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An analysis of an elastic solid incorporating a crack under the influences of surface effects in plane & anti-plane deformations

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Abstract. We review a series of crack problems arising in the general deformations of a linearly elastic solid (Mode-I, Mode-II and Mode-III crack) and, perhaps more significantly, when the contribution of surface effects are taken into account. The surface mechanics are incorporated using the continuum based surface/interface model of Gurtin and Murdoch. We show that the deformations of an elastic solid containing a single crack can be decoupled into in-plane (Mode-I and Mode-II crack) and anti-plane (Mode-III crack) parts, even when the surface mechanics is introduced. In particular, it is shown that, in contrast to classical fracture mechanics (where surface effects are neglected), the incorporation of surface elasticity leads to the more accurate description of a finite stress at the crack tip. In addition, the corresponding stress fields exhibit strong dependency on the size of crack.

Keywords: surface elasticity; mode-I, II, III; plane deformations; anti-plane deformations; crack tip analysis; complete exact solution; Cauchy singular integro-differential equations.

1. Introduction

The analysis of stresses in the general region of a crack tip is of fundamental importance in the understanding of failure and in the general deformation analysis of engineering materials. In macroscopic models, the stresses at the crack tip are found to be infinite (England 1971, Sih 1965) reflecting the fact that the crack front is usually taken to be perfectly sharp. In fact, an infinitely sharp crack in a continuum is a mathematical abstraction since, in reality, most crack tips are, in fact, blunt, with a radius of convergence of the order compatible with the nanoscale. This suggests that a more accurate analysis of the region in the vicinity of a crack tip can be achieved at the nanoscale. In the context of a continuum model this means the incorporation of surface effects into the model of deformation.

One of the most important and accessible continuum models incorporates the effects of surface mechanics using the surface elasticity model of Gurtin and Murdoch (Gurtin and Murdoch 1975, Gurtin *et al*. 1998). In this model, a surface is regarded as a thin elastic membrane perfectly bonded to the bulk solid. The additional surface stress contributed by the surface mechanics leads to highly

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unusual and nonstandard boundary conditions on the surface of the bulk solid. Consequently, the corresponding boundary value problems are not accommodated by existing classical theories and pose challenges not encountered previously in similar mathematical analyses. Nonetheless, the Gurtin-Murdoch assumptions have been used successfully in a number of studies in nano-composite mechanics (see, for example, Tian and Rajapakse (2007), Sharma and Ganti (2004), Duan *et al.* (2005), Cammarata (1997), Miller and Sheny (2000)).

In this summarized paper, we have re-addressed results first presented in Kim *et al.* (2009), Kim *et al.* (2010a), Kim *et al.* (2010b), since some of details and concepts in there are not sufficiently clear or heavily omitted which may hinder general readers from understanding of the material. Especially, we demonstrate that the deformations of an elastic solid containing a single crack can be decoupled into in-plane and anti-plane parts, even when the surface mechanics is incorporated. It is also shown that, in the case where in-plane deformations are considered (Mode-I and Mode-II crack), the surface stress σ_o gives rise to residual stresses in which case the corresponding stress field can no longer be zero even when all external loads are removed. Throughout the analysis, we maintain the assumption of a sharp crack and demonstrate how the incorporation of surface elasticity eliminates the stress singularity at the (infinitely sharp) crack tip. Surface effects are integrated using the Gurtin-Murdoch surface elasticity model with the crack occupying a finite region of the x-axis. In results, it is shown that, in contrast to classical fracture mechanics (where surface effects are neglected), the incorporation of surface elasticity leads to the more accurate description of a finite stress at the crack tip. We also demonstrate that the corresponding stress distributions derived from our analysis show clear signs of size dependency and do indeed reduce to classical solutions (England 1971, Sih 1965 and Muskhelishvili 1953), when the surface effects approach zero.

Throughout the paper, we make use of a number of well-established symbols and conventions. Thus, unless otherwise stated, Greek and Latin subscripts take the values 1, 3 and 1, 2, 3 respectively, summation over repeated subscripts is understood, (x, z) and (x, y, z) are generic points in the (x, z)-plane and $\mathbf{R}^3$, respectively. Finally, we note that the notation (x, z) and (x, y, z) may also be replaced by (x_1, x_3) and (x_1, x_2, x_3), respectively, when the reference is made to $\{\boldsymbol{e}_i\}_{i=1}^{3}$, the standard basis for $\mathbf{R}^3$.

2. Surface equation

It is well-known that in the absence of body forces, the equilibrium and constitutive relations describing the deformation of a linearly elastic, homogeneous and isotropic (bulk) solid are given by

$$\sigma^B_{ij,j} = 0, \sigma^B_{ij} = \lambda\delta_{ij}\varepsilon_{ij} + 2\mu\varepsilon_{ij}, \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1}$$

where λ and μ are the Lame constants of the material σ^B_{ij} and ε_{ij} the components of the stress (bulk) and strain tensors, respectively and u_i denotes the i^{th} component of the displacement vector $\boldsymbol{u} = u\boldsymbol{n}_1 + v\boldsymbol{n}_2 + w\boldsymbol{n}_3$ in $\mathbf{R}^3$. In addition, $()_{,j}$ denotes differentiation with respect to x_j and δ_{ij} are the kronecker delta. Although Eq. (1) remains true in the bulk material, equilibrium and constitutive relations on the surface are now described by the equations (see Gurtin and Murdoch 1975, Gurtin *et al.* 1998 and Ru 2010) for detailed derivations)

$$(div_s\sigma^s)_\alpha = \sigma^B_{\alpha j}n_j - f_\alpha \tag{2}$$

$$k_{\alpha\beta}\sigma^{s}_{\alpha\beta} - \sigma_o[v(k^2_{11} + k^2_{33}) - \nabla^2_s v] = \sigma^{B}_{ij}n_i n_j - f_\alpha \tag{3}$$

$$\sigma^{s}_{\alpha\beta} = \sigma_o\delta_{\alpha\beta} + 2(\mu^s - \sigma_o)\varepsilon^{s}_{\alpha\beta} + (\lambda^s + \sigma_o)\varepsilon^{s}_{\gamma\gamma}\delta_{\alpha\beta} + \sigma_o\nabla_s \boldsymbol{v} \tag{4}$$

Here the index "*s*" denotes the corresponding quantity resulting from the effects of surface elasticity, div_s is surface divergence, f_α is applied traction on the surface and n_i denotes the i^{th} component of the unit normal vector $\boldsymbol{n} = n_i\boldsymbol{e_i}$ to the surface, where $\{\boldsymbol{e_i}\}^3_{i=1}$, the standard basis for **R³**. The mean curvature of the surface is characterized by $k_{\alpha\beta}$ and $\nabla_s v$ is the surface Laplacian of normal deflection *v*, when *v* is a component of a vector $v\boldsymbol{n}$ normal to the surface. Finally, σ_o is the surface tension.

Remark 1.

It should be noted here that the expressions of constitutive relations for the surface stresses depend on the choice of different mathematical and physical assumptions describing the general behavior of the surface. More precisely, the surface stresses ($\boldsymbol{\sigma}^s$) can be determined by the relations

$$\sigma^{s}_{\alpha\beta} = \sigma_o\delta_{\alpha\beta} + \frac{\partial\Gamma}{\partial\varepsilon^{s}_{\alpha\beta}}$$

where Γ is the surface energy defined on the initial (Lagrangian) surface area. Currently, a number theories with different versions are available in literatures (see, for example, Gurtin and Murdoch 1975, Gurtin *et al.* 1998 and Ru 2010, Ogden *et al.* 1997) within this subject. In the presentment manuscript, we have adopted the following expression for the surface energy originally suggested in (Ru 2010)

$$\Gamma = \sigma_o(1 + \varepsilon^{s}_{\alpha\alpha}) + \frac{1}{2}\lambda_o(\varepsilon^{s}_{\alpha\alpha})^2 + \mu_o(\varepsilon^{s}_{\alpha\beta}\varepsilon^{s}_{\alpha\beta}) + \sigma_o\frac{1}{2}(\nabla_s v\boldsymbol{n})\cdot(\nabla_s v\boldsymbol{n})$$

The above expression is motivated by the well-known von Karman's large-deflection theory of elastic thin shells that the normal deflection (out-plane component) $u_2 = v$ is dominated over other two tangential (in-plane) displacement components ($u_1 = u, u_3 = w$) of the surface, especially when the initial curvatures vanishes (see more detailed descriptions in (Ru 2010)).

2.1 Equilibrium conditions on the crack surface: decomposition theory

We consider deformations of a linearly elastic and homogeneous isotropic solid occupying a region **R³** with generators parallel to the *z*-axis of a rectangular Cartesian coordinate system. We assume that a cross-section of the crack occupies the region [$-a, a$], $a \in \mathbf{R}$ of the *x*-axis as shown in the Fig. 1. Within the present setting, the Eqs. (2)-(3) can now be re-written as

$$\sigma^{s}_{xx,x} + \sigma^{s}_{xz,z} + [\sigma^{B}_{xy}] = 0 \tag{5}$$

$$\sigma^{s}_{zx,x} + \sigma^{s}_{zz,z} + [\sigma^{B}_{yz}] = 0 \tag{6}$$

$$[\sigma^{B}_{yy}] = -\sigma_o\frac{\partial^2 v}{\partial x^2} - \sigma_o\frac{\partial^2 v}{\partial z^2} \tag{7}$$

$\because f_a = 0(for\ traction-free\ boundary\ condition)$, $k_{xx} = k_{zz} = 0(for\ flat\ crack\ face)$

Here the unit normal vector $\boldsymbol{n}$ is defined in such a way that it points from the "–" side to "+" side (see Fig. 1) and ∇_s^2 is a surface gradient defined as $\nabla_s^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$. Finally, $[*] = (*)^{in} - (*)^{out}$ denotes the jump of the corresponding quantity "*" across a surface (here "*in*" and "*out*" refer, respectively, to the inside and outside of the body). Then, it can be readily verified that the above Eqs. (5)-(7) for three dimensional solid can be decomposed into two independent parts (i.e. in-plane components (u, v) and out-plane component (w)). In the case where an isotropic elastic medium undergoes in-plane deformations, the displacement vector $\boldsymbol{u}$ with component (u, v, w) admits

$$u = u(x, y), v = v(x, y), w = 0 \tag{8}$$

and for the anti-plane shear deformation case, the displacement vector $\boldsymbol{u}$ now satisfies

$$w = w(x, y), u = v = 0, \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \tag{9}$$

Apply Eqs. (8)-(9) on Eqs. (5)-(7) independently, we obtain the following equilibrium conditions on the crack surface for the in-plane deformation case (Mode-I & Mode-II crack)

$$\sigma^s_{xx,x} + [\sigma^B_{xy}] = 0 \tag{10}$$

$$[\sigma^B_{yy}] = -\sigma_o \frac{\partial^2 v}{\partial x^2} \tag{11}$$

and for the anti-plane deformation case (Mode-III crack)

$$\sigma^s_{zx,x} + [\sigma^B_{yz}] = 0 \tag{12}$$

Clearly, the deformations of an elastic solid with a single crack can be decoupled into in-plane and anti-plane parts, even when the surface mechanics is incorporated.

3. Crack problem with surface elasticity

In the above section, we have shown that the corresponding crack problem can be decomposed into two parts (in-plane and anti-plane deformations) under the condition that the displacement vector $\boldsymbol{u}$ is independent of variable "z" (either plane-strain or plane-stress assumption) which is most commonly adopted treatment in solving engineering problems of this kind. We now examine each problem and see how the surface effects alter the original stress field.

3.1 A traction-free mode-III crack problem with surface effects

Again in anti-plane shear deformation, the displacement vector $\boldsymbol{u}$ with components (u, v, w) satisfies

$$u = v = 0, w = w(x, y), \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

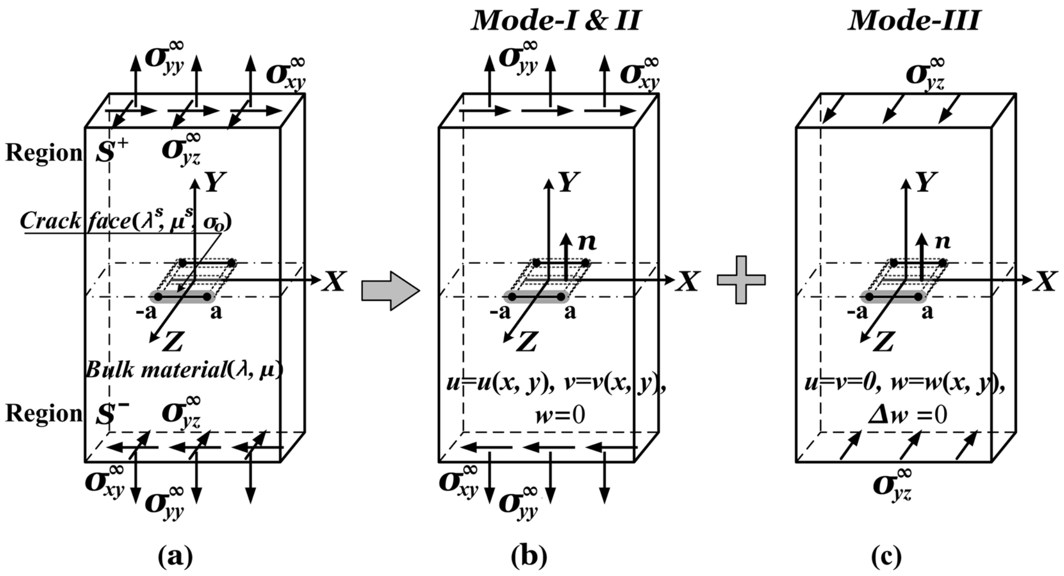


Fig. 1 Schematics of the problem

Then from Eq. (1), the strain components are now given by

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}\frac{\partial w}{\partial x},\ \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\frac{\partial w}{\partial y} \tag{13}$$

$$\varepsilon_{xy} = \varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = 0$$

From Eq. (13), the stress components can be written as

$$\sigma_{xz} = 2\mu\varepsilon_{xz} = \mu\frac{\partial w}{\partial x},\ \sigma_{yz} = 2\mu\varepsilon_{yz} = \mu\frac{\partial w}{\partial y}$$

$$\sigma_{xy} = \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0 \tag{14}$$

In addition, the relations between surface and body (bulk) stresses can be derived from Eq. (4) with successive use of Eqs. (13)-(14) as

$$\sigma_{xz}^s = 2(\mu^s - \sigma_o)\varepsilon_{xz} = \frac{\mu^s - \sigma_o}{\mu}\sigma_{xz},\ \sigma_{yz}^s = 2(\mu^s - \sigma_o)\varepsilon_{yz} = \frac{\mu^s - \sigma_o}{\mu}\sigma_{yz} \tag{15}$$

It should be noted here that for a cohere interface, the interfacial strains are equal to those in the adjoined bulk material, i.e. $\varepsilon_{xz}^s = \varepsilon_{xz}$ and $\varepsilon_{yz}^s = \varepsilon_{yz}$.

Since $w(x, y)$ is a harmonic function, we denote by $\psi(x, y)$ its conjugate harmonic function. Introducing the complex variable $z = x + iy$, we can now write

$$w = \mathrm{Re}[\Omega(z)],\ \Omega(z) = w(x, y) + i\psi(x, y) \tag{16}$$

where Ω(z) is an analytic function of z in the plane $S^+ \cup S^- = S$ outside the crack (see Fig. 1.). From Eq. (16), we then have that

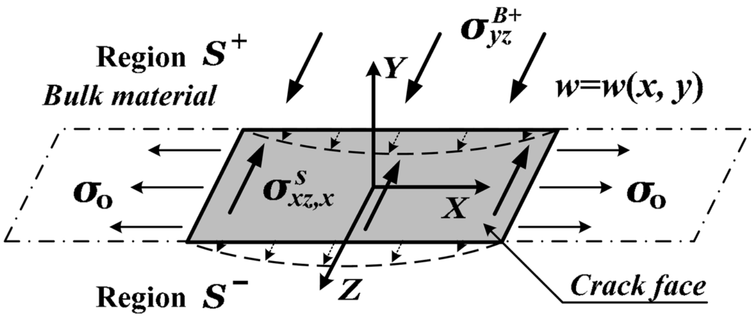


Fig. 2 Equilibrium on upper crack face under anti-plane shear motion

$$\frac{d\Omega(z)}{dz} = \Omega'(z) = \frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y} = \frac{1}{\mu}(\sigma_{xz} - i\sigma_{yz}) \tag{17}$$

and

$$\sigma_{yz} = \frac{\mu i}{2}[\Omega'(z) - \overline{\Omega'(z)}],\ \sigma_{xz} = \frac{\mu}{2}[\Omega'(z) + \overline{\Omega'(z)}] \tag{18}$$

Now, in view of Eq. (12), the boundary conditions on the upper and bottom crack faces are given by (see Fig. 2.)

$$\frac{\partial\sigma_{zx}^{s}}{\partial x} + (\sigma_{yz}^{B})^{+} = 0 \text{ on the upper face} \tag{19}$$

$$\frac{\partial\sigma_{zx}^{s}}{\partial x} - (\sigma_{yz}^{B})^{-} = 0 \text{ on the bottom face} \tag{20}$$

where, subscripts "+" and "–" denotes the upper ($y > 0$) and lower ($y < 0$) side of the crack, respectively.

It is clear from Eqs. (19)-(20) that surface stress σ_o does not contribute the boundary condition for Mode-III crack case. From Eqs. (15) and (18)-(20), the surface condition on either side of the crack [$-a < x < a$], ($y = \pm 0$) can be formulated as follows

$$\frac{\mu i}{2}[\Omega'(z) - \overline{\Omega'(z)}]^{+} = -\frac{\mu^{s} - \sigma_o}{2}[\Omega''(z) + \overline{\Omega''(z)}]^{+} \text{ on the upper face} \tag{21}$$

$$\frac{\mu i}{2}[\Omega'(z) - \overline{\Omega'(z)}]^{-} = \frac{\mu^{s} - \sigma_o}{2}[\Omega''(z) + \overline{\Omega''(z)}]^{-} \text{ on the lower face.} \tag{22}$$

In anti-plane deformations (Mode-III crack), it is clear that $w = -w^{-}$ (the right side of Eqs. (21)-(22)). Therefore, adding Eqs. (21) and (22) yields

$$\frac{\mu i}{2}([\Omega'(z) - \bar{\Omega}'(z)]^{+} + [\Omega'(z) - \bar{\Omega}'(z)]^{-}) = -(\mu^{s} - \sigma_o)(\Omega''(z)^{-} - \bar{\Omega}'(z)^{+}) \tag{23}$$

$$\because \overline{\Omega'(z)}^{+} = \bar{\Omega}'(z)^{-} \text{ on } y = \pm 0$$

Since we have assumed uniform remote stress $\sigma_{yz}^{\infty} = \sigma_{yz}$, we necessarily have that

$$\Omega'(z) + \overline{\Omega'}(z) = 0, \Omega'(z) = -\overline{\Omega'}(z) \tag{24}$$

Consequently, from Eqs. (23)-(24), we derive the following Hilbert-problem in terms of the derivatives of the unknown function Ω(z) defined by Eq. (15) as

$$\mu i(\Omega'(z)^{+} + \Omega'(z)^{-}) = (\mu^{s} - \sigma_{o})(\Omega''(z)^{-} - \Omega''(z)^{+}) \tag{25}$$

Next, if we express the unknown $\Omega'(z)$ as a Cauchy integral (Muskhelishvili 1953), noting the requirement that the stresses be bounded at the crack tips, we have that

$$\Omega'(z) = \frac{1}{2i\pi}\int_{-a}^{+a}\frac{f(t)}{t-z}dt + \frac{1}{\mu i}[\sigma_{yz}^{\infty}] \tag{26}$$

$$\Omega''(z) = \frac{1}{2i\pi}\int_{-a}^{+a}\frac{f(t)}{(t-z)^2}dt = -\left[\frac{f(t)}{t-z}\right]_{-a}^{+a} + \frac{1}{2i\pi}\int\frac{f'(t)}{t-z}dt = \frac{1}{2i\pi}\int\frac{f'(t)}{t-z}dt \tag{27}$$

where,

$$f(t) = \Omega'(z)^{+} - \Omega'(z)^{-}, \; f(a) = f(-a) = 0$$

Finally, from Eqs. (25)-(27), we obtain the following first-order Cauchy singular integro-differential equation for the unknown $f(t), t \in [a, -a]$

$$\frac{\mu}{\pi}\int_{-a}^{+a}\frac{f(t)}{t-t_o}dt + 2[\sigma_{yz}^{\infty}] = -(\mu^{s} - \sigma_o)f'(t_o), \quad -a < t_o < a$$

$$f(a) = f(-a) = 0 \tag{28}$$

The solution of the above equation can be numerically found via Chebychev polynomials and the collocation method (see more details in Kim *et al*. (2009), Kim *et al*. (2010a), Kim *et al*. (2010b) and Chakrabarti (1999)). In addition, the numerical method guarantees rapid convergence (see Fig. 3).

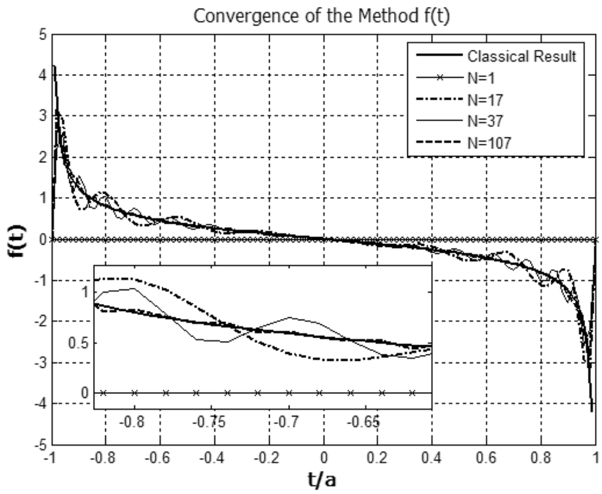


Fig. 3 Convergence of the method with respect to number of iteration (N)

3.2 A traction-free plane-strain crack problem (Mode-I & Mode-II) with surface effects

For the in-plane deformations, the displacement vector $\boldsymbol{u}$ with component u, v, w satisfies (see Eq. (8))

$$u = u(x, y), v = v(x, y), w = 0$$

In the absence of body forces, the corresponding governing equations of two-dimensional elasticity are described by (England 1971)

$$2\mu(u + iv) = \kappa\Omega(z) - z\overline{\Omega'(z)} - \overline{\omega(z)} \tag{29}$$

$$\sigma_{yy} - i\sigma_{xy} = \Omega'(z) + \overline{\Omega'(z)} + z\overline{\Omega''(z)} + \overline{\omega'(z)} \tag{30}$$

Here $\Omega(z)$ and $\omega(z)$ are analytic functions of the complex variable $z = x + iy$ in the cut plane $S^+ \cup S^- = S$ outside the crack (see. Fig. 1) and κ is defined as

$$\kappa = \frac{\lambda + 3\mu}{\lambda + \mu} = 3 - 4\nu \text{ (for plane-strain)}$$

where ν is Poisson's ratio taking values in the range $0 < \nu < 1/2$. Since the displacements and stresses are continuous across $y = 0,\ x > |a|$ (outside the crack), from Eq. (30), following (England 1971), we can define an analytic function also analytic in the whole plane cut along $L = -a < x < a$ by

$$\Omega(z) - z\overline{\Omega'(\bar{z})} - \overline{\omega'(\bar{z})} = \theta(z) \tag{31}$$

Therefore, Eqs. (29)-(30) can be re-written with the aid of Eq. (31) as

$$2\mu(u + iv) = \kappa\Omega(z) - \Omega(\bar{z}) - (\bar{z} - z)\overline{\Omega'(z)} + \theta(\bar{z}) \tag{32}$$

$$\sigma_{yy} - i\sigma_{xy} = \Omega'(z) + \Omega'(\bar{z}) + (z - \bar{z})\overline{\Omega''(z)} - \theta(\bar{z}) \tag{33}$$

From Eqs. (10) and (11), we obtain

$$[\sigma_{yy}^B - i\sigma_{xy}^B] = i\sigma_{xx,x}^s - \sigma_o\frac{\partial^2 v}{\partial x^2} \tag{34}$$

Under the assumption $\varepsilon_{xx}^s = \varepsilon_{xx}$ (coherent interface), Eq. (4) can now be re-written with the use of Eqs. (1) and (8) as

$$\sigma_{xx}^s = \sigma_o + 2\mu(\mu^s - \sigma_o)\varepsilon_{xx} + (\lambda^s + \sigma_o)\varepsilon_{xx}$$

In addition, with the aid of Eq. (1), the surface stress now can be expressed explicitly in terms of displacements (bulk)

$$\sigma_{xx,x}^s = \frac{\partial\left(\sigma_o + (2\mu^s - \sigma_o + \lambda^s)\frac{\partial u}{\partial x}\right)}{\partial x} = 2(\mu^s - \sigma_o + \lambda^s)\frac{\partial^2 u}{\partial x^2} \tag{35}$$

Finally, by substituting Eq. (35) into Eq. (34) and applying Eq. (33), we derive the equilibrium conditions on the upper and bottom crack faces as (see Fig. 4.)

$$\Omega'(z)^+ + \Omega'(z)^- - \theta'(z)^- = (\sigma_{yy}^B - i\sigma_{xy}^B)^+ = iJ_o\frac{\partial^2 u}{\partial x^2} - \sigma_o\frac{\partial^2 v}{\partial x^2}, \text{ on the upper face} \tag{36}$$

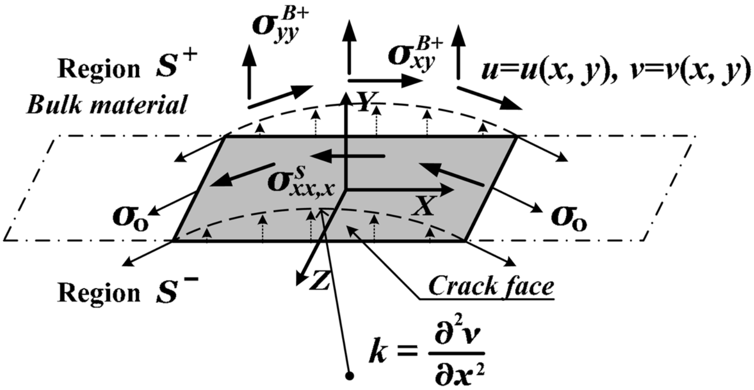


Fig. 4 Equilibrium on upper crack face under in-plane deformations

$$\Omega'(z)^- + \Omega'(z)^+ - \theta'(z)^+ = (\sigma_{yy}^B - i\sigma_{xy}^B)^- = -iJ_o\frac{\partial^2 u}{\partial x^2} + \sigma_o\frac{\partial^2 v}{\partial x^2}, \text{ on the bottom face} \tag{37}$$

where, $J_o \equiv 2\mu^s - \sigma_o + \lambda^s$ and $\Omega'(\bar{z})^+ = \Omega'(z)^-, \bar{z} = z$ on $y = \pm 0$.

Remark 2.

It is obvious from Eqs. (36)-(37) and Fig. 4. that, different from the Mode-III crack case, surface stress σ_o does indeed alter the equilibrium condition. Therefore, the residual stress is induced by the non-zero surface stress σ_o, and the corresponding stress field (residual) can no longer be zero even when all external loads are removed. The result further suggests that the complete solution of the present problem is actually composed of two separate stress fields (the residual stress induced by surface tension and another induced by the external loading). In this study, we limit our attention only to the stress field induced by the external loading (when the external loading is set to zero, the solution reduces to the zero solution). The main motivation for doing so is to be able to draw direct comparison with the analogous results from the linear elastic fracture mechanics. An analysis of the separate problem concerning the stress field induced by the surface tension can be found in, for example, author's reference (Wu 1999, Wu and Wang 2000), where it is stated that, depending on the particular description of the crack tip adopted, stress singularity may occur (infinite stress at the crack tip).

Now, adding and subtracting Eqs. (36) and (37) yields

$$[\theta'(z)^+ - \theta'(z)^-] = iJ_o\left(\frac{\partial^2 u^+}{\partial x^2} + \frac{\partial^2 u^-}{\partial x^2}\right) - \sigma_o\left(\frac{\partial^2 v^+}{\partial x^2} + \frac{\partial^2 v^-}{\partial x^2}\right) \tag{38}$$

$$[2\Omega'(z)^+ - \theta'(z)^+] + [2\Omega'(z)^- - \theta'(z)^-] = iJ_o\left(\frac{\partial^2 u^+}{\partial x^2} - \frac{\partial^2 u^-}{\partial x^2}\right) - \sigma_o\left(\frac{\partial^2 v^+}{\partial x^2} - \frac{\partial^2 v^-}{\partial x^2}\right) \tag{39}$$

In addition, from Eq. (32), we have that

$$\left(\frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2}\right) = \frac{1}{2\mu}[\kappa\Omega''(z) - \Omega''(\bar{z}) - (\bar{z} - z)\overline{\Omega'''(z)} + \theta''(\bar{z})] \tag{40}$$

Therefore, by applying the relations $\Omega''(\bar{z})^+ = \Omega''(z)^-, \bar{z} = z$ on $y = \pm 0$, we have

$$\left[\left(\frac{\partial^2 u^+}{\partial x^2} + \frac{\partial^2 u^-}{\partial x^2}\right) + i\left(\frac{\partial^2 v^+}{\partial x^2} + \frac{\partial^2 v^-}{\partial x^2}\right)\right] = \frac{1}{2\mu}[(\kappa - 1)\{\Omega''(z)^+ + \Omega''(z)^-\} + \{\theta''(z)^+ + \theta''(z)^-\}] \tag{41}$$

$$\left[\left(\frac{\partial^2 u^+}{\partial x^2} - \frac{\partial^2 u^-}{\partial x^2}\right) + i\left(\frac{\partial^2 v^+}{\partial x^2} - \frac{\partial^2 v^-}{\partial x^2}\right)\right] = \frac{1}{2\mu}[(\kappa + 1)\{\Omega''(z)^+ - \Omega''(z)^-\} - \{\theta''(z)^+ - \theta''(z)^-\}] \tag{42}$$

Consequently, Eqs. (38)-(39) takes the following forms with the use of Eqs. (41)-(42)

$$[\theta'(z)^+ - \theta'(z)^-] = \frac{iJ_o}{2\mu}\mathrm{Re}[(\kappa - 1)\{\Omega''(z)^+ + \Omega''(z)^-\} + \{\theta''(z)^+ + \theta''(z)^-\}]$$
$$-\frac{\sigma_o}{2\mu}\mathrm{Im}[(\kappa - 1)\{\Omega''(z)^+ + \Omega''(z)^-\} + \{\theta''(z)^+ + \theta''(z)^-\}] \tag{43}$$

$$[2\Omega'(z)^+ - \theta'(z)^+] + [2\Omega'(z)^- - \theta'(z)^-] = \frac{iJ_o}{2\mu}\mathrm{Re}[(\kappa + 1)\{\Omega''(z)^+ - \Omega''(z)^-\} - \{\theta''(z)^+\theta''(z)^-\}]$$
$$-\frac{\sigma_o}{2\mu}\mathrm{Im}[(\kappa + 1)\{\Omega''(z)^+ - \Omega''(z)^-\} - \{\theta''(z)^+ - \theta''(z)^-\}] \tag{44}$$

Next, if we express the unknowns $\Omega'(z)$ and $\theta'(z)$ as Cauchy integral (similar to those in Mode-III case) with given the behavior of $\theta'(z)$ (England 1971), we have that

$$\Omega'(z) = \frac{1}{2i\pi}\int_{-a}^{+a}\frac{f(t) + ig(t)}{t - z}dt + \frac{1}{\mu i}[\sigma_{yy}^\infty - i\sigma_{xy}^\infty]$$

$$\Omega''(z) = \int_{-a}^{+a}\frac{f(t) + ig(t)}{(t - z)^2}dt = -\left[\frac{f(t) + ig(t)}{t - z}\right]_{-a}^{+a} + \frac{1}{2i\pi}\int\frac{f'(t) + ig'(t)}{t - z}dt = \frac{1}{2i\pi}\int\frac{f'(t) + ig'(t)}{t - z}dt \tag{45}$$

where, $f(t) + ig(t) = \Omega'(z)^+ - \Omega'(z)^-$, $f(a) = f(-a) = g(-a) = g(a) = 0$

$$\theta'(z) = \frac{1}{2i\pi}\int_{-a}^{+a}\frac{\alpha(t) + i\beta(t)}{t - z}dt$$

$$\theta''(z) = \int_{-a}^{+a}\frac{\alpha(t) + i\beta(t)}{(t - z)^2}dt = -\left[\frac{\alpha(t) + i\beta(t)}{t - z}\right]_{-a}^{+a} + \frac{1}{2i\pi}\int\frac{\alpha'(t) + i\beta'(t)}{t - z}dt = \frac{1}{2i\pi}\int\frac{\alpha'(t) + i\beta'(t)}{t - z}dt \tag{46}$$

Finally, from Eqs. (43)-(44), we derive the following coupled first-order Cauchy singular integro-differential equations for the unknowns $f(t), g(t), \alpha(t)$ and $\beta(t), t \in [-a, a]$ via successive use of Eqs. (45)-(46) and by separating the real and imaginary part of the corresponding terms in Eqs. (43)-(44)

$$\alpha(t_o) = \frac{\sigma_o}{2\mu}\left[\frac{(\kappa - 1)}{\pi}\int_{-a}^{+a}\frac{f'(t)}{t - t_o}dt + \frac{1}{\pi}\int_{-a}^{+a}\frac{\alpha'(t)}{t - t_o}dt\right. \tag{47}$$

$$\frac{2}{\pi}\int_{-a}^{+a}\frac{f(t)}{t - t_o}dt + 2\sigma_{xy}^\infty - \frac{1}{\pi}\int_{-a}^{+a}\frac{\alpha(t)}{t - t_o}dt = -\frac{J_o}{2\mu}[(\kappa + 1)f'(t_o) - \alpha'(t_o)] \tag{48}$$

$$\beta(t_o) = \frac{\sigma_o}{2\mu}\left[\frac{(\kappa - 1)}{\pi}\int_{-a}^{+a}\frac{g'(t)}{t - t_o}dt + \frac{1}{\pi}\int_{-a}^{+a}\frac{\beta'(t)}{t - t_o}dt\right. \tag{49}$$

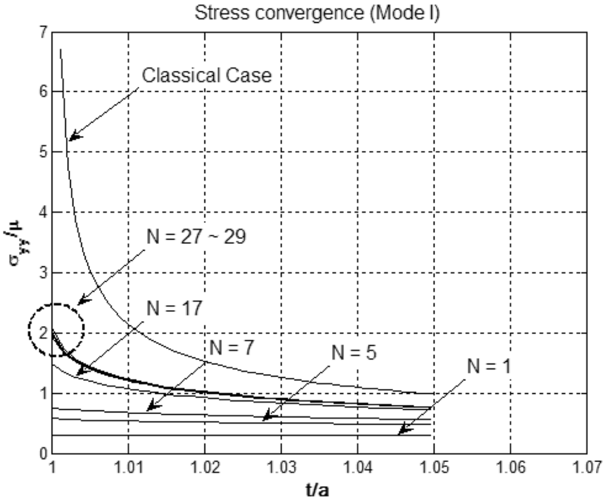


Fig. 5 Stress convergence in Mode-I case

$$\frac{2}{\pi}\int_{-a}^{+a}\frac{g(t)}{t-t_o}dt+2\sigma_{xy}^{\infty}-\frac{1}{\pi}\int_{-a}^{+a}\frac{\beta(t)}{t-t_o}dt = -\frac{J_o}{2\mu}[(\kappa+1)g'(t_o)-\beta'(t_o)] \tag{50}$$

The above series of equations can be numerically solved using analogous method as adopted in Mode-III crack analysis, yet more comprehensive style (Kim *et al.* 2010a). The method ensures fast convergence for this case as well (within 30 iterations (N), see, for example, the results in Fig. 5).

4. Results and discussions

In this section, the numerical solution of Eqs. (28, for Mode-III crack) and (47-50, for Mode-I & II crack) is performed for a range of surface parameters. The listed values are estimated properties of "GaN" obtained from the work of Sharma in (Sharma and Ganti 2004). GaN is composed of a mixture of nitrified aluminum (AI), gallium (Ga) and indium (In) and used in the manufacture of a semiconductor.

$$\mu^s = 161.73[J/m^2],\ \sigma_o = 1.3[J/m^2],\ J_o = 400[J/m^2],\ \mu = 168[Gpa] \tag{51}$$

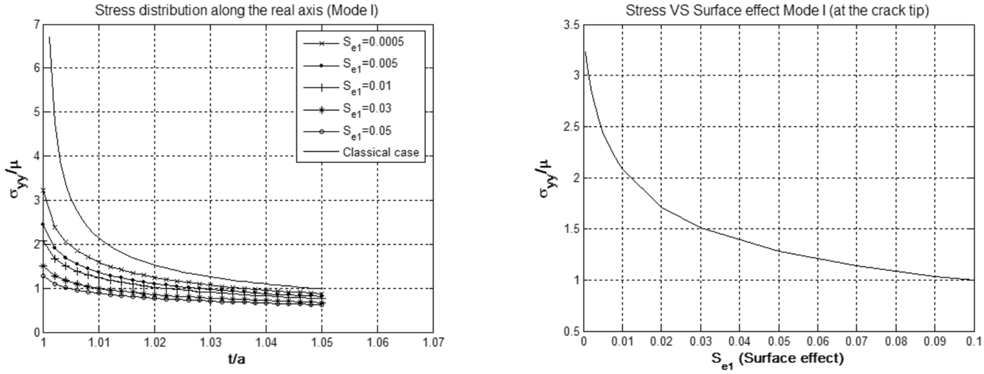
In addition, the following surface parameters (dimensionless) are introduced throughout the analysis

$$S_{e1} = \frac{\sigma_o}{2a\mu}:0.0005 < S_{e1} < 0.05\text{ , for Mode-I crack (axial tension)}$$

$$S_{e2} = \frac{J_o}{2a\mu}:0.0005 < S_{e2} < 0.1\text{ , for Mode-II crack (in-plane shear)}$$

$$S_e = \frac{\mu^s\sigma_o}{a\mu}:0.001 < S_e < 0.1\text{ , for Mode-III crack (out-plane shear)} \tag{52}$$

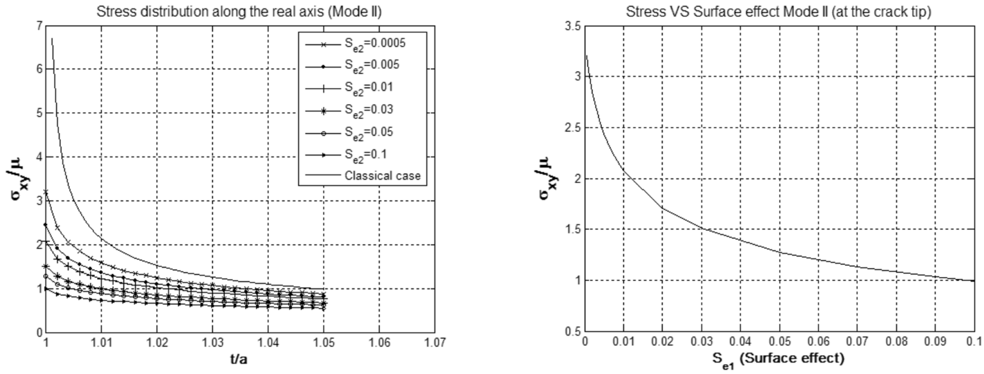
Mode-I crack



(a) **(b)**

Fig. 6 (a) Stress distribution (Mode-I) with respect to surface parameter (S_{e1}), when $\sigma_{yy}^{\infty}/\mu = 0.3$, (b) Stress (Mode-I, at the crack tip) versus surface effect, when $\sigma_{yy}^{\infty}/\mu = 0.3$

Mode-II crack



(a) **(b)**

Fig. 7 (a) Stress distribution (Mode-II) with respect to surface parameter (S_{e2}), when $\sigma_{xy}^{\infty}/\mu = 0.3$, (b) Stress (Mode-II, at the crack tip) versus surface effect, when $\sigma_{xy}^{\infty}/\mu = 0.3$

The estimated stress distributions (along x-axis) for each case are presented through Figs. 6(a)-8(a).

It can be seen from Figs. 6(a)-8(a) that the obtained solutions in Mode-I, II and III case reduce to those of the classical case when the surface effects become negligible. Further, we found that, in plane-strain case (Mode-I and Mode-II), the general solution for the mixed mode problem (Mode-I + Mode-II, in this case, tension and in-plane shear are applied on the remote boundary simultaneously) reduces to those of the Mode-I and Mode-II case, separately, even with the incorporation of the surface effects (see, the compatible results from the linear elastic fracture mechanics, (England 1971) and (Muskhelishvili 1953)). Finally, Figs. 6(b)-8(b) display the relation between crack tip stresses and the surface effects for Mode-I, II and III cases, respectively. The results clearly indicate that the surface effects can effectively reduce the corresponding stress at the crack tip. Further, the surface parameters (see, Eq. (52)) are controlled by variations of the crack

Mode-III crack

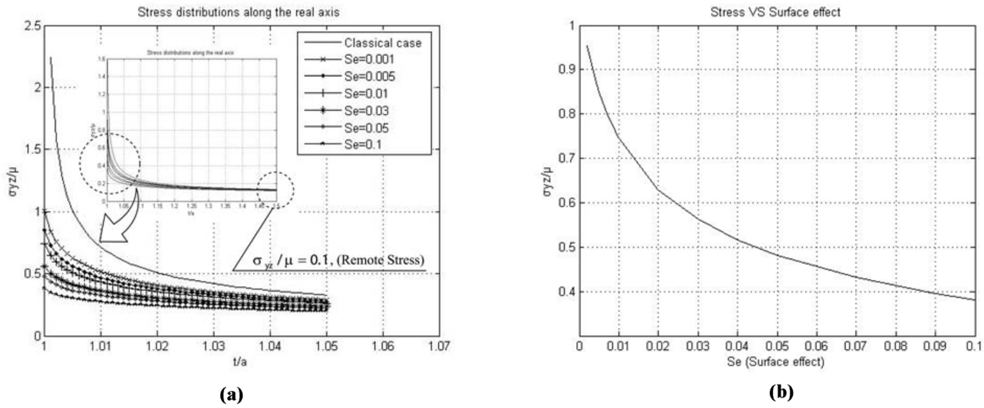


Fig. 8 (a) Stress distribution (Mode-III) with respect to surface parameter (S_e), when $\sigma_{yz}^{\infty}/\mu = 0.1$, (b) Stress (Mode-III, at the crack tip) versus surface effect, when $\sigma_{yz}^{\infty}/\mu = 0.1$

$$\sigma^{s}{}_{xx} = h\sigma_{xx}$$

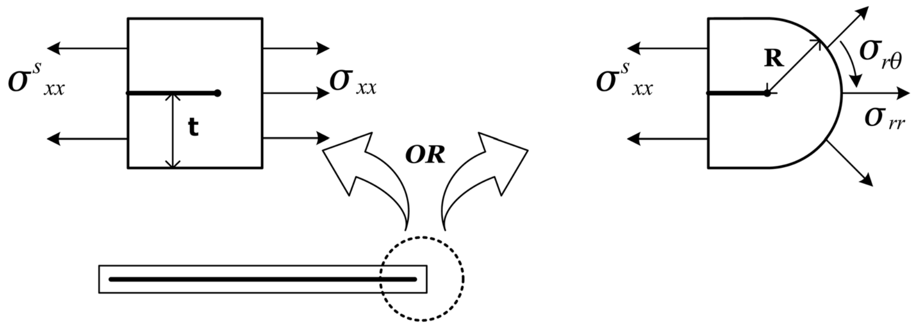


Fig. 9 Schematic of crack tip region and surrounding surface layer

size, the present results also indicate that the corresponding stresses exhibit strong dependency on crack size. In fact, size dependency of the stress fields are the dominant phenomena in study of sufficiently small scale structures such as in micro and nano sized structures (see, for example, (Tian and Rajapakse 2007) and (Wang and Wang 2006)).

Remark 3.

We have estimated the thickness of the surface layer surrounding the crack via appropriate balance equation given by (see Fig. 9.)

Using the values of σ_{xx}^{s} and σ_{xx} obtained above yields

$$56.62[J/m^2] = h \times 0.6577 \times 168 \times 10^9[N/m^2],\ h = 0.5124 \times 10^{-9}[m]$$

This is indeed consistent with the results in the literature (see, for example Suzuki *et al*. (2008), Sugiyama *et al*. (2010)) that the thickness (h) of the 'surface layer' is reported as being in the range of a half to a few nanometers. Varying the material properties and the distribution of the estimated stresses as we approach the crack tip, continues to yields results consistent with the results reported independently in the literature.

5. Conclusions

In this summarized paper, we have reviewed the general deformations of a linearly elastic solid in the case where a Mode-I, Mode-II or Mode-III crack is present and, especially, when the surface effects are incorporated into the system of analysis. Throughout the rigorous derivation, we show that the deformations of an elastic solid containing a single crack can be decoupled into in-plane and anti-plane parts, even when the surface mechanics is incorporated. We also demonstrate that, in the case where in-plane deformations are considered (Mode-I and Mode-II crack), the surface stress σ_o gives rise to residual stresses in which case the corresponding stress field can no longer be zero even when all external loads are removed. This also may result in stress singularity at the crack tips depending on the particular description of the crack tip adopted. As we mentioned previously, in the present study, we exclude this situation in an attempt to draw direct comparison with the analogous results from the linear elastic fracture mechanics. In results, it is shown that, in contrast to classical fracture mechanics (where surface effects are neglected), the incorporation of surface elasticity leads to the more accurate situation of a finite stress at the crack tip. It should be noted here that finite stresses at the crack tip can also be achieved using other ‘means’, for example, by directly considering molecular interactions near the crack tip (see (Philip 2009) and the references therein), yet, eventually, they share the same idea: the incorporation of the role of surfaces into analysis.

Finally, we demonstrate that the corresponding stress distributions derived from our analysis show clear signs of size dependency and do indeed reduce to classical solutions, when the surface effects approach zero. Although, in the summary, we deliberately omitted details of numerical analysis of the system of integro-differential equations, this does not necessarily mean that the corresponding analysis is not of interest or important. One reason for doing so is that the author would like to put more attention on formatting governing equation in which the effects of surface mechanics is incorporated and examining basic rules from the classical mechanics (for example, decomposition theory) within the current problem setting. Comprehensive details of the corresponding numerical analysis are available in Kim *et al*. (2009), Kim *et al*. (2010a), Kim *et al*. (2010b).

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