Steel and Composite Structures, *Vol. 17, No. 1 (2014) 1-19*
DOI: http://dx.doi.org/10.12989/scs.2014.17.1.001

Buckling and vibration of laminated composite circular plate on winkler-type foundation

B. Afsharmanesh, A. Ghaheri and F. Taheri-Behrooz *

School of Mechanical Engineering, Iran University of Science and Technology, Tehran, 16846-13114, Iran

(*Received July 10, 2013, Revised January 18, 2014, Accepted February 02, 2014*)

Abstract. Buckling and vibration characteristics of circular laminated plates under in-plane edge loads and resting on Winkler-type foundation are solved by the Ritz method. Inclusive numerical data are presented for the first three eigen-frequencies as a function of in-plane load for different classical edge conditions. Moreover, the effects of fiber orientation on the natural frequencies and critical buckling loads of laminated angle-ply plates with stacking sequence of $[(\beta/-\beta/\beta/-\beta)]_s$, are studied. Also, selected deformation mode shapes are illustrated. The correctness of results is established using finite element software as well as by comparison with the existing results in the literature.

Keywords: natural vibrations; buckling loads; composite laminates; elastic foundation; variational method; in-plane force

1. Introduction

The transverse vibration and buckling of composite laminated plate structures are the most typical dynamic problems come into aerospace industries, transportation, underwater applications, civil structures, etc., where weight saving is of main importance. Circular plates are widely used as structural components for diaphragms and deckplates in launch vehicles (Houmat 2009, Gupta 2006). Laminated composite plates have the benefit of controllability of their mechanical properties by changing the fiber orientation and the number of plies. In the most of applications the mid-plane symmetry exists, which is avoided the coupling between transverse bending and in-plane stretching.

The laminated circular plates resting on an elastic foundation finds applications in diverse areas of engineering such as foundation of deep wells (Van Niekerka *et al*. 1995, Zhang *et al*. 2011, Ponnusamy *et al*. 2012), reinforced concrete pavements of high runways, storage tanks and slabs of buildings, etc (Kim *et al*. 2005, Thomasa *et al*. 2003). Therefore there is a great interest in vibration analysis of the circular plates on elastic foundations. In various engineering applications, plates are often subjected to in-plane forces due to compressive loads which may induce buckling, a phenomenon which is highly undesirable. Thus, the study of stability of plates assumes great significance. An excellent review on stress and vibration analysis of composite plates was presented by Sharma and Mittal (2010).

*Corresponding author, Assistant Professor, E-mail: taheri@iust.ac.ir

http://www.techno-press.org/?journal=scs&subpage=8 ISSN: 1229-9367 (Print), 1598-6233 (Online)

A considerable amount of work dealing with vibration and buckling of rectangular laminated composite plates under the initial in-plane forces is available in literature. Dawe and Craig (1986) employed first-order shear deformation plate theory (FSDT) to create a model for vibration and buckling of thin and moderately thick rectangular symmetrically-laminated composite plates with various classical boundary conditions, subjected to different in-plane stress. They used the finite strip and Rayleigh-Ritz methods (RRM) to produce numerical results. Xiang *et al*. (1996) presented closed-form solutions for vibration and buckling of moderately thick, simply supported cross-ply laminates on Pasternak foundations. They used the variational principle in total potential energy functional to find the governing differential equations of motion for the free vibration of plate, Based on the FSDT. Aiello and Ombres (1999) used FSDT with the RRM to evaluate the buckling load and free vibration for simply supported, moderately thick unsymmetric rectangular composite laminates under the initial in-plane compression and shear forces, resting on the elastic foundations. In a series of publications, Matsunaga (2000, 2001, 2002) employed the method of power series expansion through Hamilton's principle, to derive the dynamic equations of vibration for thick rectangular cross-ply/angle-ply laminated composite plates under the in-plane stress, by using a global two-dimensional higher-order plate theory. Leung *et al*. (2005) used a new trapezoidal p-element with analytical integration to investigate free vibration of polygonal laminated composite plates subjected to in-plane forces, based on the FSDT. Won *et al*. (2006) presented an analytical solution for free vibration of some structural members with simply supported all edges, under lateral and in-plane forces. Lu and Li (2009) presented the precise integration method to study the free vibrations and buckling of initially stressed cross-ply, angle-ply and hybrid rectangular laminated plates with all boundaries simply supported, based on exact elasticity theory. Malekzadeh *et al*. (2010) used the Lindstedt–Poincare perturbation technique to analytically solve the natural vibration of laminated rectangular composite plates with non-ideal simply supported edge conditions under the initial in-plane stresses and on the Pasternak foundation. Chen *et al*. (2011) used the Galerkin method and Runge-Kutta method to numerically solve the nonlinear vibration of hybrid composite plates on Pasternak and Winkler foundations subjected to the initial in-plane force.

In theoretical study, for polar orthotropic circular/annular plates resting on elastic foundation, Gupta *et al*. (1986) used Hamilton's energy principle and spline technique to analyzed the axisymmetric vibrations and buckling of polar orthotropic circular annular plate of variable thickness on Winkler-type elastic foundation subjected to the hydrostatic peripheral in-plane load, on the basis of classical plate theory. Gupta and Ansari (1998) obtained an approximate solution for the asymmetric vibration and buckling of polar orthotropic circular plate of variable profile with elastically restrained edge and subjected to in-plane load, based upon classical theory of plates, by using approximating functions in the Ritz method. Subsequently, Gupta *et al*. (2006) used basis functions based on the static deflection of polar orthotropic circular plates in the Ritz method to analyze the natural vibration and buckling behavior of plates of linearly varying thickness resting on Winkler-type foundation, with elastically restrained edge.

For laminated angle-ply or cross-ply circular plates, Sivakumaran (1989) used the Rayleigh-Ritz energy method to calculate the natural frequencies of asymmetric composite annular and circular laminated thin plate based on Kirchhoff's hypothesis, considering bending-stretching coupling and rotatory inertia with free boundary conditions. Krizhevsky and Stavsky (1996) employed Hamilton's variational principle to obtain a closed form solution for asymmetric linearized vibrations and buckling of transversally isotropic annular laminated plates elastically restrained against rotation, including effects of transverse shear and rotatory inertia. Narita *et al*.

(2002) presented an analytical approach for the free vibration of symmetrically laminated, composite elliptical and circular thick plates, employing FSDT. Nallim and Grossi (2008) used the Rayleigh–Ritz approach with general polynomial type shape function to solve the free vibration of symmetrically laminated angle-ply and cross-ply, solid and annular, circular and elliptical plates with the existence of internal ring support, concentrated masses and the elastically restrained edge conditions. Viswanathan *et al*. (2009) used spline function and point collocation method to study the asymmetric vibrations of symmetric and anti-symmetric laminated cross-ply annular plates considering the effects of shear deformation and rotary inertia. Malekzadeh *et al*. (2010) employed the Hamilton's principle in conjunction with layerwise-finite element method based on three-dimensional elasticity theory to derive the discretized governing equations for free vibration of composite laminated thick circular and annular plates, supported on Pasternak-type foundation. Nguyen-Van *et al*. (2011) presents a smoothed quadrilateral flat shell element to study the vibration and buckling of laminated composite plate/shell structures of various shapes, boundary conditions, and stacking sequence within the framework of the FSDT. Seifi *et al*. (2012) used energy method and Trefftez rule in the stability equations to investigate the critical buckling loads of symmetrically laminated cross-ply annular plates under uniform external and internal radial edge forces.

The above review clearly indicates to be no rigorous analytic, approximate solutions for the vibration and buckling of composite symmetrically laminated circular plates under uniform in-plane stress and resting on Winkler foundations. The main purpose of present document is to utilize the classical theory of plates in conjunction with the Ritz energy method, to calculate critical buckling loads and natural frequencies of laminates with different boundary conditions. It can well assist the design engineers in evaluating the effects of changing the fiber orientation, foundation parameter and in-plane force on the dynamic characteristics of laminated circular plates in a wide range of physical and industrial applications.

2. Formulation

2.1 Basic relations

Fig. 1 shows a symmetrically laminated composite plate made of L plies. Each ply of the laminated plateconsists of unidirectional fiber reinforced composite material. The ply orientation is indicated by an angle β_m in Fig. 1 measured from the x axis to the fiber direction. The constitutive relation for the m-th ply is written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix} = \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{yx} & Q_{yy} & Q_{ys} \\ Q_{sx} & Q_{sy} & Q_{ss} \end{bmatrix}_m \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{Bmatrix}, \tag{1}$$

where Q_{ij} ($i, j = x, y, s$) are the stiffness constants defined in the x-y off-axis coordinate system. The relationships between the stiffness components of on-axis and off-axis are given in (Tsai and Hahn 1980), where for the on-axis stiffness parameters we have

$$Q_{11} = \frac{E_1^m}{1-\nu_{22}^m \nu_{21}^m}, \quad Q_{22} = \frac{E_2^m}{1-\nu_{12}^m \nu_{21}^m}, \quad Q_{12} = \frac{\nu_{21}^m E_1^m}{1-\nu_{12}^m \nu_{21}^m}, \quad Q_{66} = E_6^m, \tag{2}$$

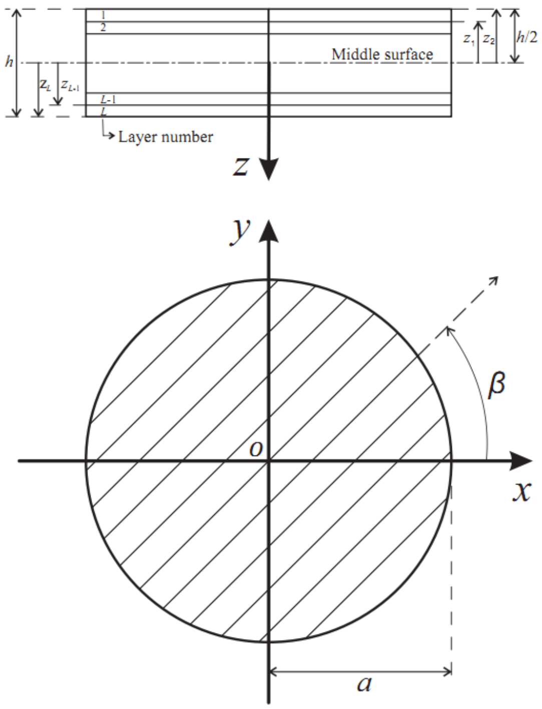

Fig. 1 Problem geometry

that defined in the material on-axis coordinates. E_1^m and E_2^m are the longitudinal and transverse Young's of m-th ply in the direction of the fibers, E_6^m is Longitudinal shear modulus and ν_{12}^m and ν_{21}^m are transverse and longitudinal Poisson's ratios.

In the classical laminated plate theory (CLPT) (Reddy 2004), the extensional and bending stiffness of plate is obtained by integrating the stress-strain relations and its first-order moment relations over the thickness. We study symmetric laminate that have not coupling stiffness so $[B] = 0$. Extensional and bending stiffness matrices are defined as

$$[A] = 2\sum_{m=1}^{L/2}[Q]_m(z_m - z_{m-1}), \ \ [D] = \frac{2}{3}\sum_{m=1}^{L/2}[Q]_m({z_m}^3 - {z_{m-1}}^3), \tag{3}$$

where and $[Q]_m$ is stiffness matrix in Eq. (1). Also, the displacements of an arbitrary point in the plate in x, y, and z directions are given as

$$\begin{aligned} U(x,y,z,t) &= u(x,y,t) - z\frac{\partial w(x,y,t)}{\partial x}, \\ V(x,y,z,t) &= v(x,y,t) - z\frac{\partial w(x,y,t)}{\partial y}, \\ W(x,y,z,t) &= w(x,y,t), \end{aligned} \tag{4}$$

where $u(x, y, t)$, $v(x, y, t)$ and $w(x, y, t)$ are the displacements on the middle surface. Substitution of

Eq. (4) into the linear strain-displacement relations, the strain and curvature components are obtained as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_s^0 \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_s^0 \end{Bmatrix},$$

$$\varepsilon_x^0 = \frac{\partial u}{\partial x}, \quad \varepsilon_y^0 = \frac{\partial v}{\partial y}, \quad \varepsilon_s^0 = \frac{\partial v}{\partial y} + \frac{\partial v}{\partial x}, \tag{5}$$

$$k_x = -\frac{\partial^2 w}{\partial x^2}, \quad k_y = -\frac{\partial^2 w}{\partial y^2}, \quad k_s = -2\frac{\partial^2 w}{\partial x \partial y},$$

2.2 Energy expressions

Assumed that the Donnell-Mushtari-Vlasov’s assumptions (Soedel 2004), the kinetic energy of the laminated plate is as follows

$$T = \frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}} \iint_A \rho(z)\dot{w}^2 \ dxdydz = \frac{1}{2}\iint_A \rho\dot{w}^2 \ dxdy, \tag{6}$$

where

$$\rho = 2\sum_{m=1}^{L/2} \rho_m (z_m - z_{m-1})$$

and A is the plate domain ($A = \{(x, y), x^2 + y^2 \leq a^2; x, y \in R\}$), and ρ_m is the mass per unit volume of m-th ply. The total strain energy of laminate can be expressed as

$$U_s = \frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}} \iint_A [\varepsilon]^T [\sigma] \, dxdydz, \tag{7}$$

where $[\varepsilon] = [\varepsilon_x \ \ \varepsilon_y \ \ \varepsilon_s]^T$, and $[\sigma] = [\sigma_x \ \ \sigma_y \ \ \sigma_s]^T$. The potential energy due to subjected external in-plane radial tensile force N can be obtained as

$$U_{ex} = \frac{1}{2}\iint_A N\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] dxdy, \tag{8}$$

and for the potential energy due to elastic foundation we have

$$U_f = \frac{1}{2}\iint_A K_w w^2 \ dxdy, \tag{9}$$

where K_w is parameter of Winkler-type foundation. Finally, the total energy of the elastic structure is defined as

$$\pi = U_s + U_f + U_{ex} - T. \tag{10}$$

2.3 Ritz method

The variational approach in conjunction with the Ritz method is used to obtain the frequencies and critical buckling loads of the laminated plates (Reddy 2007). The weak form related to Eq. (10) is given by

$$\delta\pi = \frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\iint_A \delta\left([\varepsilon]^T[Q][\varepsilon]\right)dxdydz + \iint_A \left[K_w w\delta w + N\left(\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y}\right) - \rho\dot{w}\delta\dot{w}\right]dxdy, \quad (11)$$

In order to finding the natural frequencies we suppose the harmonic motion for the deflection of laminate in the form

$$w(x,y,t) \approx w_0(x,y)e^{i\omega t} \quad (12)$$

where ω indicates natural frequency and $w_0\,(x,\ y)$ is the amplitude. Next with integration in z direction and substituting Eqs. (1) and (12) into Eq. (11), we obtain

$$\begin{aligned}\delta\pi = \iint_A &\left[\left(N_x\delta\varepsilon_x^0 + N_y\delta\varepsilon_y^0 + N_s\delta\varepsilon_s^0 + M_x\delta k_x + M_y\delta k_y + M_s\delta k_s\right)\right]\\ &+ K_w w_0\delta w_0 + N\left(\frac{\partial w_0}{\partial x}\frac{\partial \delta w_0}{\partial x} + \frac{\partial w_0}{\partial y}\frac{\partial \delta w_0}{\partial y} - \rho\omega^2 w_0\delta w_0\right)dxdy,\end{aligned} \quad (13)$$

The pertinent moment resultants, stress resultants within the plies, appearing in the relations are expressed as

$$\begin{Bmatrix} N_x \\ N_y \\ N_s \\ M_x \\ M_y \\ M_s \end{Bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} & A_{xs} & 0 & 0 & 0 \\ & A_{yy} & A_{ys} & 0 & 0 & 0 \\ & & A_{ss} & 0 & 0 & 0 \\ & & & D_{xx} & D_{xy} & D_{xs} \\ & Sym & & & D_{yy} & D_{ys} \\ & & & & & D_{ss} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_s^0 \\ k_x \\ k_y \\ k_s \end{Bmatrix}, \quad (14)$$

Next, since we study lateral vibration of laminate with uncoupling of in-plane and lateral equations, because of $[B] = 0$, we ignore in-plane equations for this study. In the Ritz method, the unknown displacement, w_0, is approximated by a finite linear combination of w_{ij} in the form

$$w_0(x,y) = \sum_{i=0}^{I}\sum_{j=0}^{J} c_{ij}w_{ij}(x,y), \quad (15)$$

Substituting Eqs. (14) and (15) into Eq. (13), after some manipulation, we have

$$\delta\pi = \sum_{p=0}^{I}\sum_{q=0}^{J}\delta c_{pq}\sum_{i=0}^{I}\sum_{j=0}^{J}c_{ij}\iint_A\left[\left(D_{xx}\frac{\partial^2 w_{ij}}{\partial x^2} + D_{xy}\frac{\partial^2 w_{ij}}{\partial y^2} + 2D_{xs}\frac{\partial^2 w_{ij}}{\partial x\partial y}\right)\frac{\partial^2 w_{pq}}{\partial x^2}\right. \quad (16)$$

$$
\begin{aligned}
&+\left(D_{xy}\frac{\partial^2 w_{ij}}{\partial x^2}+D_{yy}\frac{\partial^2 w_{ij}}{\partial y^2}+2D_{ys}\frac{\partial^2 w_{ij}}{\partial x\partial y}\right)\frac{\partial^2 w_{pq}}{\partial y^2}\\
&+2\left(D_{xs}\frac{\partial^2 w_{ij}}{\partial x^2}+D_{ys}\frac{\partial^2 w_{ij}}{\partial y^2}+2D_{ss}\frac{\partial^2 w_{ij}}{\partial x\partial y}\right)\frac{\partial^2 w_{pq}}{\partial x\partial y}\\
&\left.+K_w w_{ij}w_{pq}+N\left(\frac{\partial w_{ij}}{\partial x}\frac{\partial w_{pq}}{\partial x}+\frac{\partial w_{ij}}{\partial y}\frac{\partial w_{pq}}{\partial y}\right)-\rho\omega^2 w_{ij}w_{pq}\right]dxdy,
\end{aligned}
\tag{16}
$$

Since δc_{pq} are arbitrary and linearly independent, we conclude that the coefficients of them must be zero. So we obtain

$$\left(\boldsymbol{K}_t+\boldsymbol{K}_{ex}N-\boldsymbol{M}\omega^2\right)\boldsymbol{C}=0, \tag{17}$$

where $\boldsymbol{C}=[c_{00}, c_{01}, \ldots c_{0J}; c_{10}, c_{11}, \ldots c_{1J}; c_{I0}, c_{I1}, \ldots c_{IJ}]^T$, and $\boldsymbol{K}_{ex}$ is stiffness matrix which contain terms relative to potential energy of external force, $\boldsymbol{K}_t$ is stiffness matrix associated with strain energy and foundation's potential energy, and $\boldsymbol{M}$ is mass matrix related to kinetic energy, and they are given in Appendix. In the Ritz method a complete set of approximated functions that satisfy the geometrical boundary conditions is sufficient. Also, the corresponding geometrical boundary conditions are given as

$$
\begin{aligned}
&\text{Clamped:} && w\big|_{r=a}=\frac{\partial w}{\partial r}\bigg|_{r=a}=0,\\
&\text{Simply supported:} && w\big|_{r=a}=0,\\
&\text{Free:} && -,
\end{aligned}
\tag{18}
$$

The natural boundary conditions will be satisfied (Reddy 2007) if the number of approximation functions approaches infinity. So, the assumed shape functions that satisfy essential boundary conditions, is expressed as (Nallim and Grossi 2008)

$$w_{ij}(x,y)=x^i, y^j(x^2+y^2-a^2)^\alpha \tag{19}$$

To satisfy the boundary conditions, $\alpha=0$ is adopted when the plate is free, $\alpha=1$ when it is simply supported which refers to soft simple support (Babuška and Pitkäranta 1990), $\alpha=2$ when it is clamped. Thus, the eigenvalue problem of laminated platecan be formulated as

$$\left(\boldsymbol{K}_t+\boldsymbol{K}_{ex}N-\boldsymbol{M}\omega^2\right)\boldsymbol{C}=0, \tag{20}$$

where it is a system of linear equations of $(I+1)\cdot(J+1)\times(I+1)\cdot(J+1)$ order. A nontrivial solution ($\boldsymbol{C}\neq 0$) to Eq. (20) exists only if the determinant of the coefficients matrix is zero, which is evaluating frequencies of laminate. Moreover, the analysis of critical buckling load is done by the same procedure described for the natural frequencies.For evaluating the critical buckling load, N_{cr}, one should omit the inertia terms in Eq. (20) to obtain

$$\left(\boldsymbol{K}_t+\boldsymbol{K}_{ex}N\right)\boldsymbol{C}=0, \tag{21}$$

This completes the necessary background required for analysis of the problem. Next we consider some numerical examples.

3. Numerical results and discussion

In order to calculate the natural frequency and critical buckling load of laminated composite circular plates and also evaluating the effect of various parameters on them, different numerical examples are presented here. A circular plate made of unidirectional carbon/epoxy (E_1 = 138 GPa, E_2 = 8.96 GPa, ν_{12} = 0.3, E_6 = 7.1 GP) with layup sequence of $[(\beta/-\beta/\beta/-\beta)]_s$ and $[(0/\pm 45/90)]_s$ under simply supported, clamped and free boundary conditions are considered. The radius of composite circular plate ($a = 0.1$ m), the thickness of each layer ($h_m = 125\,\mu$m) and the dimensionless stiffness of elastic foundation ($\bar{K}_w = K_w a^4/D = 0, 10$), in which $D = E_1 h^3/12\,(1-\nu_{12}\nu_{21})$, are considered.

The convergence analysis of first three non-dimensional natural frequencis, $\Omega = \omega a^2\sqrt{ph/D}$, of laminated circular plates with stacking sequence of $[(\beta/-\beta/\beta/-\beta)]_s$ (β = 0, 45), and selected in-plane load parameters ($N = 0, N_{cr}/2$), under different edge conditions are performed and results to four decimal places are depicted in Table 1. The parameter of elastic foundation is considered to be zero ($\bar{K}_w = 0$) in this analysis. Here, it can clearly be seen that convergence rate reduces with increasing the mode number. Moreover, Table 2 tabulates the convergence analysis results of

Table 1 Convergence study of the first three dimensionless natural frequencies of laminated circular plates with layup sequence of $[(\beta/-\beta/\beta/-\beta)]_s$ (β = 0, 45), under zero elastic foundation parameter for different edge conditions

	N	$I = J$	Simply supported			Clamped			Free		
			Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
$\beta = 0$	0	4	4.4740	6.2631	11.7454	7.3124	9.7222	14.5174	3.7453	3.7725	6.3149
		5	4.4616	6.2616	9.5116	7.3079	9.7220	13.4151	3.6368	3.7725	5.6627
		6	4.4616	6.2024	9.5116	7.3079	9.7023	13.4151	3.6368	3.7691	5.5638
		7	4.4616	6.2024	9.2911	7.3079	9.7023	13.3462	3.6318	3.7691	5.5627
		8	4.4616	6.2018	9.2911	7.3079	9.7020	13.3462	3.6318	3.7691	5.4891
		9	4.4615	6.2018	9.2860	7.3079	9.7020	13.3446	3.6318	3.7691	5.4891
		10	4.4615	6.2018	9.2860	7.3079	9.7020	13.3446	3.6318	3.7691	5.4885
	$0.5\,N_{cr}$	4	3.5391	4.4934	10.6294	5.9245	7.2291	12.0541	2.7078	3.0787	5.3386
		5	3.5093	4.4925	7.3419	5.8687	7.2290	10.3547	2.6688	3.0787	4.7327
		6	3.5093	4.3972	7.3419	5.8687	7.1399	10.3547	2.6688	3.0785	4.3269
		7	3.5086	4.3972	7.0609	5.8660	7.1399	10.1951	2.6660	3.0785	4.3267
		8	3.5086	4.3953	7.0609	5.8660	7.1367	10.1951	2.6660	3.0785	4.2893
		9	3.5085	4.3953	7.0526	5.8659	7.1367	10.1889	2.6660	3.0785	4.2893
		10	3.5085	4.3953	7.0526	5.8659	7.1367	10.1889	2.6660	3.0785	4.2891

Table 1 Continued

	N	$I = J$	Simply supported			Clamped			Free		
			Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
$\beta = 45$	0	4	4.5071	8.5708	10.5786	7.4009	12.8340	15.5693	3.8614	5.1585	7.8356
		5	4.5069	8.5674	10.5782	7.4003	12.8326	15.5690	3.8197	5.1060	6.7964
		6	4.5068	8.5499	10.5242	7.4003	12.8272	15.5573	3.8197	5.0925	6.7890
		7	4.5068	8.5498	10.5242	7.4003	12.8272	15.5573	3.8174	5.0869	6.7288
		8	4.5068	8.5497	10.5237	7.4003	12.8271	15.5572	3.8174	5.0868	6.7288
		9	4.5068	8.5497	10.5237	7.4003	12.8271	15.5572	3.8174	5.0868	6.7283
		10	4.5068	8.5496	10.5237	7.4003	12.8271	15.5572	3.8174	5.0868	6.7283
	0.5 N_{cr}	4	3.1875	6.9008	9.2804	5.3020	10.0467	13.3433	2.7381	4.3752	6.9364
		5	3.1875	6.8967	9.2798	5.2993	10.0439	13.3431	2.7227	4.2772	5.9536
		6	3.1875	6.8800	9.2185	5.2993	10.0348	13.3222	2.7227	4.2716	5.9450
		7	3.1875	6.8799	9.2185	5.2993	10.0348	13.3222	2.7220	4.2673	5.9039
		8	3.1875	6.8798	9.2179	5.2993	10.0347	13.3219	2.7220	4.2673	5.9039
		9	3.1875	6.8798	9.2171	5.2993	10.0347	13.3219	2.7220	4.2672	5.9035
		10	3.1875	6.8798	9.2179	5.2993	10.0347	13.3219	2.7220	4.2672	5.9035

non-dimensional critical buckling load, $\bar{N}_{cr} = |N_{cr}|a^2 / D$, of laminated composite circular plates with layup sequence of $[(\beta / -\beta / \beta / -\beta)]_s$ (β = 0, 15, 30 and 45), and selected foundation parameter ($\bar{K}_w$ = 10) under various boundary conditions. Using a maximum truncation constant of I, J = 10 was found to yield acceptable results for the first three natural frequencies and critical buckling loads.

Before presenting the key results, we shall exhibit the overall validity of the formulation. To this end the results of proposed technique are compared with calculated results obtained from commercial finite element package (ABAQUS) for circular laminated composite plate. The present simulation is focused on laminated plate with stacking sequence of $[(0 / \pm 45 / 90)]_s$, under various boundary conditions (simply supported, clamped and free) and in-plane loading (N = 0, ± 0.5 $|N_{cr}|$).The mesh element type used in the modeling of plate is "S8R5" (eight-node doubly curved thin shell, reduced integration) and the least number of elements required for globalconvergence of results is found to be 1500. Elastic foundation for this analysis is considered to be linear with two different magnitudes ($\bar{K}_w$ = 0, 10). To model such a foundation, the "Elastic foundation" procedure in "Interaction" module of ABAQUS software is used. Well agreements between the present FEM numerical data and the calculated results using Ritz method is obtained in Table 3. The maximum percentage of error between the results is seen to be 0.44%. Also, Table 4 shows the first three lowest non-dimensional natural frequencies, $\Omega = \omega a^2 \sqrt{\rho h / D}$ ($D_0 = Eh^3 / 12\,(1 - \nu^2)$, ν = 0.3), calculated for an isotropic circular plate under different boundary conditions and zero elastic foundation parameter. Calculated results are compared with the exact results of Zhoua *et al*. (2011) as well as FEM data. Good agreements are obtained between different methods.

Table 2 Convergence study of dimensionless critical buckling load of laminated circular plates with layup sequence of $[(\beta/-\beta/\beta/-\beta)]_s$ (β = 0, 15, 30 and 45), under $\bar{K}_w$ = 10 elastic foundation parameter for different edge conditions

$I=J$	Simply supported				Clamped				Free			
	$\beta=0$	$\beta=15$	$\beta=30$	$\beta=45$	$\beta=0$	$\beta=15$	$\beta=30$	$\beta=45$	$\beta=0$	$\beta=15$	$\beta=30$	$\beta=45$
4	2.6465	3.0878	3.4937	3.4994	4.8904	5.7284	7.2305	7.5573	0.8788	0.9232	1.0774	1.2425
5	2.6464	3.0862	3.4931	3.4991	4.7308	5.5961	7.1157	7.5382	0.8728	0.9184	1.0718	1.2424
6	2.5733	3.0338	3.4931	3.4991	4.6255	5.5023	7.1153	7.5381	0.8728	0.9183	1.0717	1.2424
7	2.5733	3.0337	3.4931	3.4991	4.6255	5.5019	7.1093	7.5379	0.8502	0.9002	1.0620	1.2356
8	2.5694	3.0315	3.4931	3.4991	4.6344	5.4839	7.1093	7.5379	0.8502	0.9002	1.0620	1.2356
9	2.5694	3.0315	3.4931	3.4991	4.6017	5.4839	7.1092	7.5379	0.8502	0.9002	1.0619	1.2356
10	2.5693	3.0315	3.4931	3.4991	4.6008	5.4833	7.1092	7.5379	0.8502	0.9002	1.0619	1.2356

Table 3 Comparisons of dimensionless natural frequencies of circular laminated plate with layup sequence of $[(0/\pm45/90)]_s$, for various edge conditions, foundation parameter and in-plane loads, with the FEM data

	$\bar{K}_w$		$\bar{N}_{cr}=-0.5\,\lvert N_{cr}\rvert$			$N=0$			$N_{cr}=0.5\,\lvert N_{cr}\rvert$		
			present	FEM	Error%	present	FEM	Error%	present	FEM	Error%
Clamped	0	Ω_1	4.7916	4.8035	0.24	6.6577	6.6753	0.26	8.0594	8.0808	0.26
		Ω_2	8.8172	8.8561	0.44	11.3594	11.4027	0.38	13.3907	13.4385	0.36
		Ω_3	14.2214	14.2685	0.33	15.9103	15.9612	0.32	17.4201	17.4754	0.32
	10	Ω_1	5.3508	5.3608	0.19	7.3708	7.3865	0.21	8.8851	8.9045	0.22
		Ω_2	8.8040	8.8423	0.43	11.7917	11.8331	0.35	14.1095	14.1554	0.32
		Ω_3	14.2195	14.2660	0.32	16.2213	16.2716	0.31	17.9800	18.0340	0.30
Simply-supported	0	Ω_1	2.2670	2.2682	0.05	3.2057	3.2076	0.05	3.9264	3.9282	0.05
		Ω_2	6.1852	6.1952	0.17	7.1628	7.1729	0.14	8.0224	8.0324	0.12
		Ω_3	9.8872	9.9029	0.16	10.5281	10.5432	0.14	11.1313	11.1464	0.13
	10	Ω_1	3.1900	3.1856	0.14	4.5032	4.5044	0.03	5.5116	5.5160	0.08
		Ω_2	5.9697	5.9734	0.06	7.8301	7.8389	0.12	9.3255	9.3387	0.14
		Ω_3	9.7528	16.0479	0.12	10.9924	11.0075	0.13	12.1052	12.1221	0.14
Free	0	Ω_1	1.8523	1.8542	0.08	2.5987	2.6006	0.08	3.1598	3.1623	0.08
		Ω_2	2.8852	2.8903	0.18	3.3439	3.3489	0.15	3.7442	3.7492	0.13
		Ω_3	5.9640	5.9684	0.07	6.3265	6.3309	0.07	6.6690	6.6740	0.07
	10	Ω_1	2.9600	2.9613	0.04	4.0929	4.0948	0.04	4.9222	4.9241	0.03
		Ω_2	3.7982	3.8026	0.11	4.6024	4.6062	0.08	5.2754	5.2785	0.06
		Ω_3	6.0934	6.1029	0.16	7.0724	7.0768	0.06	7.7780	7.7824	0.06

Table 4 Comparisons of first three calculated dimensionless natural frequencies of isotropic circular plate (v = 0.3), under zero elastic foundation parameter for different edge conditions with the results of Zhoua *et al*. (2011) as well as FEM data

	Clamped			Simply supported			Free		
	Zhoua *et al.* (2011)	FEM	present	Zhoua *et al.* (2011)	FEM	present	Zhoua *et al.* (2011)	FEM	present
Ω_1	10.216	10.213	10.216	4.9352	4.9348	4.9351	5.3583	5.3563	5.3584
Ω_2	21.260	21.249	21.260	13.898	13.892	13.898	9.0031	9.0020	9.0031
Ω_3	34.877	34.850	34.877	25.615	25.592	25.613	12.439	12.431	12.439

Table 5 Comparisons of normalized critical buckling load of an isotropic circular plate (v = 0.3), under zero elastic foundation parameter for different edge conditions with the results of Reddy (2007) as well as FEM data

	Reddy (2007)	present	FEM
Clamped	14.6820	14.6820	14.6774
Simply supports	4.1978	4.1978	4.1976
Free	-	2.6026	2.5978

Based on the authors' knowledge, there isno published data regarding the critical buckling load of laminated circular composite plates with general fiber orientations in Cartesian coordinate system. Hence, the validity of the critical buckling load ($\bar{N}_{cr} = |N_{cr}|a^2 / D_0$) calculated by the current research are compared with exact results of Reddy (2007) and FEM for an isotropic circular plate (v = 0.3), under simply supported and clamped boundary edges in Table 5. For free boundary condition, results are just compared with FEM and effect of elastic foundation is ignored in this example.There is a good correlation between present and exact results.

Fig. 2 displays the variation of first three lowest natural frequencies of composite laminated circular plates $[(\beta / -\beta / \beta / -\beta)]_s$ ($\beta \rightarrow 0, 45$), with selected foundation parameters ($\bar{K}_w$ = 0, 10) under simply supported, clamped and free edge conditions. The variation of fiber direction has less increasing effect on first natural frequency but it has considerable increasing effect on the second natural frequency. Also, third natural frequency increased up to a certain fiber orientation and then decreased. Elastic foundation in all curves showed an increasing effect on natural frequencies, which is related to increasing in stiffness of structural system. Furthermore elastic foundation has more increasing effect on plates with free edge conditions since free plates are generally softer than plates with simply supported and clamped boundary conditions. The variation of critical buckling load of laminated circular plates $[(\beta / -\beta / \beta / -\beta)]_s$, with various fiber directions and boundary conditions are represented in Fig. 3. There is an increase of critical buckling load with increasing the angle of fibers ($\beta \rightarrow 0, 45$) for clamed and free boundary conditions. On the other hand, critical buckling loads of simply supported plates increased up to 25 degree and then didn't changed. Elastic foundation ($\bar{K}_w$ = 0, 1 and 10), in all curves showed an increasing effect on critical buckling load, nearly regardless of the edge condition. Moreover, Fig. 4 shows some representative three dimensional transverse displacement mode shapes in conjunction with nodal

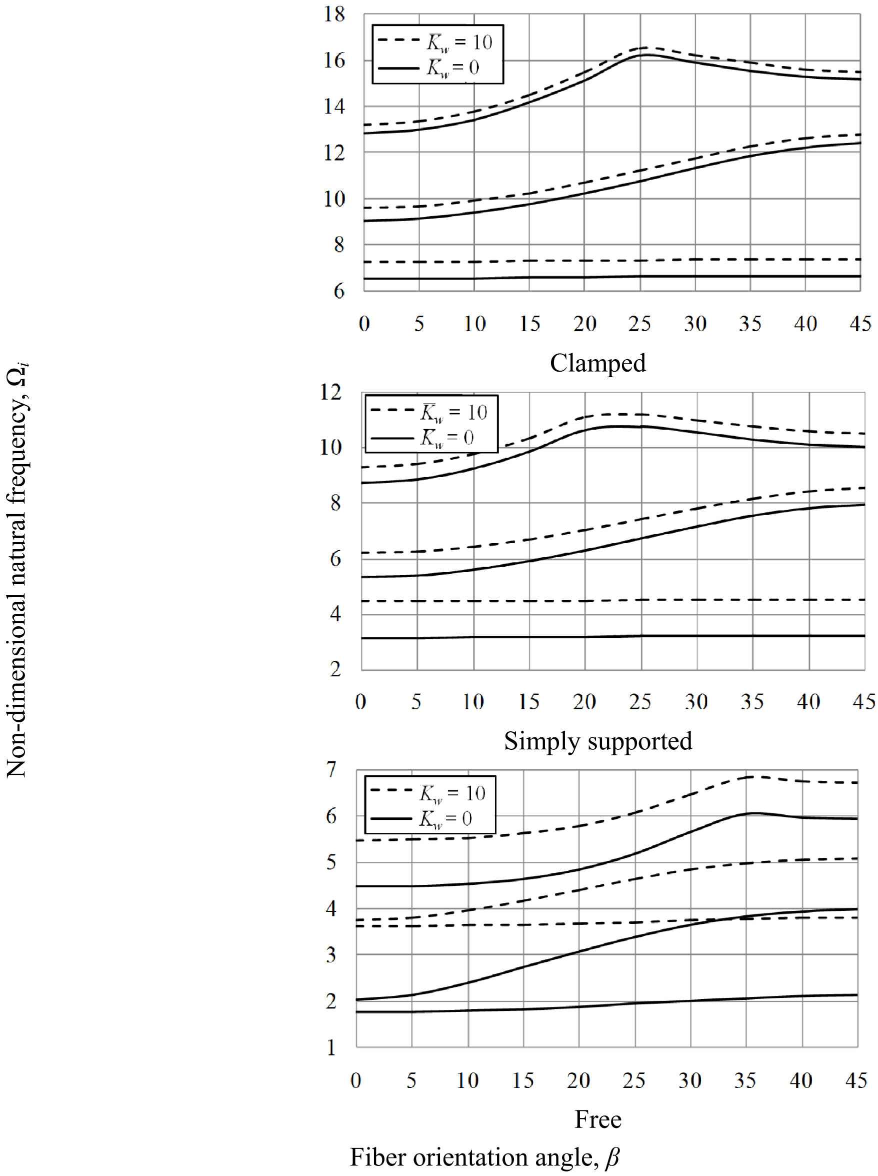

Fig. 2 Variation of first three calculated normalized natural frequencies of angle-ply circular plate of $[(\beta/-\beta/\beta/-\beta)]_s$, with fiber orientation angle for various boundary conditions and elastic foundation parameter ($\bar{K}_w$ = 0, 10)

lines of corresponding mode shapes related to first three natural frequencies of angle-ply laminated circular plates of $[(\beta/-\beta/\beta/-\beta)]_s$ (β = 0, 45), and zero Winkler foundation parameter under different edge conditions. By variation of fiber orientation the nodal lines of composite plates are changed. It can be seen that first mode shapes of simply supported and clamped boundary conditions are approximately axisymmetric and variation of fiber orientation doesn't have much

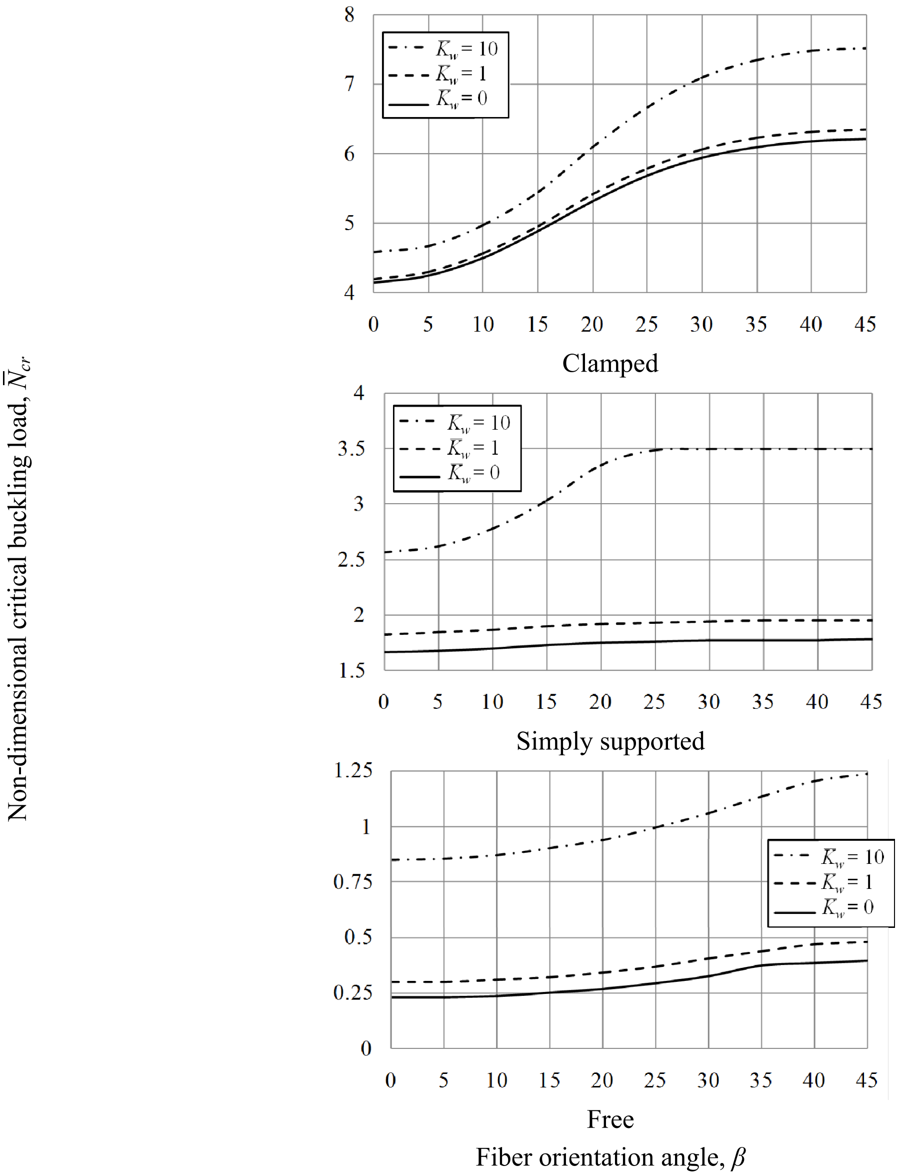

Fig. 3 Variation of calculated normalized critical buckling load of angle-ply circular plate of $[(\beta/-\beta/\beta/-\beta)]_s$, with fiber orientation angle for various boundary conditions and elastic foundation parameter ($\bar{K}_w = 0$, 1 and 10)

effect on them. This could be related to the less increasing effect on first natural frequency verse the variation of fiber angle in Fig. 2. Furthermore, for the second modes of simply supported and clamped edge conditions, the nodal lines aligned at the same direction as fibers, however, for the third modes the two nodal lines aligned at the same as fiber direction up to nearly 25 degree and then number of nodal lines decreased to one. This may be linked to trend of variation in second and third natural frequencies shown in Fig. 2.

Fig. 5 displays the variation of first three natural frequencies of circular composite laminates with stacking sequence of $[(0/\pm 45/90)]_s$ with a wide range of initial in-plane stresses for selected foundation parameter ($\bar{K}_w$ = 1, 10) under different edge conditions. As depicted in this figure, compressive in-plane loading reduced the natural frequencies and made them zero at critical buckling loads, while tensile loading increased the frequencies. Also, elastic foundation had an

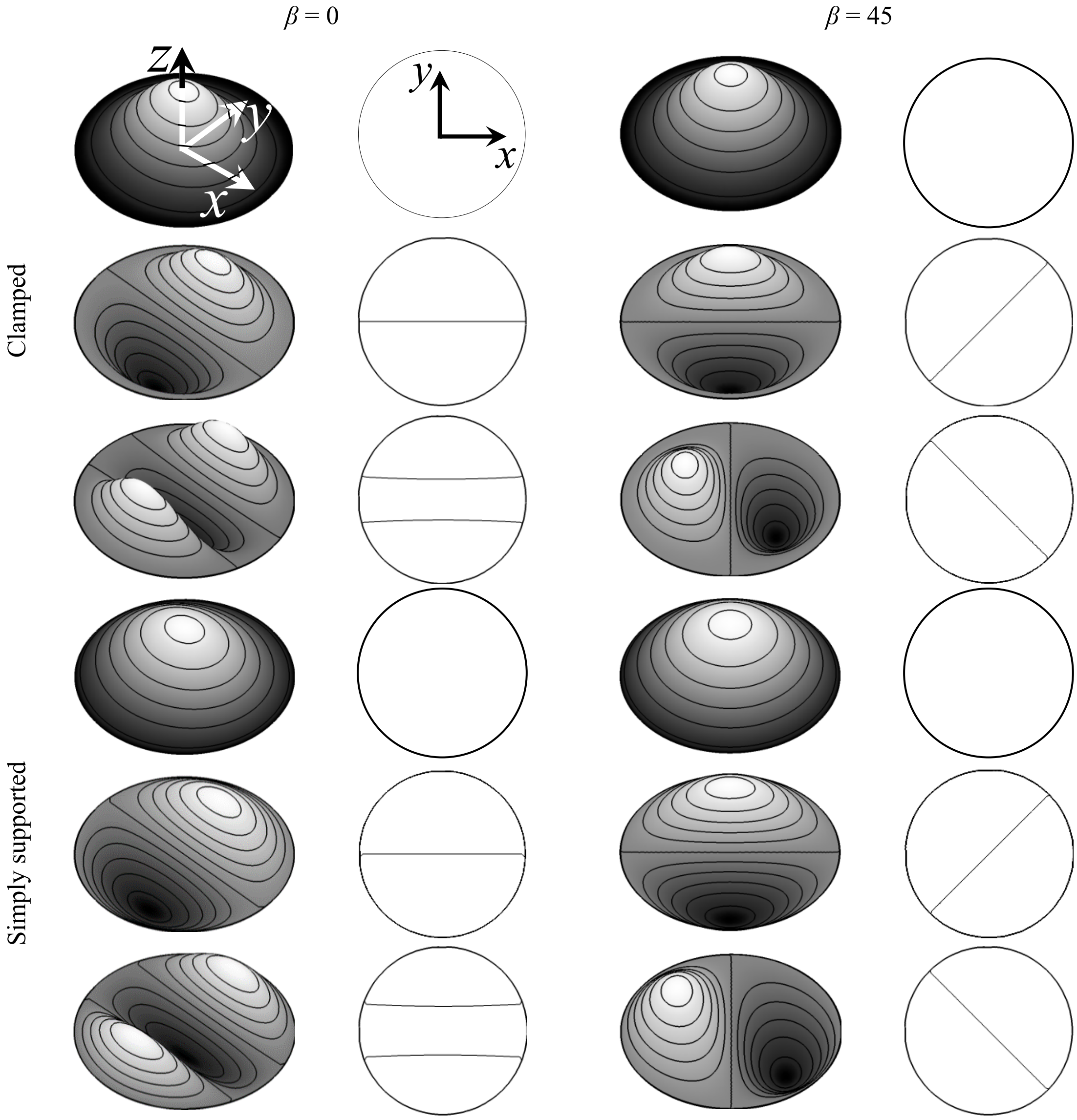

Fig. 4 The transverse displacement mode shapes corresponding to the first three calculated natural frequencies of angle-ply laminated circular plate of $[(\beta/-\beta/\beta/-\beta)]_s$, for ($\beta$ = 0, 45) under zero elastic foundation parameter various edge conditions

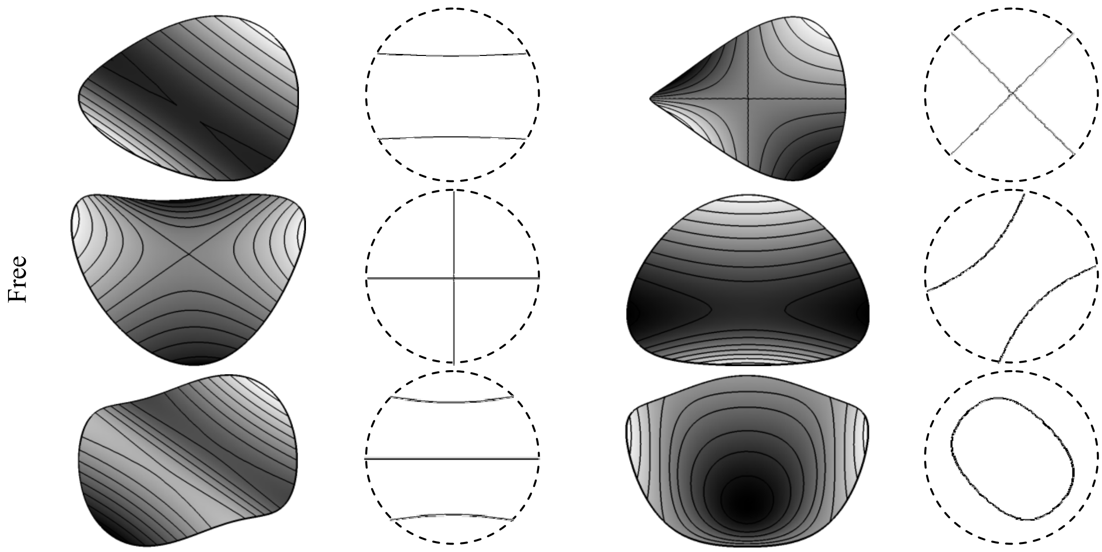

Fig. 4 Continued

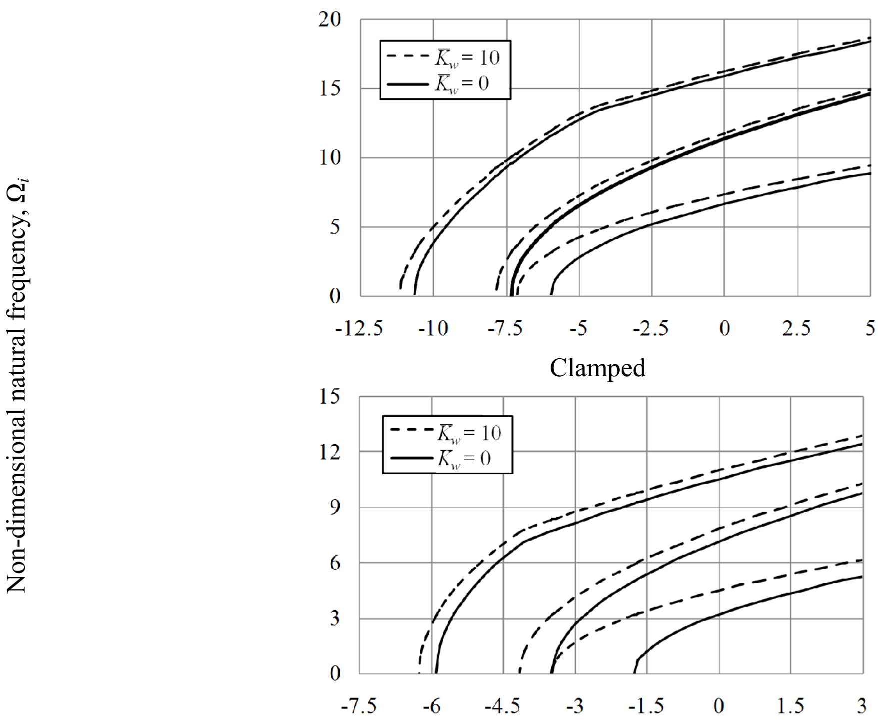

Fig. 5 Variation of first three calculated normalized natural frequencies of circular laminated plate of $[(0/\pm 45/90)]_s$, with initial in-plane stresses for various boundary conditions and elastic foundation parameter ($\bar{K}_w$ = 0, 1 and 10)

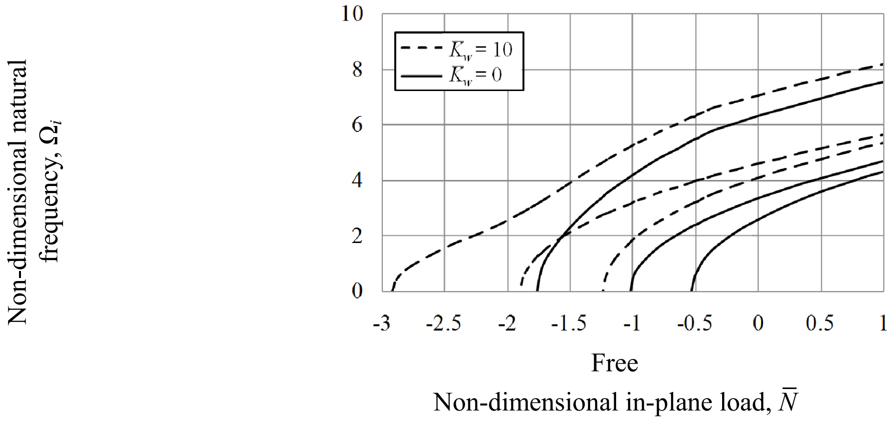

Fig. 5 Continued

increasing effect on natural frequencies in all curves. The critical buckling load $\bar{N}_{cr}$ ($\Omega_1 = 0$) of plate with clamped (simply supported) edge condition is always greater than critical buckling load of simply supported (free) plate.

4. Conclusions

In this paper the total potential energy functional in conjunction with the Ritz energy method is used to extract the transverse natural frequencies and critical buckling load along with selected deformed mode shapes for composite symmetrically laminated thin circular plates under radial membrane force and resting on the Winkler-type elastic foundation, based on the classical laminate plate theory. The general approximated solutions are presented and the related buckling load and frequency determinants are achieved and solved for different classical boundary conditions (free, clamped and simply supported). The presented results show that the dynamic and static characteristics of laminate are significantly influenced by the fiber orientation of plies, stacking sequence and foundation parameter. In particular, it is observed that:

- The variation of fiber orientation ($\beta \rightarrow 0, 45$) for angle-ply laminate $[(\beta/-\beta/\beta/-\beta)]_s$, has an increasing effect on first and second natural frequencies but for third natural frequency it has decreasing effect after a distinct peak in the certain fiber orientation angle. Also, elastic foundation parameter has an increasing effect on the frequencies.
- There is an overall increase of critical buckling load for laminated circular plate $[(\beta/-\beta/\beta/-\beta)]_s$, with variation of fiber direction ($\beta \rightarrow 0, 45$) and elastic foundation stiffness, regardless of boundary condition.
- There is a raise in the frequencies of circular laminated plate with applying the in-plane tensile force, while compressive force decrease the frequencies and made them zero at structural buckling loads.
- Variation of fiber direction changes the nodal lines of composite circular plates $[(\beta/-\beta/\beta/-\beta)]_s$, especially for the 2nd mode shapes the nodal lines align at the same as fiber direction for the simply supported and clamped edge conditions.

References

Aiello, M.A. and Ombre, L. (1999), "Buckling and vibrations of unsymmetric laminates resting on elastic foundations under in-plane and shear forces", *Compos. Struct.*, **44**(1), 31-41.

Babuška, I. and Pitkäranta, J. (1990), "The plate paradox for hard and soft simple support", *SIAM J. Math. Anal.*, **21**(3), 551-576.

Chen, W.R., Chen, C.S. and Yu, S.Y. (2011), "Nonlinear vibration of hybrid composite plates on elastic foundations", *Struct. Eng. Mech., Int. J.*, **37**(4), 367-383.

Dawe, D.J. and Craig, T.J. (1986), "The vibration and stability of symmetrically-laminated composite rectangular plates subjected to in-plane stresses", *Compos.* Struct., **5**(4), 281-307.

Gupta, U.S., Lal, R. and Verma, C.P. (1986), "Buckling and vibrations of polar orthotropic annular plates on elastic foundation subjected to hydrostatic peripheral loading", *J. Sound. Vib.*, **109**(3), 423-434.

Gupta, U.S. and Ansari, A.H. (1998), "Asymmetric vibrations and elastic stability of polar orthotropic circular plates of linearly varying profile", *J. Sound. Vib.*, **215**(2), 231-250.

Gupta, U.S., Ansari, A.H. and Sharma, S. (2006), "Buckling and vibration of polar orthotropic circular plate resting on Winkler foundation", *J. Sound. Vib.*, **297**(3-6), 457-476.

Houmat, A. (2009), "Nonlinear free vibration of a shear deformable laminated composite annular elliptical plate", *Acta Mech.*, **208**(3-4), 281-297.

Kim, S., Clark, W.W. and Wang, Q.M. (2005), "Piezoelectric energy harvesting with a clamped circular plate: Experimental study", *J. Intell. Mater. Syst. Struct.*, **16**(10), 855-863.

Krizhevsky, G. and Stavsky, Y. (1996), "Refined theory for vibrations and buckling of laminated isotropic annular plates", *Int. J. Mech. Sci.*, **38**(5), 539-555.

Leung, A.Y.T., Xiao, C., Zhu, B. and Yuan, S. (2005), "Free vibration of laminated composite plates subjected to in-plane stresses using trapezoidal *p*-element", *Compos. Struct.*, **68**(2), 167-175.

Lu, H.X. and Li, J.Y. (2009), "Analysis of an initially stressed laminated plate based on elasticity theory", *Compos. Struct.*, **88**(2), 271-279.

Malekzadeh, K., Khalili, S.M.R. and Abbaspour, P. (2010), "Vibration of non-ideal simply supported laminated plate on an elastic foundation subjected to in-plane stresses", *Compos. Struct.*, **92**(6), 1478-1484.

Malekzadeh, P., Afsari, A., Zahedinejad, P. and Bahadori, R. (2010), "Three-dimensional layerwise-finite element free vibration analysis of thick laminated annular plates on elastic foundation", *Appl. Math. Model.*, **34**(3), 776-790.

Matsunaga, H. (2000), "Vibration and stability of cross-ply laminated composite plates according to a global higher-order plate theory", *Compos. Struct.*, **48**(4), 231-244.

Matsunaga, H. (2001), "Vibration and stability of angle-ply laminated composite plates subjected to in-plane stresses", *Int. J. Mech. Sci.*, **43**(8), 1925-1944.

Matsunaga, H. (2002), "Vibration of cross-ply laminated composite plates subjected to initial in-plane stresses", *Thin-Wall. Struct.*, **40**(7-8), 557-571.

Nallim, L.G. and Grossi, R.O. (2008), "Natural frequencies of symmetrically laminated elliptical and circular plates", *Int. J. Mech. Sci.*, **50**(7), 1153-1167.

Narita, Y., Fukushi, K. and Zhao, X. (2002), "Vibration analysis of shear-deformable circular and elliptical laminated composite plates", *JSME Int. Ser. C.*, **45**(1), 113-120.

Nguyen-Van, H., Mai-Duy, N., Karunasena, W. and Tran-Cong, T. (2011), "Buckling and vibration analysis of laminated composite plate/shell structures via a smoothed quadrilateral flat shell element with in-plane rotations", *Comput. Struct.*, **89**(7-8), 612-625.

Ponnusamy, P. and Selvamani, R. (2012), "Wave propagation in a generalized thermo elastic plate embedded in elastic medium", *Interact. Multisc. Mech., Int. J.*, **5**(1), 13-26.

Reddy, J.N. (2004), *Mechanics of Laminated Composite Plates and Shells: Theory and Analysis*, (2nd Edition), CRC Press, Boca Raton, FL, USA.

Reddy, J.N. (2007), *Theory and Analysis of Elastic Plates and Shells*, (2nd Edition), CRC Press, Boca Raton, FL, USA.

Seifi, R., Khoda-Yari, N. and Hosseini, H. (2012), “Study of critical buckling loads and modes of cross-ply laminated annular plates”, *Compos. Part B*, **43**(2), 422-430.

Sharma, A.K. and Mittal, N.D. (2010), “Review on stress and vibration analysis of composite plates”, *J. Appl. Sci.*, **10**(23), 3156-3166.

Sivakumaran, K.S. (1989), “Free vibration of annular and circular asymmetric composite laminates”, *Compos. Struct.*, **11**(3), 205-226.

Soedel, W. (2004), *Vibrations of Shells and Plates*, (3rd Edition), Marcel Dekker, Inc., CRC Press, Boca Raton, FL, USA.

Thomasa, O., Touzé, C. and Chaigneb, A. (2003), “Asymmetric non-linear forced vibrations of free-edge circular plates. Part II: experiments”, *J. Sound Vib.*, **265**(5), 1075-1101.

Tsai, S.W. and Hahn, H.T. (1980), *Introduction to Composite Materials*, Technomic, Connecticut, Westport.

Van Niekerka, J.L., Tongueb, B.H. and Packardc, A.K. (1995), “Active control of a circular plate to reduce transient noise transmission”, *J. Sound Vib.*, **183**(4), 643-662.

Viswanathan, K.K., Kim, K.S. and Lee, J.H. (2009), “Asymmetric free vibrations of laminated annular cross-ply circular plates including the effects of shear deformation and rotary inertia: Spline method”, *Forsch im Igenieurwesen*, **73**(4), 205-217.

Won, C.M., Yun, K.K. and Lee, J.H. (2006), “A simple but exact method for vibration analysis of composite laminated plates with all edges simply supported and under lateral and in-plane loading”, *Thin-Wall. Struct.*, **44**(2), 247-253.

Xiang, Y., Kitipornchai, S. and Liew, K.M. (1996), “Buckling and vibration of thick laminates on Pasternak foundations”, *J. Eng. Mech.*, **122**(1), 54-63.

Zhang, L., Zhao, M.H., Xiao, Y. and Ma, B.H. (2011), “Nonlinear analysis of finite beam resting on Winkler foundation with consideration of beam-soil interface resistance effect”, *Struct. Eng. Mech., Int. J.*, **38**(5), 573-592.

Zhoua, Z.H., Wongb, K.W., Xua, X.S. and Leung, A.Y.T. (2011), “Natural vibration of circular and annular thin plates by Hamiltonian approach”, *J. Sound. Vib.*, **330**(5), 1005-1017.

CC

Appendix

$$K_t = \begin{bmatrix} \overline{K}^t_{(00,00)} & \overline{K}^t_{(00,01)} & \cdots & \overline{K}^t_{(00,0J)} & \cdots & \overline{K}^t_{(00,IJ)} \\ \overline{K}^t_{(01,00)} & \overline{K}^t_{(01,01)} & \cdots & \overline{K}^t_{(01,0J)} & \cdots & \overline{K}^t_{(01,IJ)} \\ \vdots & & \vdots & & & \vdots \\ \overline{K}^t_{(0J,00)} & \overline{K}^t_{(0J,01)} & \cdots & \overline{K}^t_{(0J,0J)} & \cdots & \overline{K}^t_{(0J,IJ)} \\ \vdots & & \vdots & & & \vdots \\ \overline{K}^t_{(IJ,00)} & \overline{K}^t_{(IJ,01)} & \cdots & \overline{K}^t_{(IJ,0J)} & \cdots & \overline{K}^t_{(IJ,IJ)} \end{bmatrix}$$

$$K_{ex} = \begin{bmatrix} \overline{K}^{ex}_{(00,00)} & \overline{K}^{ex}_{(00,01)} & \cdots & \overline{K}^{ex}_{(00,0J)} & \cdots & \overline{K}^{ex}_{(00,IJ)} \\ \overline{K}^{ex}_{(01,00)} & \overline{K}^{ex}_{(01,01)} & \cdots & \overline{K}^{ex}_{(01,0J)} & \cdots & \overline{K}^{ex}_{(01,IJ)} \\ \vdots & & \vdots & & & \vdots \\ \overline{K}^{ex}_{(0J,00)} & \overline{K}^{ex}_{(0J,01)} & \cdots & \overline{K}^{ex}_{(0J,0J)} & \cdots & \overline{K}^{ex}_{(0J,IJ)} \\ \vdots & & \vdots & & & \vdots \\ \overline{K}^{ex}_{(IJ,00)} & \overline{K}^{ex}_{(IJ,01)} & \cdots & \overline{K}^{ex}_{(IJ,0J)} & \cdots & \overline{K}^{ex}_{(IJ,IJ)} \end{bmatrix}$$

$$M = \begin{bmatrix} \overline{M}_{(00,00)} & \overline{M}_{(00,01)} & \cdots & \overline{M}_{(00,0J)} & \cdots & \overline{M}_{(00,IJ)} \\ \overline{M}_{(01,00)} & \overline{M}_{(01,01)} & \cdots & \overline{M}_{(01,0J)} & \cdots & \overline{M}_{(01,IJ)} \\ \vdots & & \vdots & & & \vdots \\ \overline{M}_{(0J,00)} & \overline{M}_{(0J,01)} & \cdots & \overline{M}_{(0J,0J)} & \cdots & \overline{M}_{(0J,IJ)} \\ \vdots & & \vdots & & & \vdots \\ \overline{M}_{(IJ,00)} & \overline{M}_{(IJ,01)} & \cdots & \overline{M}_{(IJ,0J)} & \cdots & \overline{M}_{(IJ,IJ)} \end{bmatrix}$$

where

$$\begin{aligned} \overline{K}^t_{(p,q,i,j)} = \iint_A \Bigg[& \left(D_{xx} \frac{\partial^2 w_{ij}}{\partial x^2} + D_{xy} \frac{\partial^2 w_{ij}}{\partial y^2} + 2D_{xs} \frac{\partial^2 w_{ij}}{\partial x \partial y} \right) \frac{\partial^2 w_{pq}}{\partial x^2} \\ & + \left(D_{xy} \frac{\partial^2 w_{ij}}{\partial x^2} + D_{yy} \frac{\partial^2 w_{ij}}{\partial y^2} + 2D_{ys} \frac{\partial^2 w_{ij}}{\partial x \partial y} \right) \frac{\partial^2 w_{pq}}{\partial y^2} \\ & + 2\left(D_{xs} \frac{\partial^2 w_{ij}}{\partial x^2} + D_{ys} \frac{\partial^2 w_{ij}}{\partial y^2} + 2D_{ss} \frac{\partial^2 w_{ij}}{\partial x \partial y} \right) \frac{\partial^2 w_{pq}}{\partial x \partial y} + K_w w_{ij} w_{pq} \Bigg] dxdy, \end{aligned}$$

$$\overline{K}^{ez}_{(p,q,i,j)} = \iint_A N \left(\frac{\partial w_{ij}}{\partial x} \frac{\partial w_{pq}}{\partial x} + \frac{\partial w_{ij}}{\partial y} \frac{\partial w_{pq}}{\partial y} \right) dxdy,$$

$$\overline{M}_{(p,q,i,j)} = \iint_A \rho \omega^2 w_{ij} w_{pq} \, dxdy,$$