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On the large amplitude free vibrations of axially loaded Euler-Bernoulli beams

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Abstract. In this paper Hamiltonian Approach (HA) have been used to analysis the nonlinear free vibration of Simply-Supported (S-S) and for the Clamped-Clamped (C-C) Euler-Bernoulli beams fixed at one end subjected to the axial loads. First we used Galerkin's method to obtain an ordinary differential equation from the governing nonlinear partial differential equation. The effect of different parameter such as variation of amplitude to the obtained on the non-linear frequency is considered. Comparison of HA with Runge-Kutta 4th leads to highly accurate solutions. It is predicted that Hamiltonian Approach can be applied easily for nonlinear problems in engineering.

Keywords: non-linear vibration, analytical solution, beam vibration, Runge-Kutta 4th

1. Introduction

Beam vibration is still an interesting area in civil engineering and mechanical engineering. Beams widely use in many engineering applications such as bridges, tall buildings, truss structures and many other engineering applications.

To obtain better performance of these structures and improve their life time, it is very necessary to have an accurate analysis by considering all the aspects in the design of them. Dynamic response of beams is one the most important parts in the design process of structures. The nonlinear vibration of beams and distributed and continuous systems are governed by linear and nonlinear partial differential equations in space and time. Generally, it has lots of difficulties to solve nonlinear partial differential equations analytically. Consequently, many scientific have been worked on numerical solutions and asymptotic approaches to solve the initial boundary-value problems. Perturbation methods have lots of shortcoming for solving high nonlinear differential equations. Therefore, the partial-differential equations are discrete to non-linear ordinary-differential equations by using the Galerkin approach and then we can apply the direct techniques to solve them analytically in time domain.

In recent years, many approximate analytical methods have been proposed for studying nonlinear vibration equations of beams and shells and etc such as; homotopy perturbation (Ganji *et al*. 2009, Bayat *et al*. 2010), improved amplitude-frequency formulation (He 2008), energy balance (Bayat *et al*. 2011, Bayat and Pakar 2011c,d), variational approach (He 2007, Bayat 2011c, Shahidi

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et al. 2011, Xu and Zhang 2009), max-min approach (Bayat and pakar 2011a, Shen and Mo 2009), iteration perturbation method (Bayat *et al*. 2011) and other analytical and numerical methods (Ghasemi and Bayat 2011, Soleimani *et al*. 2011, Bayat 2011b, Bayat *et al*. 2011, Bayat 2012a, Bayat and Pakar 2012b).

Padovan (1980) analyzed the nonlinear vibration of structures by using perturbation method and finite element method. Sathyamoorthy (1982) investigated on classical methods for the analysis of nonlinear vibration of beams under harmonic loading. Biondi and Caddemi (2005) studied the flexural stiffness and slope discontinuities for uniform Euler-Bernoulli beam and tried to apply a close form solution for the governing equation. Lai *et al.* (2008) considered the nonlinear vibration of Euler-Bernoulli beam with different supporting conditions by applying the Adomian decomposition method (ADM). Naguleswaran (2003) developed the work on the changes of cross section of an Euler-Bernoulli beam resting on elastic end supports. Pirbodaghi *et al.* (2009) presented an analytical expression for geometrically free vibration of Euler-Bernoulli beam by using homotopy analysis method (HAM). They illustrated that the amplitude of the vibration has a great effect on the nonlinear frequency and buckling load of the beams. Liu *et al.* (2009) applied He's variational iteration method to obtain an analytical solution for an Euler-Bernoulli beam with different supporting conditions. Bayat *et al.* (2011f) obtained the natural frequency of the nonlinear equation of Euler-Bernoulli beam by using energy balance method. In this study, we try to asses an analytical expression for non-linear vibration of Simply-Supported (S-S) and for the Clamped-Clamped (C-C) buckled Euler-Bernoulli beams fixed at one end by using a new analytical approach called Hamiltonian Approach (HA) in time domain.

First we used Galerkin method for discretization to obtain an ordinary nonlinear differential equation from the governing non- linear partial differential equation. It was then assumed that only fundamental mode was excited. Finally, Hamiltonian Approach is compared with other researcher's results. The Hamiltonian Approach results are accurate and only one iteration leads to high accuracy of solutions for whole domain and can be a powerful approach for solving high nonlinear engineering problems.

2. Description of the problem

The equation of motion for an axially loaded Euler-Bernoulli beam by considering the mid-plane stretching effect is

$$m\frac{\partial^2 w'}{\partial t'^2}+EI\frac{\partial^4 w'}{\partial x'^4}+\bar{P}\frac{\partial^2 w'}{\partial x'^2}-\frac{EA}{2L}\frac{\partial^2 w'}{\partial x'^2}\int_0^L\left(\frac{\partial^2 w'}{\partial x'^2}\right)^2 dx'=0 \tag{1}$$

We introduce these new non-dimensional variables for convenience;

$$x=x'/L, w=w'/\rho, t=t'(EI/ml^4)^{1/2}, P=\bar{P}L^2/EI$$

Where $\rho=(I/A)^{1/2}$ is the radius of gyration of the cross-section. Then Eq. (1) can be written as follows

$$\frac{\partial^2 w}{\partial t^2}+\frac{\partial^4 w}{\partial x^4}+P\frac{\partial^2 w}{\partial x^2}-\frac{1}{2}\frac{\partial^2 w}{\partial x^2}\int_0^L\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx=0 \tag{2}$$

If we assume $w(x, t)=W(t)\,\phi(x)$ in which $\phi(x)$ is the first Eigen mode of the beam (Tse 1987)

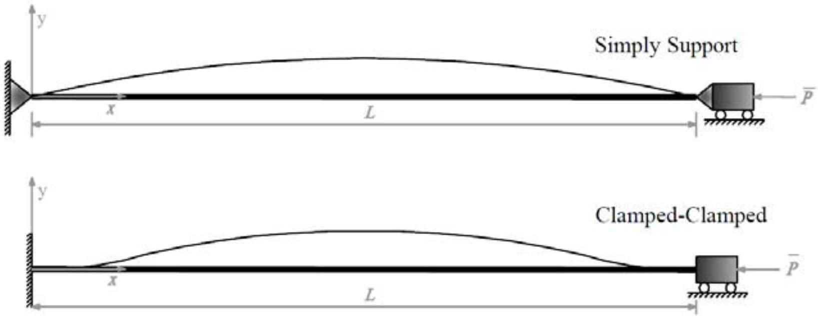

Fig. 1 Schematic representation of an axially loaded Euler-Bernoulli beam

and using the Galerkin method, then we will have the following governing nonlinear vibration equation of motion for an axially loaded Euler-Bernoulli beam

$$\frac{d^2W(t)}{dt^2}+(\alpha_1+P\alpha_2)W(t)+\alpha_3 W^3(t)=0 \tag{3}$$

The initial conditions for center of the beam are

$$W(0)=\Delta,\ \frac{dW(0)}{dt}=0 \tag{4}$$

The value of the α_1, α_2 and α_3 can be obtained as follow

$$\alpha_1=\left(\int_0^1\left(\frac{\partial^4\phi(x)}{\partial x^4}\right)\phi(x)\,dx\right)\Bigg/\int_0^1\phi^2(x)\,dx \tag{5a}$$

$$\alpha_2=\left(\int_0^1\left(\frac{\partial^2\phi(x)}{\partial x^2}\right)\phi(x)\,dx\right)\Bigg/\int_0^1\phi^2(x)\,dx \tag{5b}$$

$$\alpha_3=\left(-\frac{1}{2}\int_0^1\left(\frac{\partial^2\phi(x)}{\partial x^2}\int_0^1\left(\frac{\partial\phi(x)}{\partial x}\right)^2dx\right)\phi(x)\,dx\right)\Bigg/\int_0^1\phi^2(x)\,dx \tag{5c}$$

3. Basic idea of He's Hamiltonian Approach

The Hamiltonian Approach is a novel method which was proposed by He (2002, 2010). The Hamiltonian Approach is one of the simple and effective approaches for conservative oscillatory systems. Here we give an introduction of this approach, consider the following general oscillator

$$\ddot{W}+f(W)=0 \tag{6}$$

With initial conditions

$$W(0)=\Delta,\qquad \dot{W}(0)=0. \tag{7}$$

The variation principle for the Eq. (6) can be obtained easily by using the semi-inverse method (He 2002)

$$J(W)=\int_0^{T/4}\left\{-\frac{1}{2}\dot{W}^2+F(W)\right\}dt \tag{8}$$

Where T is period of the nonlinear oscillator, $\partial F/\partial W=f(W)$.

The first term of Eq. (8), $\frac{1}{2}\dot{W}^2$ is kinetic energy and $F(W)$ is the potential energy, so the Eq. (8) is the least Lagrangian action, from which we can obtain its Hamiltonian, which reads

$$H(W)=\frac{1}{2}\dot{W}^2+F(W)=\text{constant} \tag{9}$$

From Eq. (9), we have

$$\frac{\partial H}{\partial \Delta}=0 \tag{10}$$

Introducing a new function, $\bar{H}(W)$, defined as

$$\bar{H}(W)=\int_0^{T/4}\left\{\frac{1}{2}\dot{W}^2+F(W)\right\}dt=\frac{1}{4}TH \tag{11}$$

Eq. (10) is, then, equivalent to the following one

$$\frac{\partial}{\partial \Delta}\left(\frac{\partial \bar{H}}{\partial T}\right)=0 \tag{12}$$

Or

$$\frac{\partial}{\partial \Delta}\left(\frac{\partial \bar{H}}{\partial (1/\omega)}\right)=0 \tag{13}$$

From Eq. (13) we can obtain approximate frequency–amplitude relationship of a nonlinear oscillator.

4. Basic idea of Runge-Kutta's Method (RKM)

For such a boundary value problem given by boundary condition, some numerical methods have been developed. Here we apply the fourth-order Runge-Kutta algorithm to solve governing equation subject to the given boundary conditions. Runge-Kutta iterative formulae for the second-order differential equation are

$$\begin{aligned}\dot{W}_{(i+1)}&=\dot{W}_i+\frac{\Delta t}{6}\left(k_1+2k_2+2k_3+k_4\right),\\ W_{(i+1)}&=W_i+\Delta t\left[\dot{W}_i+\frac{\Delta t}{6}\left(k_1+k_2+k_3\right)\right],\end{aligned} \tag{14}$$

where Δt is the increment of the time and k_1, k_2, k_3 and k_4 are determined from the following formula

$$\begin{aligned} k_1 &= f\left(t_i, W_i, \dot{W}_i\right), \\ k_2 &= f\left(t_i + \frac{\Delta t}{2}, W_i + \frac{\Delta t}{2}, \dot{W}_i + \frac{\Delta t}{2} k_1\right), \\ k_3 &= f\left(t_i + \frac{\Delta t}{2}, \dot{W}_i + \frac{\Delta t}{2} \dot{W}_i, \frac{1}{4} \Delta t^2 k_1, \dot{W}_i + \frac{\Delta t}{2} k_2\right), \end{aligned} \tag{15}$$

The numerical solution starts from the boundary at the initial time, where the first value of the displacement function and its first-order derivative is determined from initial condition. Then, with a small time increment [Δt], the displacement function and its first-order derivative at the new position can be obtained using Eq. (15). This process continues to the end of time.

5. Solution using Hamiltonian Approach

The Hamiltonian of Eq. (3) is constructed as

$$H = \frac{1}{2}\dot{W}^2 + \frac{1}{2}\alpha_1 W^2 + \frac{1}{2} p\,\alpha_2 W^2 + \frac{1}{4}\alpha_3 W^4 \tag{16}$$

Integrating Eq. (16) with respect to t from 0 to $T/4$, we have

$$\bar{H} = \int_0^{T/4} \left(\frac{1}{2}\dot{W}^2 + \frac{1}{2}\alpha_1 W^2 + \frac{1}{2} p\,\alpha_2 W^2 + \frac{1}{4}\alpha_3 W^4 \right) dt \tag{17}$$

We use the following trial function

$$W(t) = \Delta \cos(\omega t) \tag{18}$$

If we Substitute Eq. (18) into Eq. (17), its results are

$$\begin{aligned} \bar{H} &= \int_0^{T/4} \left(\frac{1}{2}\Delta^2\omega^2 sin^2(\omega t) + \frac{1}{2}\alpha_1\Delta^2 cos^2(\omega t) + \frac{1}{2} p\alpha_2\Delta^2 cos^2(\omega t) + \frac{1}{4}\,\alpha_3\,\Delta^4 cos^4(\omega t) \right) dt \\ &= \int_0^{\pi/2} \left(\frac{1}{2}\Delta^2\omega^2 sin^2 t + \frac{1}{2}\alpha_1\Delta^2 cos^2 t + \frac{1}{2} p\alpha_2\Delta^2 cos^2 t + \frac{1}{4}\,\alpha_3\,\Delta^4 cos^4 t \right) dt \\ &= +\frac{1}{8}\Delta^2\omega\pi + \frac{1}{8}\Delta^2\alpha_1\frac{\pi}{\omega} + \frac{1}{8}\Delta^2 p\alpha_2\frac{\pi}{\omega} + \frac{3}{64}\Delta^4\alpha_3\frac{\pi}{\omega} \end{aligned} \tag{19}$$

Setting

$$\frac{\partial}{\partial\Delta}\left(\frac{\partial\bar{H}}{\partial(1/\omega)} \right) = \frac{1}{4}\Delta\pi\omega^2 + \frac{1}{4}\Delta\pi\alpha_1 + \frac{1}{4}\Delta\pi p\alpha_2 + \frac{3}{16}\Delta^3\pi\alpha_3 \tag{20}$$

If we solve Eq. (20) the approximate frequency of the system is

$$\omega = \frac{1}{2}\sqrt{4\left(\alpha_1 + p\alpha_2\right) + 3\alpha_3\Delta^2} \tag{21}$$

Hence, the approximate solution can be readily obtained

$$W(t) = \Delta \cos\left(\frac{1}{2}\sqrt{4\left(\alpha_1 + p\alpha_2\right) + 3\alpha_3\Delta^2}\ \ t \right) \tag{22}$$

The ration of the non-linear to linear frequency is

$$\frac{\omega_{NL}}{\omega_L} = \frac{1}{2}\frac{\sqrt{4\left(\alpha_1 + p\alpha_2\right) + 3\alpha_3\Delta^2}}{\sqrt{\alpha_1 + p\alpha_2}} \tag{23}$$

6. Results and discussion

The Hamiltonian Approach is used to obtain an analytical solution for simply supported and clamped- clamped beams.

To obtain numerical solution we must specify the parameter β. This parameter depends on value of α_1, α_2, α_3 and p, then we have

$$\beta = \frac{\alpha_3}{\left(p\alpha_2 + \alpha_1\right)} \tag{24}$$

So Eq. (21) become

$$\frac{\omega_{NL}}{\omega_L} = \sqrt{1 + \frac{3}{4}\beta\Delta^2} \tag{25}$$

From the reference (Azrar 1999) for a simply supported beam we had $\Delta = \delta / \sqrt{12}$ and for clamped- clamped beam $\Delta = \delta / \sqrt{12}\ w_1^*(1/2)$ that δ is maximum amplitude parameter and $w_1^*(1/2)$ is first mod of beam in middle of beam. Tables 1 and 2 represent the comparisons of nonlinear to linear frequency ratio (ω_{NL} / ω_L) for Simply-Supported Beam and for the Clamped-Clamped Beams with the Hamiltonian Approach (HA) and the numerical solutions and other researchers results for different parameters of Δ and β. Azrar (1999) and Lewandowski (1987) ignored to consider the mid-plane effect in their study therefore for large amplitude the ratio of nonlinear to linear frequency increases. To show the accuracy of the HA results, Runge-Kutta 4th is used to consider the effect of the variation of non-dimensional amplitude ratio versus t for the beam center. Figs. 2 and 3 represent a comparison of analytical solution of $W(t)$ based on time with the

Table 1 Comparison of nonlinear to linear frequency ratio (ω_{NL}/ω_L) for Simply-Supported Beams

δ	Δ	β	Present Study (HA)	Azrar[28]	Lewandowski[29]
1	0.2886	3	1.0897	1.0891	1.0897
2	0.5773	3	1.3228	1.3177	1.3229
3	0.8660	3	1.6393	1.6256	1.6394
4	1.1547	3	2	-	1.9999

Table 2 Comparison of nonlinear to linear frequency ratio (ω_{NL} / ω_L) for Clamped-Clamped Beams

δ	$w_1^*(1/2)$	Δ	β	Present Study (HA)	Azrar[28]	Lewandowski[29]
1	1.58815	0.18177	1.81421	1.0222	1.0222	1.0222
1.5	1.58815	0.27265	1.81421	1.0494	1.0492	1.0492
2	1.58815	0.36354	1.81421	1.0862	1.0857	1.0858
2.5	1.58815	0.45442	1.81421	1.1318	1.1307	1.1308
3	1.58815	0.54531	1.81421	1.1852	1.1831	1.1832
3.5	1.58815	0.63619	1.81421	1.2453	1.2420	1.2422
4	1.58815	0.72707	1.81421	1.3112	1.3064	1.3063
4.5	1.58815	0.81796	1.81421	1.3822	1.3756	1.3751

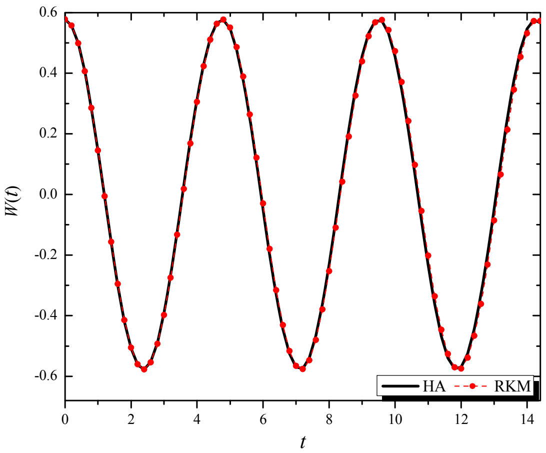

Fig. 2 Comparison of analytical solution of $W(t)$ based on time with the RKM solution for simply supported beam, $\delta = 2$, $\beta = 3$

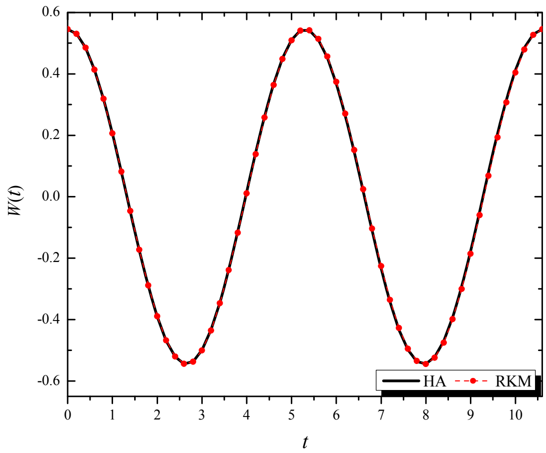

Fig. 3 Comparison of analytical solution of $W(t)$ based on time with the RKM solution for clamped-clamped beam, $\delta = 3$, $\beta = 1.81421$

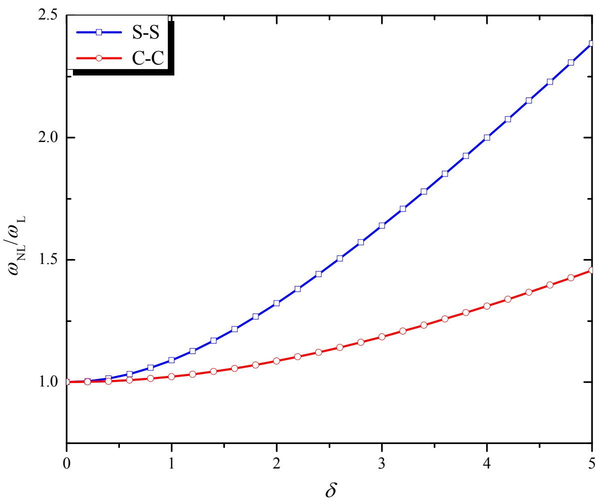

Fig. 4 Nonlinear to linear frequency ratio versus non-dimensional amplitude ratio

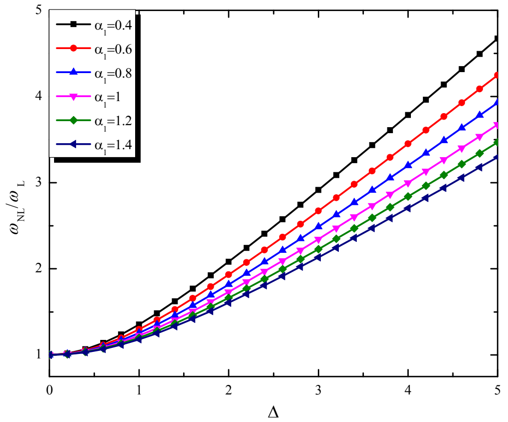

Fig. 5 Influence of α_1 on nonlinear to linear frequency base on Δ for $\alpha_2 = 0.5$, $\alpha_3 = 1$, $p = 1$

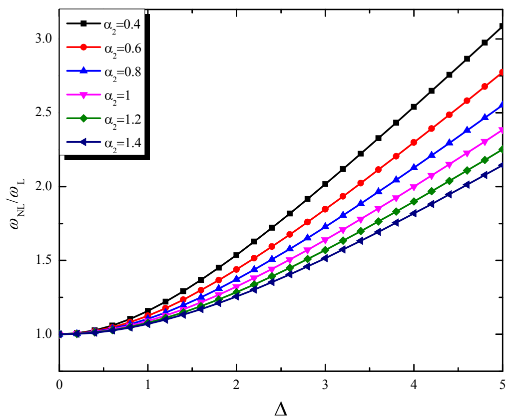

Fig. 6 Influence of α_2 on nonlinear to linear frequency base on Δ for $\alpha_1 = 1$, $\alpha_2 = 1$, $p = 3$

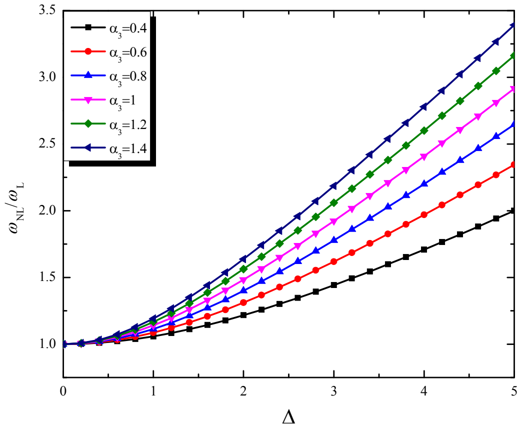

Fig. 7 Influence of α_3 on nonlinear to linear frequency base on Δ for $\alpha_1 = 1$, $\alpha_2 = 0.5$, $p = 3$

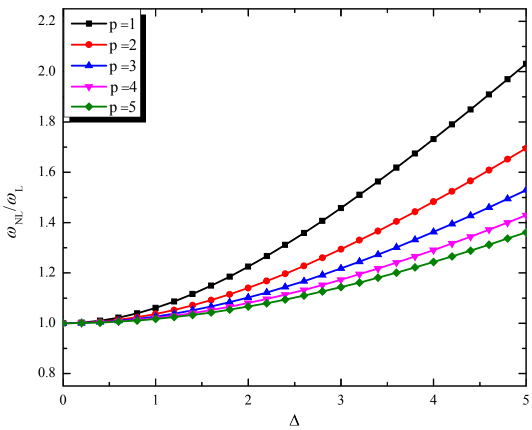

Fig. 8 Influence of p on nonlinear to linear frequency base on Δ for $\alpha_1 = 1$, $\alpha_2 = 2$, $p = 0.5$

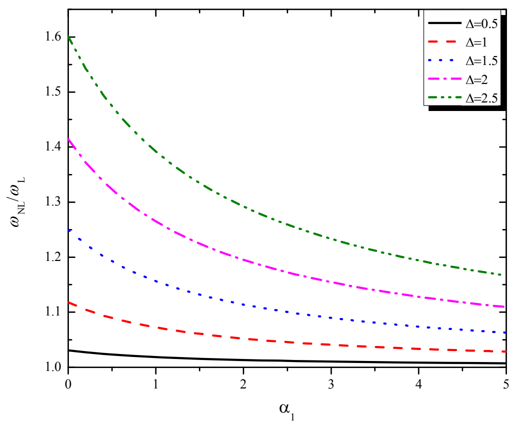

Fig. 9 Influence of Δ on nonlinear to linear frequency base on α_1 for $\alpha_2 = 0.5$, $\alpha_3 = 0.5$, $p = 3$

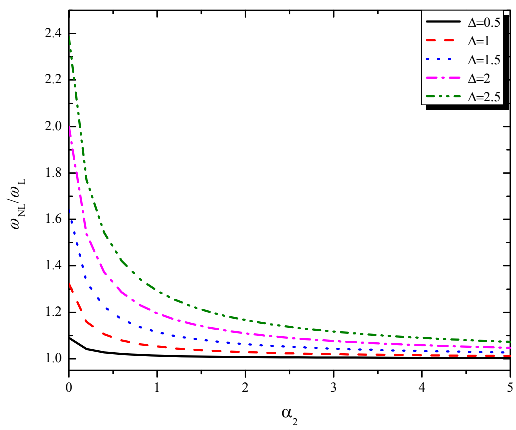

Fig. 10 Influence of Δ on nonlinear to linear frequency base on α_2 for $\alpha_1 = 0.5$, $\alpha_3 = 0.5$, $p = 3$

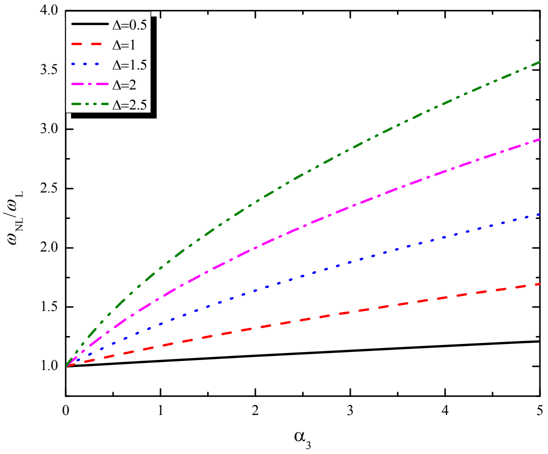

Fig. 11 Influence of Δ on nonlinear to linear frequency base on α_3 for α_1 = 0.5, α_2 = 0.5, p = 3

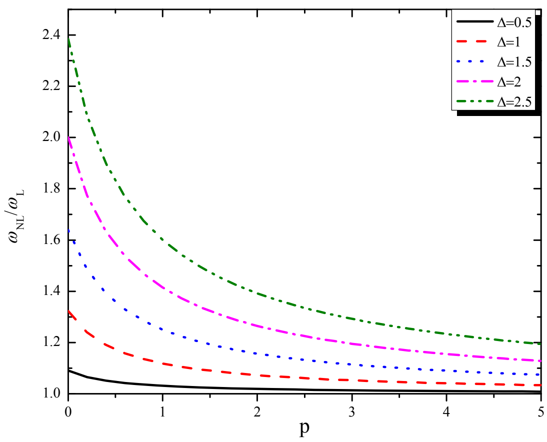

Fig. 12 Influence of Δ on nonlinear to linear frequency base on p for α_1 = 0.5, α_2 = 1, α_3 = 0.5

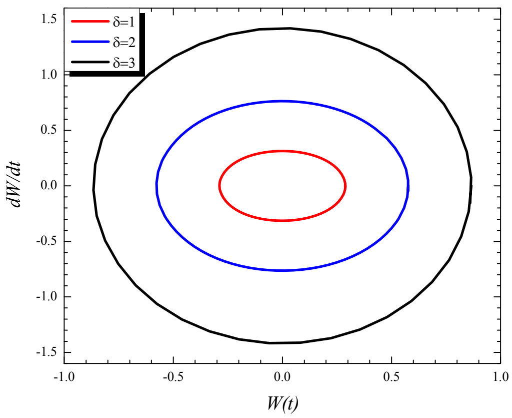

Fig. 13 The phase plane of simply supported beam for different variations of amplitude

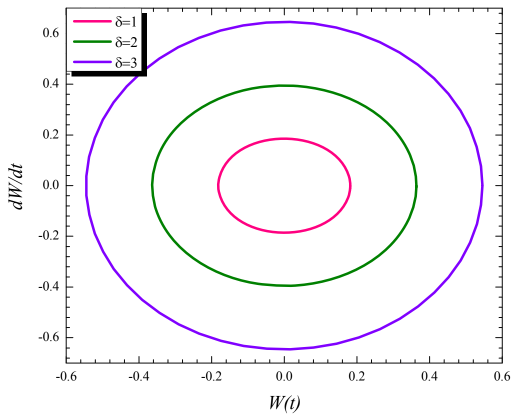

Fig. 14 The phase plane of clamped-clamped beam for different variations of amplitude

numerical solution. From Figs. 2-3, the motion of the system is a periodic motion and the amplitude of vibration is a function of the initial conditions. In clamped beams the eigenmodes of them involve hyperbolic component and simply supported beams have only sinusoidal component in their eigenmodes, in this case the HA provides more accurate solution as it is indicated in Fig. 4. Figs. 5 to 7 show the effect of α_1, α_2, α_3 on the ratio of non-linear to linear frequency versus non-dimensional amplitude ratio. For further illustration, the variation of the buckling load parameter (P) versus the non-dimensional amplitude ratio is shown in Fig. 8. The Influence of Δ on nonlinear to linear frequency base on α_1, α_2, α_3 and p show in Figs. 9 to 12. In small amplitudes the rate of increase in non-linear fundamental frequency is too low with increasing displacement. The effect of non-linearity becomes more obvious when the maximum amplitude increases. The phase plans of the problem for different variations of amplitude are also considered in Figs. 13 and 14.

7. Conclusions

In this paper, nonlinear responses of the Simply-Supported and the Clamped-Clamped buckled Euler-Bernoulli beams fixed at one end are investigated mathematically. The Galerkin method was used for discretization, the governing non-linear partial differential equation to a single non-linear ordinary differential equation. The Hamiltonian Approach (HA) has been successfully applied to obtain an accurate analytical solution for the non-linear vibration of axially loaded Euler-Bernoulli beams. The results and errors of the method are compared with Runge-Kutta 4th order and the other researchers results. It has been indicated that HA is very powerful mathematical tool for providing an accurate analytical solutions. The HA solutions are quickly convergent and its components can be simply calculated .This method contrary to the perturbation method does not need small parameters and are applicable for whole range of parameters.

References

Azrar, L., Benamar, R. and White, R.G. (1999), "A semi-analytical approach to the non-linear dynamic response problem of S–S and C–C beams at large vibration amplitudes part I: General theory and application to the single mode approach to free and forced vibration analysis", *J. Sound Vib.*, **224**(2), 183-207.

Bayat, M., Pakar, I. and Shahidi, M. (2011), "Analysis of nonlinear vibration of coupled systems with cubic nonlinearity", *Mechanika*, **17**(6), 620-629.

Bayat, M., Barari, A. and Shahidi, M. (2011), "Dynamic response of axially loaded Euler-Bernoulli beams", *Mechanika*, **17**(2), 172-177.

Bayat, M., Bayat, M., and Bayat, M. (2011) "An analytical approach on a mass grounded by linear and nonlinear springs in series", *Int. J. Phy. Sci.*, **6**(2), 229-236.

Bayat, M., Pakar, I., and M, Bayat. (2011a) "Analytical study on the vibration frequencies of tapered beams", *Latin American J. Solids Struct.*, **8**(2), 149-162.

Bayat, M. and Pakar, I. (2011b), "Nonlinear free vibration analysis of tapered beams by Hamiltonian Approach", *J. Vibroengineering.*, **13**(4), 654-661.

Bayat, M. and Pakar, I. (2011c), "Application of He's energy balance method for nonlinear vibration of thin circular sector cylinder", *Int. J. Phy. Sci.*, **6**(23), 5564-5570.

Pakar, I. and Bayat, M. (2011d) "Analytical solution for strongly nonlinear oscillation systems using energy balance method", *Int. J. Phy. Sci.*, **6**(22), 5166- 5170.

Bayat, M. and Pakar, I. (2012a), "Analytical study on the non-linear vibration of Euler-Bernoulli beams", *J. of Vibroengineering*, **14**(1), 216-224.

Pakar, I., Bayat, M. and Bayat, M. (2012b), "On the approximate analytical solution for parametrically excited nonlinear oscillators", *J. Vibroengineering*, **14**(1), 423-429.

Bayat, M. and Pakar, I. (2012c), "On the approximate analytical solution to non-linear oscillation systems", *Shock and Vibration*, DOI: 10.3233/SAV-2012-0726.

Bayat, M., Shahidi, M., Barari, A. and Domairry, G. (2010), "The approximate analysis of nonlinear behavior of structure under harmonic loading", *Int. J. Phy. Sci.*, **5**(7), 1074-1080.

Bayat, M., Shahidi, M., Barari, A. and Domairry, G. (2011) "Analytical evaluation of the nonlinear vibration of coupled oscillator systems", *Zeitschrift fur Naturforschung Section A-A Journal of Physical Sciences*, **66**(1-2), 67-74.

Bayat, M., Shahidi, M. and Bayat, M. (2011), "Application of iteration perturbation method for nonlinear oscillators with discontinuities", *Int. J. Phy. Sci.*, **6**(15), 3608-3612.

Bayat, M., Pakar, I. and Domaiirry, G. (2012), "Recent developments of some asymptotic methods and their applications for nonlinear vibration equations in engineering problems: A review", *Latin American J.*

Solids and Struct., **9**(2), 145-234.

Biondi, B. and Caddemi, S. (2005), "Closed form solutions of Euler-Bernoulli beams with singularities", *Inter. J. Solids and Struct*., **42**(9-10), 3027-3044.

Ganji, D.D., Rafei, M., Sadighi, A. and Ganji, Z.Z. (2009), "A comparative comparison of He's method with perturbation and numerical methods for nonlinear vibrations equations," *Inter. J. Nonlinear Dyn. in Eng. Sci*., **1**(1), 1-20.

Ghasemi, E., Bayat, M. and Bayat, M. (2011), "Visco-elastic MHD flow of walters liquid B fluid and heat transfer over a non-isothermal stretching sheet", *Int. J. Phy. Sci*., **6**(21), 5022-5039.

He, J.H. (2007), "Variational Approach for nonlinear oscillators", *Chaos. Soliton. Fractals*., **34**(5), 1430-1439.

He, J.H. (2010), "Hamiltonian Approach to nonlinear oscillators", *Physics Letters A*., **374**(23), 2312-2314.

He, J.H. (2002), "Preliminary report on the energy balance for nonlinear oscillations", *Mech. Res. Communications*., **29**(2-3), 107-111.

He, J.H. (2008), "An improved amplitude-frequency formulation for nonlinear oscillators", *Inter. J. of Nonlinear Sci. and Numerical Simulation*., **9**(2), 211-212.

Lai, H.Y., Hsu, J.C. and Chen, C.K. (2008), "An innovative eigenvalue problem solver for free vibration of Euler-Bernoulli beam by using the Adomian decomposition method", *Comput. Math. Appl*., **56**(12), 3204-3220.

Xu, L. and Zhang, N. (2009), "A variational approach next term to analyzing catalytic reactions in short monoliths", *Comput. Math. Appl*., **58**(11-12), 2460-2463.

Lewandowski, R. (1987), "Application of the Ritz method to the analysis of nonlinear free vibrations of beams", *J. Sound and Vib*., **114**(1), 91-101.

Liu, Y. and Gurram, S.C. (2009), "The use of He's variational iteration method for obtaining the free vibration of an Euler-Bernoulli beam", *Math. Comput. Modelling*, **50**(11-12), 1545-1552.

Naguleswaran, S. (2003), "Vibration and stability of an Euler-Bernoulli beam with up to three-step changes in cross-section and in axial force", *Inter. J. Mech. Sci*., **45**(9),1563-1579.

Padovan, J. (1980), "Nonlinear vibrations of general structures", *J. Sound Vib*., **72**, 427-441.

Pirbodaghi, T., Ahmadian, M.T. and Fesanghary, M. (2009), "On the homotopy analysis method for non-linear vibration of beams", *Mech. Res. Communications*, **36**(2), 143-148.

Sathyamoorthy, M. (1982), "Nonlinear analysis of beams, Part-I: A survey of recent advances", *Shock Vib. Dig*., **14**, 19-35.

Shahidi, M., Bayat, M., Pakar, I. and Abdollahzadeh, G.R. (2011), "On the solution of free non-linear vibration of beams", *Int. J. Phy. Sci*., **6**(7), 1628-1634.

Shen, Y.Y. and Mo, L.F. (2009), "The max-min approach to a relativistic equation", *Comput. Math. Appl*., **58**, 2131-2133.

Soleimani, Kutanaei, S., Ghasemi, E. and Bayat, M. (2011), "Mesh-free modeling of two-dimensional heat conduction between eccentric circular cylinders", *Int. J. Phy. Sci*., **6**(16), 4044-4052.

Tse, F.S., Morse, I.E. and Hinkle, R.T. (1987), *Mechanical Vibrations: Theory and Applications*, 2nd Edition, Allyn and Bacon Inc., Bosto.

		EA	axial rigidity of the beam cross section
A	cross-sectional area	EI	bending rigidity of the beam cross section
L	beam length	$\phi(x)$	trial function
w'	normal displacement	$W(t)$	time-dependent deflection parameter
E	Young's modulus	Δ	dimensionless maximum amplitude of oscillation
X	axial coordinate	δ	maximum amplitude parameter of beam
$\overline{E}$	axial load	β	parameter of boundary condition of beam
m	mass per unit length	ω_{NL}	nonlinear frequency
t	time	ω_L	linear frequency