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Accurate periodic solution for nonlinear vibration of thick circular sector slab

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Abstract. In this paper we consider a periodic solution for nonlinear free vibration of conservative systems for thick circular sector slabs. In Energy Balance Method (EBM) contrary to the conventional methods, only one iteration leads to high accuracy of the solutions. The excellent agreement of the approximate frequencies and periodic solutions with the exact ones could be established. Some patterns are given to illustrate the effectiveness and convenience of the methodology. Comparing with numerical solutions shows that the energy balance method can converge to the numerical solutions very rapidly which are valid for a wide range of vibration amplitudes as indicated in this paper.

Keywords: thick circular sector cylinder; nonlinear vibration; energy balance method

1. Introduction

Many engineering problems can be parted into linear or nonlinear according to the type of differential equations of motion. Nonlinear oscillators systems are used in many subjects of mechanical and civil engineering. In recent years many researchers have been focused on new approximate solution for nonlinear problems because of their advantages respect to numerical methods. In fact, it is too difficult to find and exact solution for nonlinear governing equations.

Perturbation technique is one of the well-known analytical methods. They are not applicable for strongly nonlinear equations, and to eliminate the imperfections, novel techniques have been developed and are well documented in open literature, for instance, for instance; homotopy perturbation method (Shaban *et al*. 2010, Bayat *et al*. 2013), hamiltonian approach (Bayat 2011, 2012, 2013a, Bayat *et al*. 2013, 2014a, b), energy balance method (He 2002, Bayat and Pakar 2011b, Pakar *et al*. 2011, Mehdipour 2010), variational iteration method (Dehghan 2010, Pakar *et al*. 2012), amplitude frequency formulation (Bayat *et al*. 2011c, 2012, Pakar and Bayat 2013a, He 2008), max-min approach (Shen *et al*. 2009, Zeng *et al*. 2009), variational approach method (He 2007, Bayat and Pakar 2013b, Bayat *et al*. 2013, 2014c, Pakar *et al*. 2012), and the other analytical and numerical (Xu 2009, Alicia *et al*. 2010, Bor-Lih *et al*. 2009, Wu 2011, Odibat *et al*. 2008).

In this paper, the basic idea of energy balance method is introduced and then its application for

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solving nonlinear vibration of thick circular sector slab is studied. Some comparisons between analytical and numerical solutions are presented; eventually we show that EBM can meet to a precise periodic solution for nonlinear systems. It has been indicated that the numerical results of other methods are trigger same conclusion; while EBM is much easier, more convenient and more efficient than other approaches.

2. Swinging oscillation of thick circular sector slab

Consider a thick circular sector slab with angle α and radius R as shown in Fig. 1. The height of mass center obtained as below

$$\bar{y} = \left(\frac{R_2^3 - R_1^3}{R_2^2 - R_1^2}\right)\frac{2\sin(\alpha)}{3\alpha} \tag{1}$$

The governing equation of the oscillation is as follow

$$\left(\Sigma - \psi\cos(\theta)\right)\left(2\frac{d^2\theta}{dt^2}\right) + \psi\sin(\theta)\left(\frac{d\theta}{dt}\right)^2 + \left(g\bar{y}\right)\sin(\theta) = 0, \tag{2}$$

where

$$\begin{aligned} \Sigma &= \left(\frac{3}{2}R_2^2 + \frac{1}{2}R_1^2\right), \\ \psi &= 2R_2\bar{y}. \end{aligned} \tag{3}$$

Eq. (2) becomes

$$\begin{aligned} &\left(X - \Delta\cos(\theta)\right)\left(2\frac{d^2\theta}{dt^2}\right) + \Delta\sin(\theta)\left(\frac{d\theta}{dt}\right)^2 + \frac{g}{\bar{y}}\sin(\theta) = 0 \\ &t = 0 \rightarrow \theta = A,\ d\theta/dt = 0. \end{aligned} \tag{4}$$

where

$$X = \frac{\Sigma}{\bar{y}^2},\ \Delta = \frac{\psi}{\bar{y}^2} \tag{5}$$

And by introducing the dimensionless geometrical parameter

$$\lambda = \frac{\bar{y}}{\bar{R}} = \frac{2\sin(\alpha)}{3\alpha}, \quad \bar{R} = \left(\frac{R_2^3 - R_1^3}{R_2^2 - R_1^2}\right) \tag{6}$$

Eq. (4) becomes (Shaban *et al*. 2010)

$$\begin{aligned} &\left(X - \Delta\cos(\theta)\right)\left(2\frac{d^2\theta}{dt^2}\right) + \Delta\sin(\theta)\left(\frac{d\theta}{dt}\right)^2 + \frac{g}{\lambda\bar{R}}\sin(\theta) = 0 \\ &t = 0 \rightarrow \theta = A,\ d\theta/dt = 0. \end{aligned} \tag{7}$$

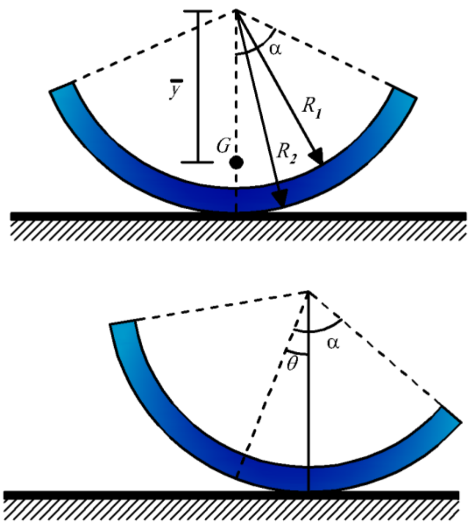


Fig. 1 Geometric parameters of the homogeneous thick circular sector cylinder

3. Basic idea of energy balance method

In the present paper, we consider a general nonlinear oscillator in the Form (He 2008)

$$\ddot{\theta} + f\left(\theta(t)\right) = 0 \tag{8}$$

In which θ and t are generalized dimensionless displacement and time variables, respectively. Its variational principle can be easily obtained

$$J(\theta) = \int_0^t \left(-\frac{1}{2}\dot{\theta}^2 + F(\theta)\right) dt \tag{9}$$

Where $T = 2\pi / \omega$ is period of the nonlinear oscillator, $F(\theta) = \int f(\theta)d\theta$.
Its Hamiltonian, therefore, can be written in the form

$$H = \frac{1}{2}\dot{\theta}^2 + F(\theta) = F(A) \tag{10}$$

or

$$\Re(t) = \frac{1}{2}\dot{\theta}^2 + F(\theta) - F(A) = 0 \tag{11}$$

Oscillatory systems contain two important physical parameters, i.e.,
The frequency ω and the amplitude of oscillation. A. So let us consider such initial conditions

$$\theta(0) = A, \qquad \dot{\theta}(0) = 0 \tag{12}$$

We use the following trial function to determine the angular frequency ω

$$\theta(t) = A\cos(\omega t) \tag{13}$$

Substituting (13) into θ term of (11), yield

$$\Re(t) = \frac{1}{2}\omega^2 A^2 \sin^2(\omega t) + F\left(A\cos(\omega t)\right) - F\left(A\right) = 0 \tag{14}$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make $\Re$ zero for all values of t by appropriate choice of ω. Since Eq. (13) is only an approximation to the exact solution, $\Re$ cannot be made zero everywhere. Collocation at $\omega t = \pi / 4$ gives

$$\omega = \sqrt{\frac{2\left(F\left(A\right)\right) - F\left(A\cos(\omega t)\right)}{A^2 \sin^2(\omega t)}} \tag{15}$$

Its period can be written in the form

$$T = \frac{2\pi}{\sqrt{\frac{2\left(F\left(A\right)\right) - F\left(A\cos(\omega t)\right)}{A^2 \sin^2(\omega t)}}} \tag{16}$$

4. Basic idea of Runge-Kutta's Method (RKM)

The Runge-Kutta method is an important iterative method for the approximation solutions of ordinary differential equations. These methods were developed by the German mathematician Runge and Kutta around 1900. For simplicity, we explain one of the important methods of Runge-Kutta methods, called forth-order Runge-Kutta method.

Consider an initial value problem be specified as follows

$$\dot{\theta} = f\,(t,\theta), \quad \theta(t_0) = \theta_0 \tag{17}$$

θ is an unknown function of time t which we would like to approximate. Then RK4 method is given for this problem as below

$$\begin{aligned} \theta_{n+1} &= \theta_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4), \\ t_{n+1} &= t_n + h. \end{aligned} \tag{18}$$

for $n = 0, 1, 2, 3, \ldots$, using

$$\begin{aligned} k_1 &= f\,\left(t_n, \theta_n\right), \\ k_2 &= f\,\left(t_n + \frac{1}{2}h,\ \theta_n + \frac{1}{2}hk_1\right), \\ k_3 &= f\,\left(t_n + \frac{1}{2}h,\ \theta_n + \frac{1}{2}hk_3\right), \\ k_4 &= f\,\left(t_n + h,\ \theta_n + hk_3\right). \end{aligned} \tag{19}$$

Where θ_{n+1} is the RK4 approximation of $\theta(t_{n+1})$. The fourth-order Runge-Kutta method requires four evaluations of the right hand side per step h.

5. Application of EBM to thick circular sector slab

Variational formulation can be readily obtained from Eq. (7) as follows

$$J(\theta)=\int_0^t\left(-X\left(\frac{d\theta}{dt}\right)^2-\Delta\left(\frac{d\theta}{dt}\right)^2\cos(\theta)-\frac{g}{\lambda\bar{R}}\cos(\theta)\right)dt\,. \tag{20}$$

Its Hamiltonian, therefore, can be written in the form

$$H=\left(X\left(\frac{d\theta}{dt}\right)^2-\Delta\left(\frac{d\theta}{dt}\right)^2\cos(\theta)-\frac{g}{\lambda\bar{R}}\cos(\theta)\right) \tag{21}$$

and

$$H_{t=0}=-\frac{g}{\lambda\bar{R}}\cos(A), \tag{22}$$

$$H_t-H_{t=0}=\left(X\left(\frac{d\theta}{dt}\right)^2-\Delta\left(\frac{d\theta}{dt}\right)^2\cos(\theta)-\frac{g}{\lambda\bar{R}}\cos(\theta)\right)-\left(-\frac{g}{\lambda\bar{R}}\cos(A)\right) \tag{23}$$

We will use the trial function to determine the angular frequency ω, i.e.,

$$\theta(t)=A\cos(\omega t) \tag{24}$$

If we substitute Eq. (24) into (23), it results the following residual equation

$$\begin{aligned}R(t)=&\left(X\left(A^2\omega^2\sin^2(\omega t)\right)\right)-\Delta A^2\omega^2\sin^2(\omega t)\cos(A\cos(\omega t))-\frac{g}{\lambda\bar{R}}\cos(A\cos(\omega t))\\&-\left(-\frac{g}{\lambda\bar{R}}\cos(A)\right)=0\end{aligned} \tag{25}$$

If we collocate at $\omega t=\pi/4$ we obtain

$$\frac{1}{2}XA^2\omega^2-\frac{1}{2}\Delta A^2\omega^2\cos\left(\frac{\sqrt{2}}{2}A\right)-\frac{g}{\lambda\bar{R}}\cos\left(\frac{\sqrt{2}}{2}A\right)+\frac{g}{\lambda\bar{R}}\cos(A)=0 \tag{26}$$

This leads to the following result

$$\omega=\frac{\sqrt{2}\sqrt{\left(X-\Delta\cos\left(\frac{\sqrt{2}}{2}A\right)\right)\frac{g}{\lambda\bar{R}}\left(\cos\left(\frac{\sqrt{2}}{2}A\right)-\cos(A)\right)}}{\left(\cos\left(\frac{\sqrt{2}}{2}A\right)\Delta-X\right)A} \tag{27}$$

Table 1 Comparison of frequency corresponding to various parameters of system

A	α	R_2	R_1	g	Energy balance method	Runge-Kutta	Error %
$\pi/12$	$\pi/6$	5	3	10	2.2940	2.2989	0.2125
$\pi/6$	$\pi/3$	2	1.5	10	2.0398	2.0474	0.3699
$\pi/4$	$\pi/3$	4	3	10	1.2452	1.2594	1.1329
$\pi/3$	$\pi/4$	5	1	10	0.9441	0.9642	2.0802
$\pi/2$	$\pi/2$	4	2	10	0.6131	0.6230	1.5925
$2\pi/3$	$\pi/2$	5	2.5	10	0.4163	0.4254	2.1341

According to Eqs. (24) and (27), we can obtain the following approximate solution

$$\theta(t) = A\cos\left(\frac{\sqrt{2}\sqrt{\left(X - \Delta\cos(\frac{\sqrt{2}}{2}A)\right)\frac{g}{\lambda\bar{R}}\left(\cos(\frac{\sqrt{2}}{2}A) - \cos(A)\right)}}{\left(\cos(\frac{\sqrt{2}}{2}A)\Delta - X\right)A}t\right) \tag{28}$$

6. Results and discussions

To illustrate and verify the accuracy of this new approximate analytical approach, some comparisons between energy balance method and numerical method (Runge-kutta algorithm) are presented in a table and some figures.

Table 1 is the comparison of obtained results with those obtained by Runge-Kutta algorithm for different values of A, α, R_1, R_2, g. The maximum relative error between the energy balance method results and numerical results is 2.1341%.

Figs. 2 to 4 represent comparison of analytical solution of time history response with the numerical solution and also the phase plan of the problem for three cases as follow

Case 1 : $A = \pi/6$, $\alpha = \pi/3$, $R_2 = 2$, $R_1 = 1.5$, $g = 10$
Case 2 : $A = \pi/2$, $\alpha = \pi/2$, $R_2 = 4$, $R_1 = 2$, $g = 10$
Case 3 : $A = 2\pi/3$, $\alpha = \pi/2$, $R_2 = 5$, $R_1 = 2.5$, $g = 10$

It is obvious from the figures the EBM has an excellent agreement with the numerical solution and quickly convergent and valid for a wide range of vibration amplitudes and initial conditions. Figs. 5(a) to 5(c) show the effect of various parameters of amplitude and angle on the nonlinear frequency of the problem.

It can be observed form Fig. 5(a) that by increasing R_1 the frequency of the oscillation is increased and opposite result is obtained by increasing R_2 in Fig. 5(b). From Fig. 5(c), the nonlinear frequency is decreased by decreasing of the amplitude for $R_2 = 5$, $R_1 = 2$, $g = 10$.

The Fig. 6 is sensitivity analysis of nonlinear frequency for three cases

(a) respect to R_1 and amplitude,
(b) respect to R_2 and amplitude,
(c) respect to α and amplitude.

The accuracy of the results shows that the EBM can be potentiality used for the analysis of strongly nonlinear oscillation problems accurately.

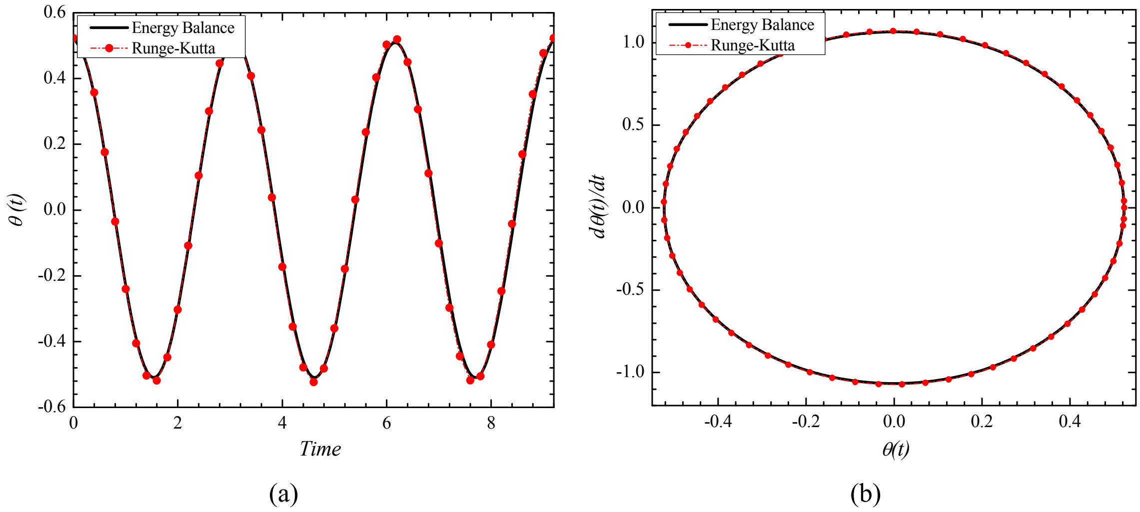


Fig. 2 Comparison of the analytical solution and numerical solution for $A = \pi/6$, $\alpha = \pi/3$, $R_2 = 2$, $R_1 = 1.5$, $g = 10$: (a) time history response; (b) phase curve

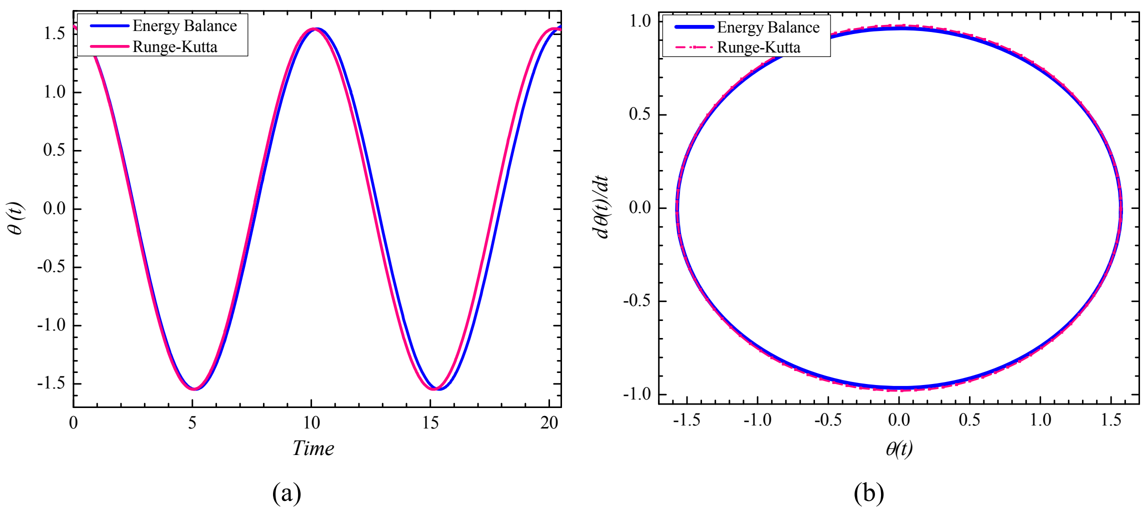
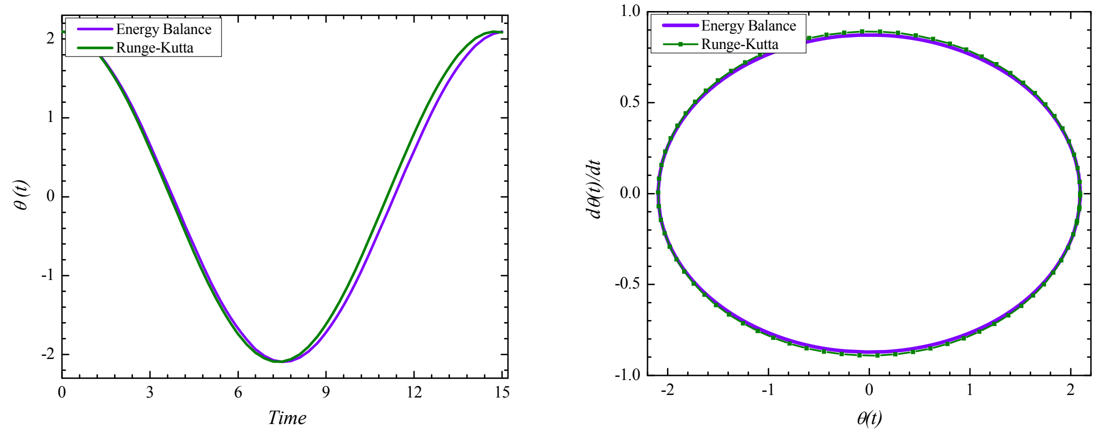
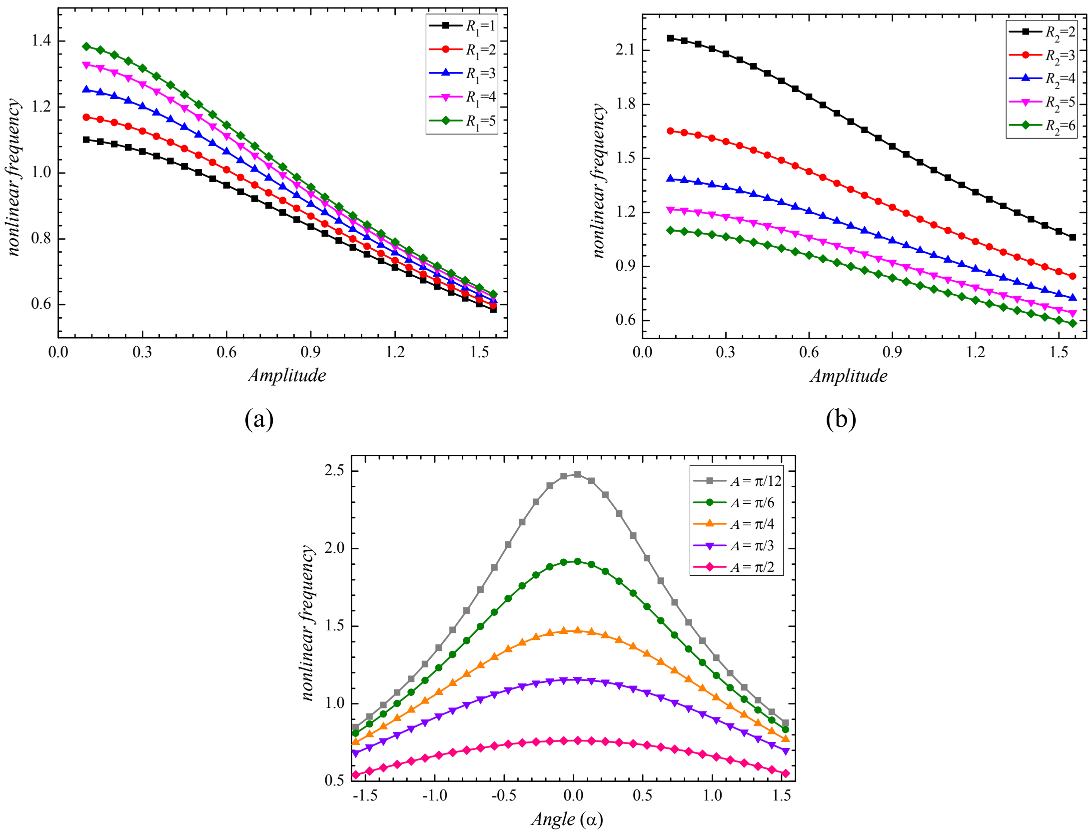


Fig. 3 Comparison of the analytical solution and numerical solution for $A = \pi/2$, $\alpha = \pi/2$, $R_2 = 4$, $R_1 = 2$, $g = 10$: (a) time history response; (b) phase curve



(a) (b)

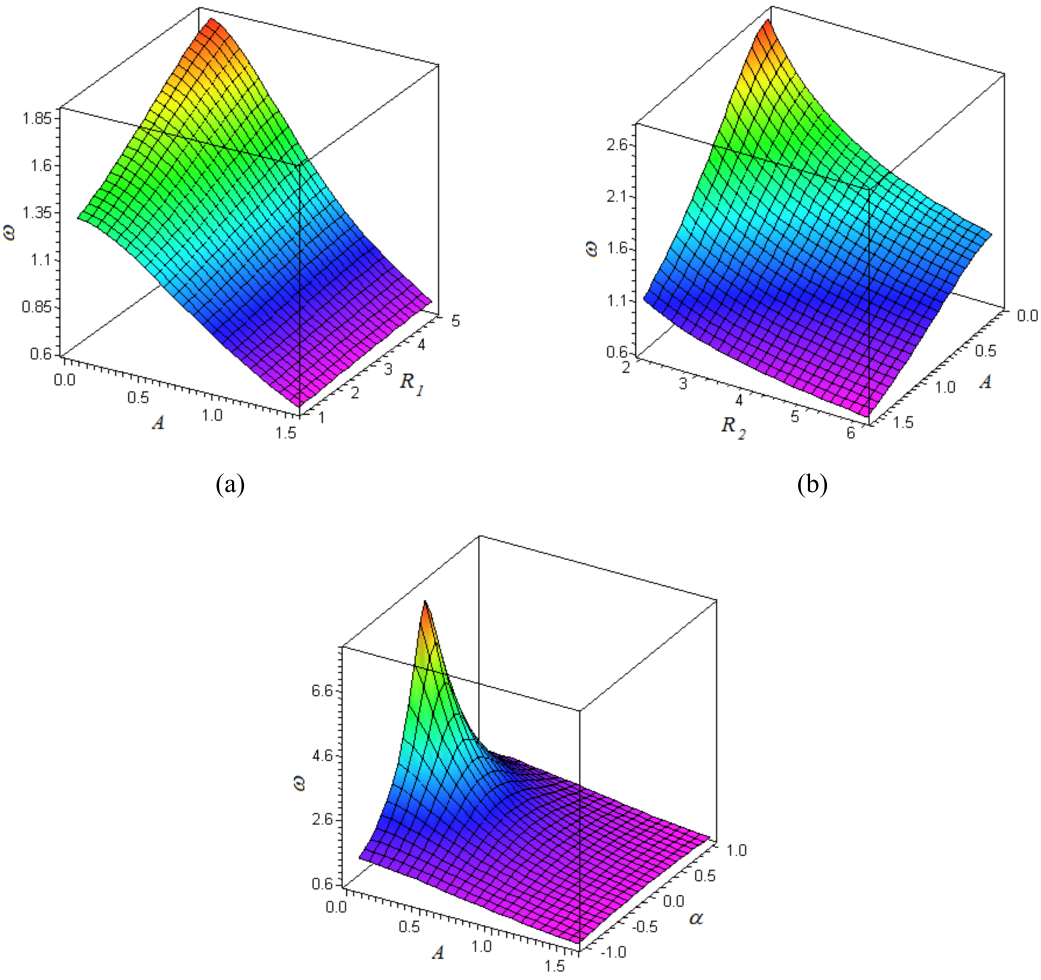
Fig. 4 Comparison of the analytical solution and numerical solution for $A = 2\pi/3$, $\alpha = \pi/2$, $R_2 = 5$, $R_1 = 2.5$, $g = 10$; (a) time history response; (b) phase curve



(a) (b)

(c)

Fig. 5 The effects of radius and amplitude on the nonlinear frequency for: (a) $R_2 = 6$, $\alpha = \pi/3$, $g = 10$; (b) $R_1 = 1$, $\alpha = \pi/3$, $g = 10$; (c): $R_2 = 6$, $\alpha = \pi/3$, $g = 10$



(a) (b)

(c)

Fig. 6 Sensitivity analysis of nonlinear frequency: (a) respect to R_1 and amplitude; (b) respect to R_2 and amplitude; (c) respect to α and amplitude

7. Conclusions

It has been used a quite uncomplicated but productive new method for non-natural oscillator called He's energy balance method. Energy balance method has been utilized on the thick circular cylinder. It has been indicated that the Energy balance method is clearly effective, comfortable and sufficiently exact in engineering problems and does not require any linearization or small perturbation, and adequately accurate to both linear and nonlinear problems in physics and engineering. The results indicated that energy balance method is extremely speedy, light, with high accuracy. The method can be easily extended to any nonlinear oscillator without any difficulty.

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