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Analytical solution for nonlinear vibration of an eccentrically reinforced cylindrical shell

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Abstract. In this study we have considered the governing nonlinear equation of an eccentrically reinforced cylindrical shell. A new analytical method called He's Variational Approach (VA) is used to obtain the natural frequency of the nonlinear equation. This analytical representation gives excellent approximations to the numerical solution for the whole range of the oscillation amplitude, reducing the respective error of angular frequency in comparison with the variation approach method. It has been proved that the variational approach is very effective, convenient and does not require any linearization or small perturbation. Additionally it has been demonstrated that the variational approach is adequately accurate to nonlinear problems in physics and engineering.

Keywords: eccentrically reinforced cylindrical shell; stringer shell; nonlinear vibration; variational approach; Runge-Kutta's Algorithm

1. Introduction

Nonlinear oscillator models have been widely used in many areas of physics and engineering and are of significant importance in mechanical and structural dynamics for the comprehensive understanding and accurate prediction of motion. One of the most important areas in the field of engineering is nonlinear vibrations of shells. Cummings (1964) was sued the Marguerre's theory to study on the large amplitude vibrations and response of freely supported cylindrical shallow shells.

Leissa and Kadi (1971) tried to achieve the nonlinear equations of motion for doubly curved shallow shells. They neglected the tangential inertia in their study. They applied Galerkin's method to reduce it to a general elliptic equation. Hui (1984) used the Donnell's shell theory to study simply supported cylindrical panels with geometric imperfection and in-plane constraints. Sinharay and Banetjee (1985) suggested a new approach based on Berger's hypothesis for the analysis of large amplitude vibrations of shallow spherical and cylindrical shells. Chia (1987) considered the nonlinear vibration and post buckling of imperfect panels by using perturbation method. Fu and Chia (1989, 1993) applied harmonic balance method to study the multimode vibrations of thick panels. Raouf and Palaxotto (1991, 1992) investigated the nonlinear problems of laminated shell panels using numerical methods.

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Kapania and Byum (1992) tried to apply the Finite Element Method (FEM) to study the imperfect laminated panels.

Many new mathematical methods have appeared in open literatures recently, for example: Homotopy perturbation (Ganji 2006, 2009, Bayat 2012a), energy balance (Pakar 2011a, 2013a, Bayat 2011a, He 2002), variational approach (He 2007, Bayat 2013a, Pakar 2011b, 2012a, Liu 2009, Shahidi 2011), max-min approach (Pakar 2013b, Bayat 2011b, c, Shen 2009), Hamiltonian approach (Bayat 2011c, 2012, 2013b) and other analytical and numerical methods (Pakar 2012b, Bayat 2011d, 2011e, Abdollahzade *et al*. 2010, He 1999, He 2010, Xu 2010).

The main object of this paper is to present a new powerful analytical method which valids for large amplitude of the vibration as well as the small. To reach this aim, we propose the variational approach method among of the mentioned methods. We apply it to study the nonlinear vibration of a structurally orthotropic stringer shell. The governing equation of the problem is presented in the next section. The application of the method and verification of it with numerical solutions are also presented completely. The results of the new approach show an excellent agreement between numerical solutions using Runge- kutta's method. Variational approach could be an easy mathematical method to analysis nonlinear problems.

2. Governing equation of a stringer shell

New nonlinear approximate equations for eccentrically reinforced cylindrical shell are obtained by considering a simplified boundary value problem (Amiro 1983, Zarutsky 1993).

The basic relations of the linear theory of shell and the detailed discussions can be found in many monographs and papers (Manevitch, 1972, Koiter, 1966) and therefore we discuss only the final results.

Appling the semi-inexensional theory the following governing equations are used (Filippov 1999, Evakin 2001, Grigolyuk 1987, Han 1965, Andrianov 2004).

$$L_1(w) = \nabla_1^4 w - R\frac{\partial^2 \phi}{\partial x^2} + \rho R^2 \frac{\partial^2 w}{\partial t_1^2} - L(w,\phi) = 0 \tag{1}$$

$$\frac{I}{B_2}\frac{\partial^4 \phi}{\partial y^4} + \frac{1}{R}\frac{\partial^2 w}{\partial x^2} = \frac{1}{2R^2}L(w,\phi) \tag{2}$$

where

$$\nabla_1^4 = \frac{1}{R^2}\left(D_1\frac{\partial^4}{\partial x^4} + 2D_3\frac{\partial^4}{\partial x^2 \partial y^2} + D_2\frac{\partial^4}{\partial y^4}\right) \tag{3}$$

$$L(w,\phi) = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} - 2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{4}$$

w is normal displacement; φ is Airy function

$$D_1 = D + \frac{NE_1 I}{(2\pi R)}, \quad D = \frac{Eh^3}{(12(1-v^2))}$$

$$\rho = \rho_0 h + \frac{N\rho_1 F}{(2\pi R)}, \quad B_1 = \frac{Eh}{(1-v^2)} + \frac{NE_1 F}{(2\pi R)}$$

E, E_1 are young modulus's of shell and rib respectively; υ is Poisson ratio; h is a shell thickness; R is shell radius; ρ_0, ρ_1 are densities of shell radius; n is number of stringer; F is square stringer cross-section; I is statically moment of stringer cross-section.

We suppose ribs symmetric with respect to shell middle surface (Grigolyuk 1987, Han 1965). Equations govern dynamics for low frequencies, the most important from practical standpoint. For simply supported shell

$$W = \frac{\partial^2 w}{\partial x^2} = 0 \qquad X = 0, L \tag{5}$$

Normal displacement w is approximated by

$$w(x, y, t_1) = f_1(t_1)\sin(m_1 x)\cos(n_1 y) + f_2(t_1)\sin^2(m_1 x) \tag{6}$$

Where: $m_1 = \pi m/L$, $n_1 = n/(Rm)$, m and n are the parameters characterizing waves into x and y directions. Note that f_1 and f_2 are not independent, and its relations is defined from a continuity condition of the displacement υ along a ring

$$\int_0^{2\pi} \frac{\partial \upsilon}{\partial y} dy = 0 \tag{7}$$

In our case $\partial\upsilon / \partial y = w$, which gives $f_2 = f_1^2 n^2/(4R)$.

The Airy function is found from the Eq. (2)

$$\begin{aligned} B_1^{-1}\phi = {} & \left(\frac{m_1}{h}\right)^2 \frac{f_1}{Rn^2}\sin(m_1 x)\cos(n_1 y) - \frac{5}{16}\left(\frac{m_1}{h}\right)^2 \times \left(\frac{f_1}{R}\right)^2 \cos(2n_1 y) \\ & + \frac{1}{2}m_1\left(\frac{f_1}{R}\right)^3 \sin(m_1 x)\cos(2m_1 x)\cos(n_1 y). \end{aligned} \tag{8}$$

Knowing φ one may apply the Bubnov-Galerkin method to governing equations to get

$$\int_0^{2\pi R}\int_0^{L} L_1(w)\sin(m_1 x)\cos(n_1 y)\, dxdy = 0 \tag{9}$$

After some computations the following second order differential equation with constant coefficient is obtained (Andrianov 2004)

$$\frac{d^2\xi}{dt^2} + \alpha\xi\left(\frac{d\xi}{dt^2}\right)^2 + \alpha\xi^2\left(\frac{d^2\xi}{dt^2}\right) + \beta\xi + \eta\xi^3 + \lambda\xi^5 = 0. \tag{10}$$

3. Basic idea of variational approach

He (2007) suggested a variational approach which is different from the known variational methods in open literature. Hereby we give a brief introduction of the method

$$\ddot{\xi} + f(\xi) = 0 \tag{11}$$

$$J(\xi) = \int_0^{T/4} \left(-\frac{1}{2}\xi^2 + F(\xi) \right) dt \tag{12}$$

Its variational principle can be easily established using the semi-inverse method.

Where T is period of the nonlinear oscillator, $\partial F / \partial \xi = f$. Assume that its solution can be expressed as

$$\xi(t) = A\cos(\omega t) \tag{13}$$

Where A and ω are the amplitude and frequency of the oscillator, respectively. Substituting Eq. (13) into Eq. (12) results in

$$\begin{aligned} J(A,\omega) &= \int_0^{T/4} \left(-\frac{1}{2}A^2\omega^2 \sin^2 \omega t + F(A\cos \omega t) \right) \\ &= \frac{1}{\omega}\int_0^{\pi/2} \left(-\frac{1}{2}A^2\omega^2 \sin^2 t + F(A\cos t) \right) dt \\ &= -\frac{1}{2}A^2\omega \int_0^{\pi/2} \sin^2 t\, dt + \frac{1}{\omega}\int_0^{\pi/2} F(A\cos t)\, dt \end{aligned} \tag{14}$$

Applying the Ritz method, we require

$$\frac{\partial J}{\partial A} = 0 \tag{15}$$

$$\frac{\partial J}{\partial \omega} = 0 \tag{16}$$

But with a careful inspection, for most cases we find that

$$\frac{\partial J}{\partial \omega} = -\frac{1}{2}A^2 \int_0^{\pi/2} \sin^2 t\, dt + \frac{1}{\omega^2}\int_0^{\pi/2} F(A\cos t)\, dt < 0 \tag{17}$$

Thus, we modify conditions Eq. (15) and Eq. (16) into a simpler form

$$\frac{\partial J}{\partial \omega} = 0 \tag{18}$$

From which the relationship between the amplitude and frequency of the oscillator can be obtained.

4. Basic idea of Runge-Kutta's Algorithm

For such a boundary value problem given by boundary condition, some numerical methods have been developed. Here we apply the fourth-order Runge-Kutta's Algorithm to solve governing equation subject to the given boundary conditions. RK iterative formulae for the second-order differential equation are

$$\dot{\xi}_{(i+1)} = \dot{\xi}_i + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4), \quad \xi_{(i+1)} = \xi_i + \Delta t\left[\dot{\xi}_i + \frac{\Delta t}{6}(k_1 + k_2 + k_3)\right], \tag{19}$$

Where Δt is the increment of the time and k_1, k_2, k_3 and k_4 are determined from the following formula

$$\begin{gathered}
k_1 = f(t_i, \xi_i, \dot{\xi}_i), \quad k_2 = f\left(t_i + \frac{\Delta t}{2},\ \xi_i + \frac{\Delta t}{2},\ \dot{\xi}_i + \frac{\Delta t}{2}k_1\right), \\
k_3 = f\left(t_i + \frac{\Delta t}{2},\ \xi_i + \frac{\Delta t}{2}\dot{\xi}_i,\ \frac{1}{4}\Delta t^2 k_1,\ \ddot{\xi}_i + \frac{\Delta t}{2}k_2\right), \\
k_3 = f\left(t_i + \Delta t,\ \xi_i + \Delta t\,\dot{\xi}_i + \frac{1}{2}\Delta t^2 k_2,\ \dot{\xi}_i + \Delta t\, k_3\right)
\end{gathered} \tag{20}$$

The numerical solution starts from the boundary at the initial time, where the first value of the displacement function and its first-order derivative is determined from initial condition. Then, with a small time increment [Δt], the displacement function and its first-order derivative at the new position can be obtained using Eq. (19). This process continues to the end of time.

5. Applications of variational approach

For differential Eq. (10) and for $t = 0$, $\xi = A$, $(d\xi / dt) = 0$, Its variational formulation can be readily obtained as follows

$$J(\xi) = \int_0^t \left(-\frac{1}{2}\left(\frac{d^2\xi}{dt^2}\right)^2 + \frac{1}{2}\alpha\,\xi^2\left(\frac{d\xi}{dt}\right)^2 + \frac{1}{2}\beta\,\xi^2 + \frac{1}{4}\eta\,\xi^4 + \frac{1}{6}\lambda\,\xi^6 \right) dt \tag{21}$$

Choosing the trial function $\xi(t) = A\cos(\omega t)$ into Eq. (21) we obtain

$$J(A,\omega) = \int_0^{T/4} \left(\begin{gathered} -\frac{1}{2}\left(A\omega\sin(\omega t)\right)^2 - \frac{1}{2}\alpha\left(A\omega\sin(\omega t)\right)^2\left(A\cos(\omega t)\right)^2 \\ +\frac{1}{2}\beta\left(A\cos(\omega t)\right)^2 + \frac{1}{4}\eta\left(A\cos(\omega t)\right)^4 + \frac{1}{6}\lambda\left(A\cos(\omega t)\right)^6 \end{gathered} \right) dt \tag{22}$$

The stationary condition with respect to A reads

$$\frac{\partial J}{\partial \omega}=\int_0^{T/4}\begin{pmatrix} -A\omega^2 \sin^2(\omega t)-2\alpha\,\omega^2 A^3 \sin^2(\omega t)\cos^2(\omega t) \\ +\beta A\cos^2(\omega t)+\eta A^3\cos^4(\omega t)+\lambda A^5\cos^6(\omega t) \end{pmatrix} dt=0 \tag{23}$$

or

$$\frac{\partial J}{\partial \omega}=\int_0^{\pi/2}\begin{pmatrix} -A\omega^2 \sin^2 t-2\alpha\,\omega^2 A^3 \sin^2 t\cos^2 t \\ +\beta A\cos^2 t+\eta A^3\cos^4 t)+\lambda A^5\cos^6 t) \end{pmatrix} dt=0 \tag{24}$$

which leads to the result

$$\omega^2=\frac{\int_0^{\pi/2}\left(\beta A\cos^2 t+\eta A^3\cos^4 t+\lambda A^5\cos^6 t\right)dt}{\int_0^{\pi/2}\left(A\sin^2 t+2\alpha\,A^3\sin^2 t\cos^2 t\right)dt} \tag{25}$$

Solving Eq. (25), according to ω, we have

$$\omega=\frac{1}{2}\sqrt{\frac{8\beta+6\eta A^2+5\lambda A^4}{\alpha A^2+2}}, \tag{26}$$

We can obtain the following approximate solution

$$\xi(t)=A\cos\left(\frac{1}{2}\sqrt{\frac{8\beta+6\eta A^2+5\lambda A^4}{\alpha A^2+2}}t\right), \tag{27}$$

6. Result and discussion

In this section to show the applicability and validation of the proposed method we have presented a table and figures to compare the obtained results with the numerical solution. Table 1 presents the comparison of time step solution of variational approach method and Runge-Kutta's Algorithm for following two different cases:

Case 1: A = 0.5, α = 0.8, β = 0.2, η = 0.6, λ = 0.3
Case 2: A = 1, α = 0.5, β = 0.4, η = 0.3, λ = 0.5

Fig. 1 shows the time history oscillatory displacement response with the numerical solution using Runge-kutta's Algorithm for two cases; (a) A = 0.2, α = 0.5, β = 0.4, η = 0.4, λ = 0.3; (b) A = 0.4, α = 0.5, β = 0.2, η = 0.4, λ = 0.3. The motion of the system is periodic and the vibrations of the amplitudes are function of the initial conditions. Fig. 2 shows the comparison of the analytical solution and numerical one by considering the β on the time history response with constant parameters. The phase plan of the problems is in the Fig. 3 for two cases. In Case 1 we have considered the effects of η parameters and for the second case for the λ parameter. The variation of amplitude of the shell is shown respect to nonlinear frequency of the problem for α and β in Fig. 4.

It is obvious from the figures and the table that the variational approach for different parameters shows an excellent agreement with the numerical ones. The accuracy of the proposed method shows that the variational approach can be potentially used for the analysis of strongly nonlinear oscillation problems accurately.

Table 1 The comparison of time history response of the Variational Approach solution and Runge-Kutta's Algorithm

	Case 1			Case 2		
Time	Variational Approach	Runge-Kutta	Error	Variational Approach	Runge-Kutta	Error
0	0.5	0.5	0	1	1	0
1	0.4281	0.4257	0.0056	0.6479	0.6399	0.0125
2	0.2331	0.2297	0.0146	-0.1606	-0.1516	0.0593
3	-0.0289	-0.0272	0.0632	-0.8559	-0.8435	0.0148
4	-0.2827	-0.2772	0.0198	-0.9484	-0.9491	0.0007
5	-0.4551	-0.4525	0.0057	-0.3730	-0.3799	0.0180
6	-0.4966	-0.4968	0.0002	0.4651	0.4405	0.0559
7	-0.3954	-0.3937	0.0043	0.9757	0.9687	0.0072
8	-0.1804	-0.1797	0.0039	0.7991	0.8048	0.0071
9	0.0864	0.0814	0.0623	0.0597	0.0855	0.3015
10	0.3284	0.3215	0.0217	-0.7217	-0.6900	0.0460
11	0.4760	0.4737	0.0049	-0.9949	-0.9975	0.0027
12	0.4866	0.4871	0.0010	-0.5673	-0.5871	0.0336

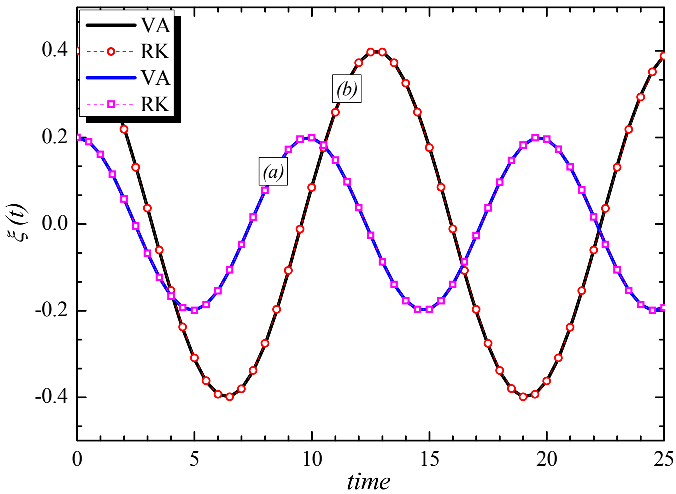

Fig. 1 Comparison of time history response of the VA solution with the RK solution for (a) $A = 0.2$, $\alpha = 0.5$, $\beta = 0.4$, $\eta = 0.4$, $\lambda = 0.3$; (b) $A = 0.4$, $\alpha = 0.5$, $\beta = 0.2$, $\eta = 0.4$, $\lambda = 0.3$

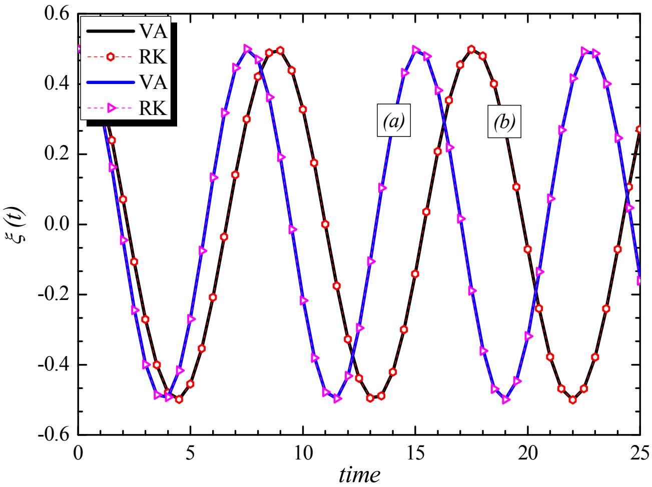

Fig. 2 Effect of parameter on time history response for cases (a) $\underline{\beta = 0.6}$, $A = 0.5$, $\alpha = 1$, $\eta = 0.8$, $\lambda = 0.6$; (b) $\underline{\beta = 0.4}$, $A = 0.5$, $\alpha = 1$, $\eta = 0.8$, $\lambda = 0.6$

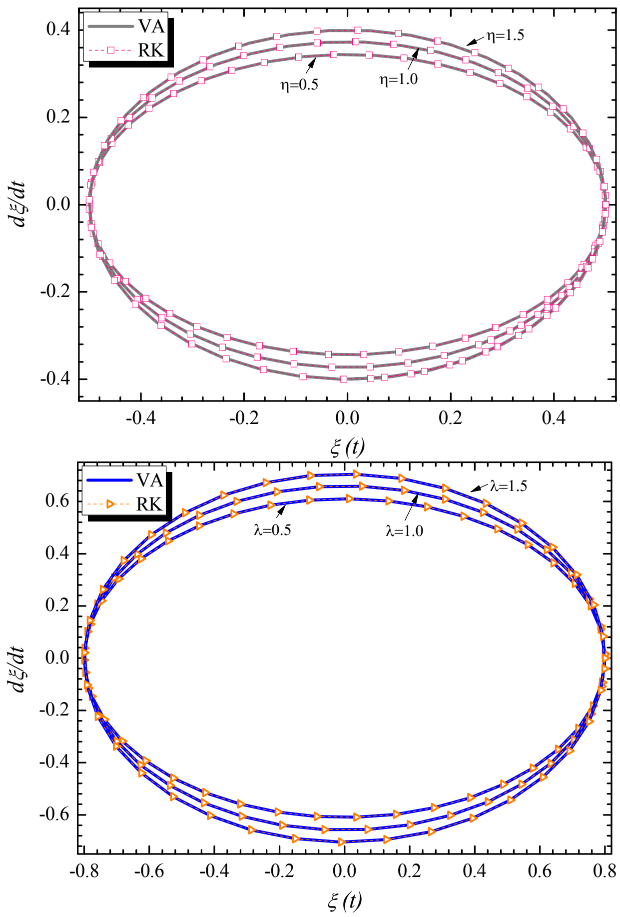

Fig. 3 Comparison of phase plan of the VA solution with the RK solution for (a) $A = 0.5$, $\alpha = 1$, $\beta = 0.4$, $\lambda = 1$; (b) $A = 0.8$, $\alpha = 1$, $\beta = 0.4$, $\lambda = 0.5$

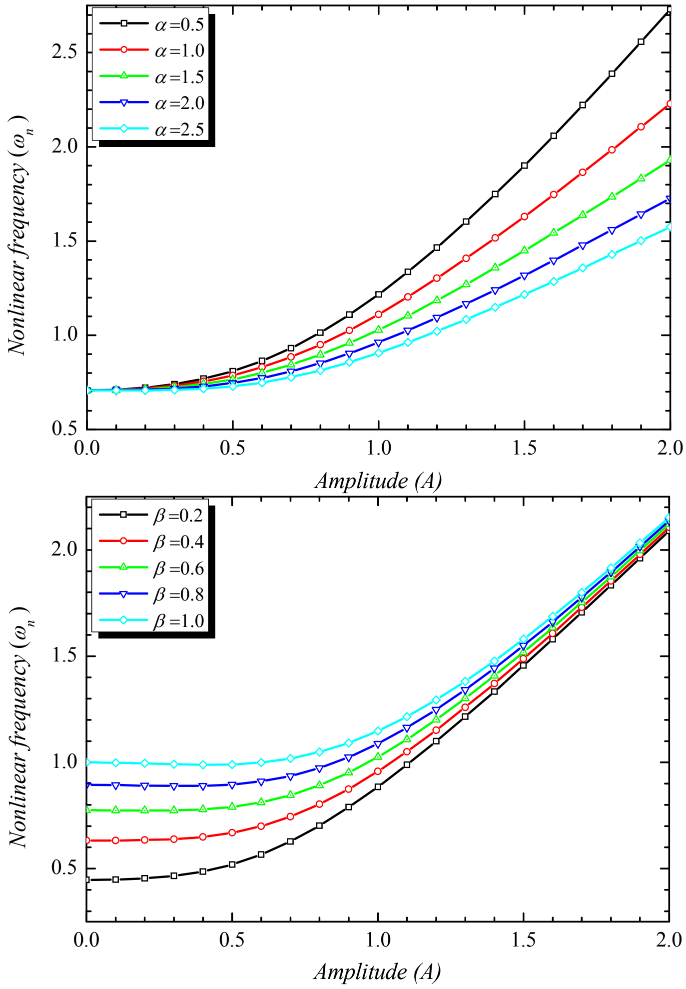

Fig. 4 Effect of amplitude on nonlinear frequency of stringer shell with various parameters (a) $\beta = 0.5$, $\eta = 0.8$, $\lambda = 1.2$; (b) $\beta = 1$, $\eta = 0.2$, $\lambda = 1.2$

7. Conclusions

In this paper, we studied the new mathematical method called Variational Approach to obtain accurate results of nonlinear vibration of a stringer shell. Different parameters on the response of the problems and also on the nonlinear frequency of the problem were considered and discussed completely. We have established that the new approach can be extends to any nonlinear conservative problems with no limitations. The method is proved to be a powerful mathematical tool for studying of nonlinear oscillators. The achieved results indicated that Variational approach is extremely simple, easy, powerful, and triggers good accuracy and leads us to high accuracy by one iteration.

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