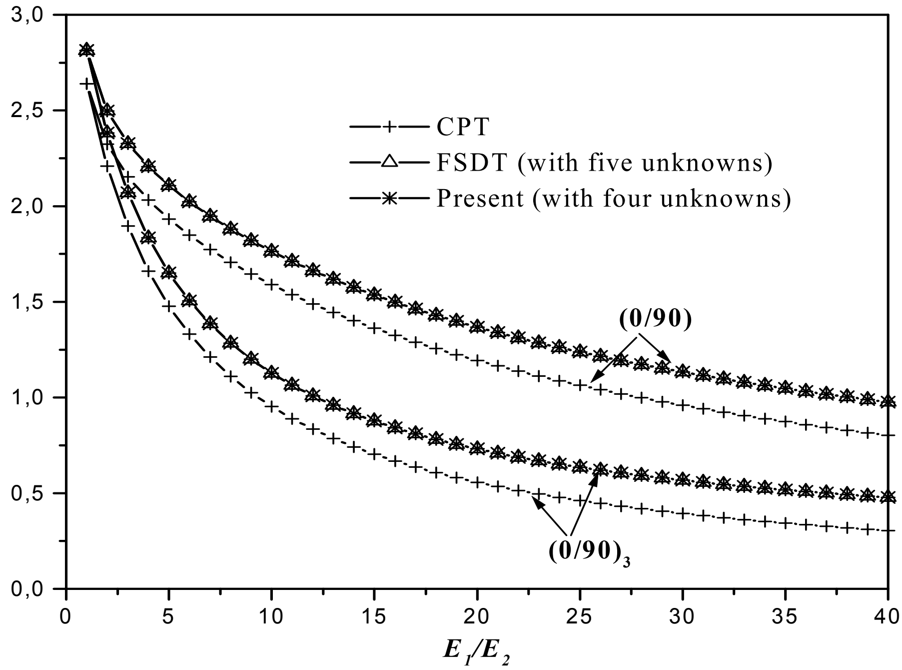

Fig. 2 Variation of dimensionless deflection of antisymmetric cross-ply $(0/90)_n$ square laminates under sinusoidal loads versus material anisotropy (Material 1, a / h = 10)

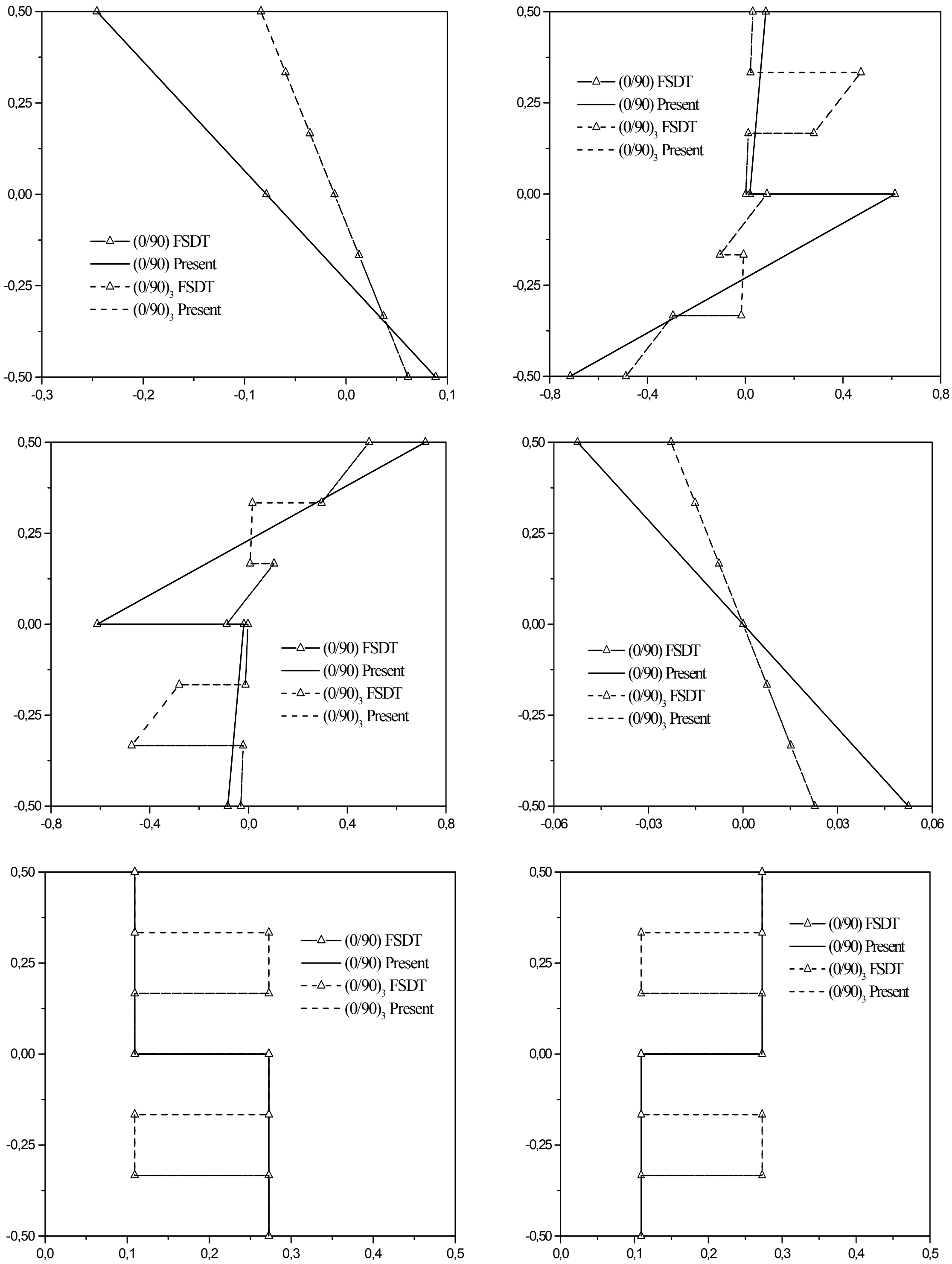

Fig. 3 Distributions of in-plane displacement and stress for antisymmetric cross-ply $(0/90)_n$ square laminates under sinusoidal loads (Material 1, a / h = 10)

Example 3

In this example, an antisymmetric angle-ply $(45/–45)_n$ square laminate under sinusoidal load is investigated using Material 1. In Table 4, dimensionless deflections of the plate are presented for different values of the thickness ratio and ply number. The obtained results are compared with those computed using CPT and the conventional FSDT. It can be seen that the present FSDT and the conventional FSDT give solutions close to each other. It should be noted that the present FSDT is simpler than the existing one due to containing a fewer number of unknowns and equations of motion (see Table 1).

To further illustrate the accuracy of the present theory for a wide range of lamination angle, Fig. 4 plots dimensionless deflections of antisymmetric angle-ply $(\theta\ /\ -\theta)_n$ square laminates under sinusoidal loads ($a\ /\ h$ = 10). The obtained results are compared with those computed using CPT

Table 4 Dimensionless deflection $\overline{w}$ of antisymmetric angle-ply $(45/\text{-}45)_n$ square laminates (Material 1)

n	Theory	a / h		
		10	20	100
1	FSDT	0.8284	0.6981	0.6564
	CPT	0.6547	0.6547	0.6547
	Present	0.8284	0.6981	0.6564
4	FSDT	0.4198	0.2896	0.2479
	CPT	0.2462	0.2462	0.2462
	Present	0.4198	0.2896	0.2479

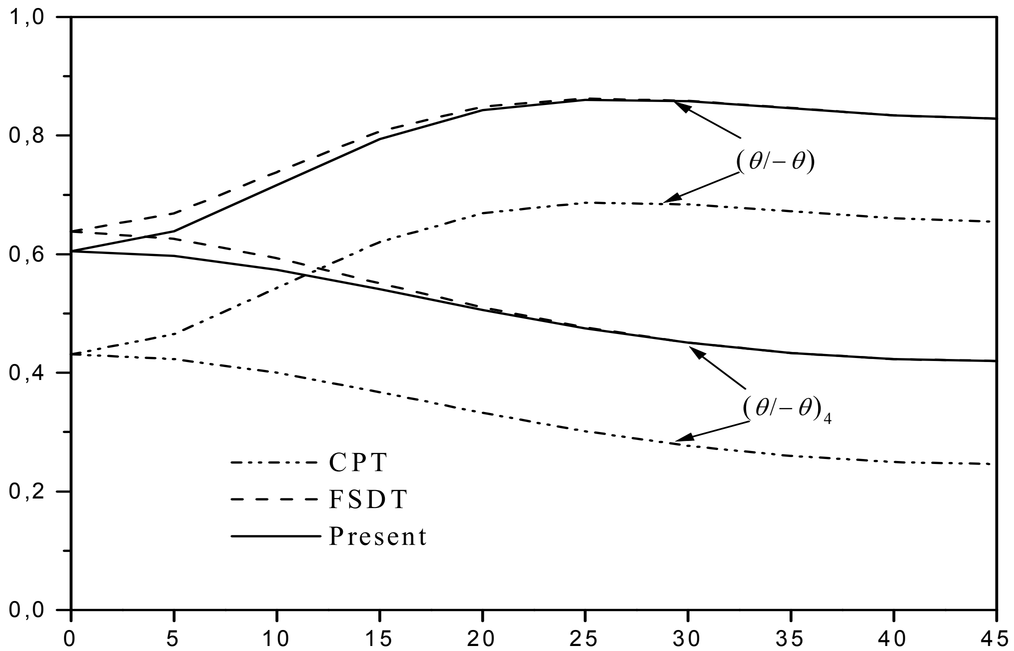

Fig. 4 Variation of dimensionless deflection of antisymmetric angle-ply $(\theta\ /-\theta)_4$ square laminates under sinusoidal loads versus lamination angle θ (Material 1, a / h = 10)

and the conventional FSDT. In general, the results of present theory and the conventional FSDT are almost identical, except in the case of low values of lamination angle ($\theta < 20°$) where a small discrepancy between the present theory and the conventional FSDT is seen.

4.2 Free vibration analysis

Example 4

In this verification, a thick antisymmetric cross-ply $(0/90)_n$ square laminate with $a / h = 5$ is analyzed using Material 2. The dimensionless fundamental frequency b is presented in Table 5 for different values of modulus ratio and ply number. The obtained results are compared with the exact 3D solutions reported by Noor (1973) and those computed using TSDT and the conventional FSDT. Again, it can be seen that the obtained results are identical with those predicted by the conventional FSDT. This can be observed also from Fig. 5 where the present solutions are compared with those predicted by the conventional FSDT for a wide range of thickness ratio a / h.

Example 5

The next example is carried out for a thin and thick 10-layer antisymmetric angle-ply $(\theta / -\theta)_5$ square laminates using Material 3. This example aims to verify the accuracy of the present theory for very thick FG plates. Table 6 shows dimensionless frequency for various values of thickness ratio and lamination angle. The obtained results are compared with the exact 3D solutions given by Noor and Burton (1990) where a very good agreement between the results is obtained.

Example 6

The last example is performed for antisymmetric angle-ply $(45/–45)_n$ square laminates with thickness ratio varied from 2 to 100. Material 2 is considered. Table 7 shows dimensionless

Table 5 Dimensionless fundamental frequency of antisymmetric cross-ply $(0/90)_n$ square laminates (Material 2, $a / h = 5$)

E_1 / E_2	Theory	n			
		1	2	3	5
3	Exact (Noor 1973)	6.2578	6.5455	6.6100	6.6458
	TSDT	6.2169	6.5008	6.5558	6.5842
	FSDT	6.2085	6.5043	6.5569	6.5837
	Present	6.2085	6.5043	6.5569	6.5837
10	Exact (Noor 1973)	6.9845	8.1445	8.4143	8.5625
	TSDT	6.9887	8.1954	8.4052	8.5126
	FSDT	6.9392	8.2246	8.4183	8.5132
	Present	6.9392	8.2246	8.4183	8.5132
40	Exact (Noor 1973)	8.5625	10.6789	11.2728	11.6245
	TSDT	9.0871	11.1716	11.5012	11.6730
	FSDT	8.8333	11.2708	11.5264	11.6444
	Present	8.8333	11.2708	11.5264	11.6444

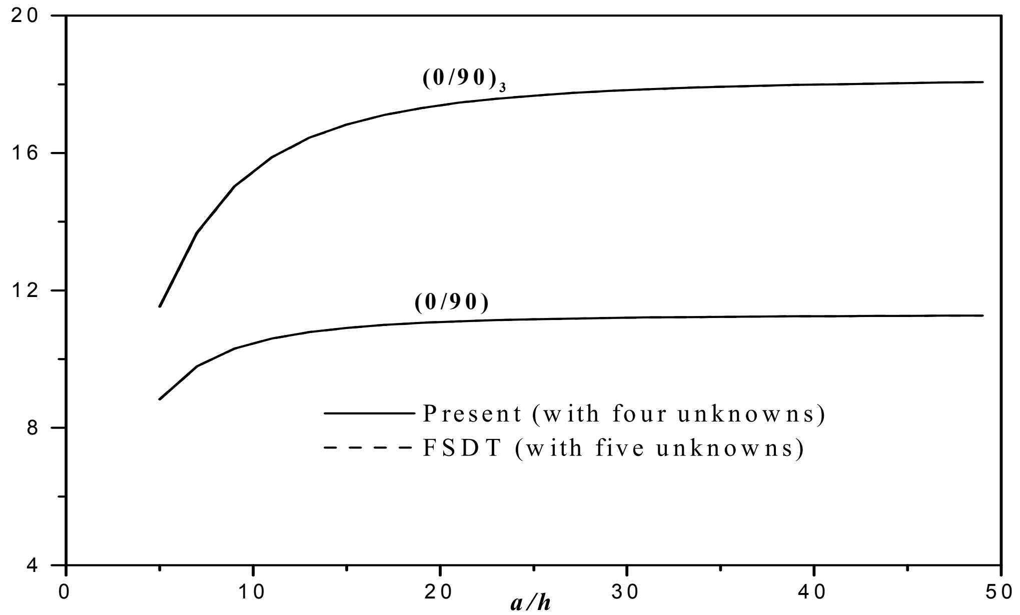

Fig. 5 Variation of dimensionless fundamental frequency of antisymmetric cross-ply $(0/90)_n$ square laminates versus thickness ratio (Material 2, $E_1 / E_2 = 40$)

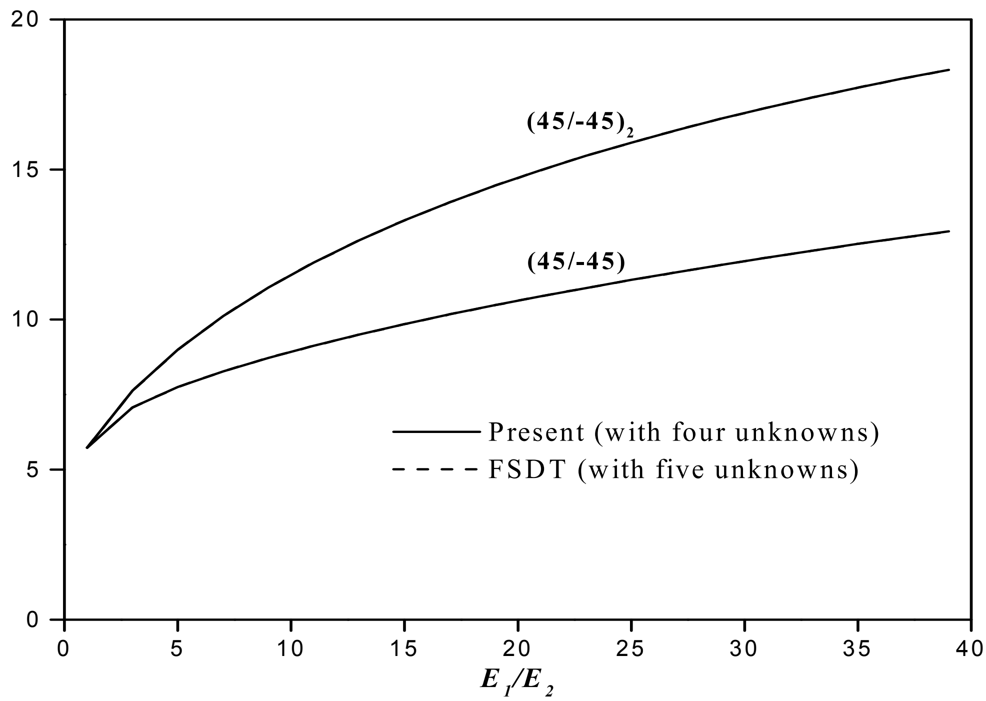

Fig. 6 Variation of dimensionless fundamental frequency of antisymmetric angle-ply $(45/\text{-}45)_n$ square laminates versus material anisotropy (Material 2, $a / h = 10$)

Table 6 Dimensionless fundamental frequency of ten-layer antisymmetric angle-ply $(\theta\ /-\theta)_5$ square laminates (Material 3)

$h\ /\ a$	Theory	θ		
		15°	30°	45°
0.01	Exact (Noor and Burton 1990)	13.2800	15.1000	15.9500
	Present	13.2795	15.1039	15.9484
0.10	Exact (Noor and Burton 1990)	11.6200	12.9600	13.5100
	Present	11.8161	13.0695	13.6140
0.15	Exact (Noor and Burton 1990)	10.2400	11.2533	11.6311
	Present	10.5338	11.4087	11.7710
0.20	Exact (Noor and Burton 1990)	8.9700	9.7225	9.9825
	Present	9.2965	9.892	10.1288
0.25	Exact (Noor and Burton 1990)	7.8944	8.4576	8.6400
	Present	8.2107	8.6185	8.7756
0.30	Exact (Noor and Burton 1990)	7.0078	7.4356	7.5667
	Present	7.2930	7.5778	7.6849

Table 7 Dimensionless fundamental frequency of antisymmetric angle-ply $(45/\text{-}45)_n$ square laminates (Material 2)

n	$E_1\ /\ E_2$	Theory	$a\ /\ h$					
			2	4	10	20	50	100
1	3	HSDT[(a)]	4.5312	6.1223	7.1056	7.3001	7.3583	7.3666
		Present	4.4556	6.0665	7.0700	7.2694	7.3291	7.3378
	10	HSDT[(a)]	4.9742	7.2647	8.9893	9.3753	9.4943	9.5123
		Present	4.9316	7.2169	8.9324	9.3173	9.4362	9.4537
	20	HSDT[(a)]	5.1817	8.0490	10.6412	11.2975	10.5074	11.5385
		Present	5.2387	8.1185	10.6265	11.2517	11.4511	11.4806
	30	HSDT[(a)]	5.2771	8.5212	11.8926	12.8422	13.1566	13.2035
		Present	5.4104	8.7213	11.9456	12.8208	13.1077	13.1505
	40	HSDT[(a)]	5.3325	8.8426	12.9115	14.1705	14.6012	14.6668
		Present	5.5205	9.1609	13.0439	14.1790	14.5608	14.6183
2	3	HSDT[(a)]	4.6498	6.4597	7.6339	7.8724	7.9442	7.9545
		Present	4.6519	6.4626	7.6293	7.8657	7.9368	7.9472
	10	HSDT[(a)]	5.2061	8.3447	11.4116	12.2294	12.4952	12.5351
		Present	5.3765	8.5634	11.4939	12.2463	12.4881	12.5239
	20	HSDT[(a)]	5.4140	9.3306	14.4735	16.2570	16.8949	16.9927
		Present	5.6542	9.7575	14.7292	16.3394	16.9008	16.9862
	30	HSDT[(a)]	5.5079	9.7966	16.4543	19.2323	20.3134	20.4839
		Present	5.7641	10.3391	16.8825	19.3944	20.3361	20.4827
	40	HSDT[(a)]	5.5674	10.0731	17.8773	21.6229	23.1949	23.4499
		Present	5.8228	10.6839	18.4633	21.8722	23.2368	23.4541

fundamental frequencies. The obtained results are compared with those reported by Swaminathan and Patil (2008) based on HSDT. A good agreement between the results is seen for various values of thickness ratio and material anisotropy. In addition, the dimensionless fundamental frequencies obtained from the present theory are in excellent agreement with those predicted by the conventional FSDT as shown in Fig. 6. It should be noted that the number of unknowns of the present theory is only four as against five in the case of the conventional FSDT and twelve in the case of HSDT (Swaminathan and Patil 2008) (see Table 1). Thus, it can be concluded that the present theory is not only accurate but also simple in predicting the response of laminated plates.

5. Conclusions

A simple FSDT was developed for bending and free vibration analysis of laminates. Unlike the other shear deformations theories, just four unknown displacement functions are used in the present theory against five unknown displacement functions used in the corresponding ones. Verification studies show that these simplifying assumptions have a minimal effect on the accuracy of results for the problem considered. Therefore, it can be concluded that the new FSDT is not only accurate but also simple in predicting the bending and vibration responses of laminates. Finally, the formulation lends itself particularly well to study the mechanical behavior of functionally graded structures (Yaghoobi and Torabi 2013a, b, c, Yaghoobi and Yaghoobi 2013, Bouderba *et al*. 2013, Bakhti *et al*. 2013, Benachour *et al*. 2011), which will be considered in the near future.

References

Bakhti, K., Kaci, A., Bousahla, A.A., Houari, M.S.A., Tounsi, A. and Adda Bedia, E.A. (2013), "Large deformation analysis for functionally graded carbon nanotube-reinforced composite plates using an efficient and simple refined theory", *Steel Compos. Struct., Int. J.*, **14**(4), 335-347.

Benachour, A., Daouadji Tahar, H., Ait Atmane, H., Tounsi, A. and Meftah, S.A. (2011), "A four variable refined plate theory for free vibrations of functionally graded plates with arbitrary gradient", *Compos. Part B: Engineering*, **42**(6), 1386-1394.

Bodaghi, M. and Saidi, A.R. (2011), "Thermoelastic buckling behavior of thick functionally graded rectangular plates", *Arch. Appl. Mech.*, **81**(11), 1555-1572.

Bouderba, B., Houari, M.S.A. and Tounsi, A. (2013), "Thermomechanical bending response of FGM thick plates resting on Winkler-Pasternak elastic foundations", *Steel Compos. Struct., Int. J.*, **14**(1), 85-104.

Fares, M.E., Elmarghany, M.K. and Atta, D. (2009), "An efficient and simple refined theory for bending and vibration of functionally graded plates", *Compos. Struct.*, **91**(3), 296-305.

Hencky, H. (1947), "Über die Berucksichtigung der Schubverzerrungen in ebenen Platen", *Ing. Arch.*, **16**(1), 344-351.

Jones, R.M. (1999), *Mechanics of Composite Materials*, (2nd Ed.), Taylor & Francis.

Lo, K.H., Christensen, R.M. and Wu, E.M. (1977), "A higher order theory of plate deformation – Part I. Homogeneous plate", *J. Appl. Mech.*, **44**(4), 663-668.

Levy, M. (1977), "Mémoire sur la théorie des plaques élastiques planes", *J. Math. Pures et Appl.*, **3**, 219-306.

Matsunaga, H. (2000), "Vibration and stability of cross-ply laminated composite plates according to a global higher-order plate theory", *Compos. Struct.*, **48**(4), 231-244.

Mindlin, R.D. (1951), "Influence of rotary inertia and shear on flexural motions of isotropic, elastic plates", *J. Appl. Mech.*, **18**, 31-38.

Moradi, S. and Mansouri, M.H. (2012), "Thermal buckling analysis of shear deformable laminated orthotropic plates by differential quadrature", *Steel Compos*. *Struct*., *Int*. *J*., **12**(2), 129-147.

Nelson, R.B. and Lorch, D.R. (1974), "A refined theory of laminated orthotropic plates", *J*. *Appl*. *Mech*., **41**(1), 177-183.

Noor, A.K. (1973), "Free vibrations of multilayered composite plates", *AIAA J*., **11**(7), 1038-1039.

Noor, A.K. and Burton, W.S. (1990), "Three-dimensional solutions for antisymmetrically solutions for antisymmetrically laminated anisotropic plates", *J*. *Appl*. *Mech*., **57**(1), 182-188.

Pagano, N.J. (1970), "Exact solutions for rectangular bidirectional composites and sandwich plates", *J*. *Compos*. *Mater*., **4**(1), 20-34.

Ray, M.C. (2003), "Zeroth-order shear deformation theory for laminated composite plates", *J*. *Appl*. *Mech*., **70**(3), 374-380.

Reddy, J.N. (1984), "A simple higher order theory for laminated composite plates", *J*. *Appl*. *Mech*., **51**(4), 745-752.

Reddy, J.N. (1997), "Mechanics of laminated composite plate: Theory and analysis", CRC Press, New York.

Reddy, J.N. (2000), "Analysis of functionally graded plates", *Int*. *J*. *Numer*. *Method*. *Eng*., **47**, 663-684.

Reddy, J.N. (2002), *Energy Principles and Variational Methods in Applied Mechanics*, John Wiley & Sons Inc.

Reddy, J.N. (2004), *Mechanics of Laminated Composite Plates and Shells*: *Theory and Analysis*, CRC Press.

Reissner, E. (1945), "The effect of transverse shear deformation on the bending of elastic plates," *J*. *Appl*. *Mech*., **12**, 69-77.

Swaminathan, K. and Patil, S. (2008), "Analytical solutions using a higher order refined computational model with 12 degrees of freedom for the free vibration analysis of antisymmetric angle-ply plates", *Compos*. *Struct*., **82**(2), 209-216.

Xiang, S. and Kang, G.W. (2013), "A nth-order shear deformation theory for the bending analysis on the functionally graded plates", *Eur*. *J*. *Mech*. – *A*/*Solids*, **37**, 336-343.

Xiang, S., Jin, Y.X., Bi, Z.Y., Jiang, S.X. and Yang, M.S. (2011), "A n-order shear deformation theory for free vibration of functionally graded and composite sandwich plates", *Compos*. *Struct*., **93**(11), 2826-2832.

Yaghoobi, H. and Torabi, M. (2013a), "Post-buckling and nonlinear free vibration analysis of geometrically imperfect functionally graded beams resting on nonlinear elastic foundation", *Appl*. *Math*. *Model*., **37**(18-19), 8324-8340.

Yaghoobi, H. and Torabi, M. (2013b), "An analytical approach to large amplitude vibration and post-buckling of functionally graded beams rest on non-linear elastic foundation", *J*. *Theor*. *Appl*. *Mech*., **51**(1), 39-52.

Yaghoobi, H. and Torabi, M. (2013c), "Exact solution for thermal buckling of functionally graded plates resting on elastic foundations with various boundary conditions", *J*. *Therm*. *Stresses*, **36**(9), 869-894.

Yaghoobi, H. and Yaghoobi, P. (2013), "Buckling analysis of sandwich plates with FGM face sheets resting on elastic foundation with various boundary conditions: an analytical approach", *Meccanica*, **48**(8), 2019-2035.

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