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Sensitivity-based reliability analysis of earth slopes using finite element method

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Abstract. For slope stability analysis, an alternative to the classical limit equilibrium method (LEM) of slices is the shear strength reduction method (SRM), which can be integrated into finite element analysis or finite difference analysis. Recently, probabilistic analysis of earth slopes has been very attractive because it is capable to take the soil uncertainty into account. However, the SRM is less commonly extended to probabilistic framework compared to a variety of probabilistic LEM analysis of earth slopes. To overcome some limitations that hinder the development of probabilistic SRM stability analysis, a new procedure based on recursive algorithm FORM with sensitivity analysis in the space of original variables is proposed. It can be used to deal with correlated non-normal variables subjected to implicit limit state surface. Using the proposed approach, a probabilistic finite element analysis of the stability of an existing earth dam is carried out in this paper.

Keywords: slope stability; SRM; finite element analysis; probabilistic analysis; FORM

1. Introduction

The stability of slopes is commonly evaluated using two different types of methods: (i) limit equilibrium methods (LEM); and (ii) strength-reduction method (SRM). The factor of safety F_s, defined as the ratio of shear strength (or resisting moment) to mobilized shear stress (or overturning moment) of a potential sliding mass, is used as an indicator of its stability. Locations of potential slip surfaces need to be assumed in LEM. The factor of safety is evaluated on all the potential slip surfaces, and a critical slip surface which produces the lowest factor of safety F_s is finally determined. In contrast, the strength-reduction technique integrated with Finite Element Method, SRFEM (e.g., Zienkiewicz *et al*. 1975, Naylor 1982, Ugai 1989, Matsui and San 1992, Griffiths and Lane 1999, Zheng and Zhao 2004) or with Finite Difference Method, SRFDM (e.g., Dawson *et al*. 1999) has been developed for slope stability analysis. Assumptions about the possible failure mode are not required a priori. Some advantages and limitations of the SRFEM/SRFDM were summarized in Cheng and Zhu (2005).

When the slope stability analyses are extended into probabilistic study, one limitation of the SRFEM/SRFDM is that the performance functions are usually implicit, i.e., the safety factor is not

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available in closed-form, and it can only be evaluated by numerical simulations. As such, direct use of reliability methods (e.g., FORM) is not easy since the optimization of reliability index requires an explicit performance function. Although one can employ Monte Carlo simulations (MCS) for probabilistic stability analysis by FEM/FDM (e.g., Griffiths and Fenton 2004), the MCS-based probabilistic slope stability analysis may require intensive computational effort especially when the potential failure probability is a very small value (e.g., 0.05%, which is common to geotechnical engineering). For such cases, the reliability method incorporating implicit performance functions is an attractive alternative.

Several useful techniques bridging the reliability methods with implicit performance functions can be found in the literature. These include response surface method (RSM) and artificial neural networks (ANN). Both methods aim at approximating the actual performance function using closed-form expressions. For example, polynomials are commonly used in RSM (Bucher and Bourgund 1990, Rajashekhar and Ellingwood 1993, Xu and Low 2006, Li *et al*. 2011, Ji and Low 2012, Ji *et al*. 2012, Jiang *et al*. 2014a, b), and multilayer functions (also called multilayer perceptrons) are used in ANN (Goh and Kulhawy 2003, Elhewy *et al*. 2006, Goh and Kulhawy 2006, Cho 2009). Another technique for reliability analysis incorporating implicit performance functions is the recursive algorithm for FORM (Rachwitz and Fiessler 1978). This is less commonly used in the literature. In this paper the recursive algorithm FORM is reformulated in the space of dimensionless variables, so that it can be easily adopted for reliability analysis of implicit performance function with correlated non-normals. Application of the proposed approach to probabilistic analysis of an earth slope using SRFEM will be presented.

2. FORM with probabilistic sensitivity analysis

2.1 First-order reliability method (*FORM*)

In classical FORM, the original correlated basic random variables $\boldsymbol{x}$ **(**in x-space**)** that define the limit state surface (LSS) or performance function $g(\boldsymbol{x})$ are transformed into uncorrelated standardized normal variables $\boldsymbol{u}$ (in u-space), and the Hasofer and Lind (1974) index β is defined as

$$\beta = \sqrt{\boldsymbol{u}^{*T}\boldsymbol{u}^{*}} \tag{1}$$

where $\boldsymbol{u}^*$ is the most probable failure point or design point, denoting the point on the limit state surface $g(\boldsymbol{u}) = 0$ closest to the origin of u-space. Usually the design point $\boldsymbol{u}^*$ is not known beforehand, and the search for $\boldsymbol{u}^*$ can be carried out using an iterative algorithm (Rackwitz 1976). The procedure is well explained in Ang and Tang (1984), Haldar and Mahadevan (2000), Baecher and Christian (2003), among others.

An alternative formulation of β in x-space or in the space of standardized correlated normal variables (n-space) is presented in Low and Tang (2004, 2007)

$$\beta = \sqrt{\left[\frac{x_i^* - \mu_i^N}{\sigma_i^N}\right]^T \boldsymbol{R}^{-1} \left[\frac{x_i^* - \mu_i^N}{\sigma_i^N}\right]} = \sqrt{\boldsymbol{n}^{*T}\boldsymbol{R}^{-1}\boldsymbol{n}^{*}} \tag{2}$$

where, x_i^* is the design point value of the ith variable evaluated in x-space, μ_i^N and σ_i^N are equivalent normal mean and standard deviation of the ith variable, respectively, $\boldsymbol{R}$ is the correlation matrix, and $\boldsymbol{n}^*$ is the design point evaluated in the space of correlated dimensionless variables (n-space). μ_i^N and σ_i^N can be calculated by the Rackwitz and Fiessler (1978) transformation.

2.2 Recursive algorithm FORM in n-space

When LSS is explicit, x_i^* and β can be easily computed using a constrained optimization approach (Low and Tang 2004, 2007); when LSS is implicit, the constrained optimization approach can still be used, provided that a closed-form approximation of the LSS is first obtained based on the response surface method (RSM) (Xu and Low 2006). It is noted that the RSM-based FORM uses an approximated polynomial as the actual performance function. Thus the efficiency of this method unavoidably depends on the accuracy of the approximated polynomial. For some problems having highly nonlinear performance functions, it may be very hard to approximate them using polynomials. Thus it is desirable to carry out reliability analysis without having to approximate the actual limit state surface *a priori*. In this circumstance, an alternative Newton-Raphson type recursive algorithm (Rachwitz and Fiessler 1978, Haldar and Mahadevan 2000) can be used to find the design point, such that (Haldar and Mahadevan 2000)

$$\boldsymbol{u}_{k+1} = \frac{1}{\left|\nabla g(\boldsymbol{u}_k)\right|^2}\left[\nabla g(\boldsymbol{u}_k)^T \boldsymbol{u}_k - g(\boldsymbol{u}_k)\right]\nabla g(\boldsymbol{u}_k) \tag{3}$$

where $\boldsymbol{u}_k$ is the kth iteration point in the space of *uncorrelated standard normal variables* (or u-space), $g(\boldsymbol{u}_k)$ and $\nabla g(\boldsymbol{u}_k)$ are the performance function and gradient vector of the performance function evaluated at $\boldsymbol{u}_k$, respectively.

Note that the original recursive algorithm given by Eq. (3) is formulated in the u-space. As a result, transformation of the basic random variables from *the original space* (or x-space) to u-space and redefinition of the performance function is required to use the recursive algorithm. However, when the performance function is implicit and basic random variables are correlated nonnormals, it is almost impossible to carry out reliability analysis by the recursive algorithm in Eq. (3) (Haldar and Mahadevan 2000). To overcome this limitation, the recursive algorithm is reformulated in n-space, such that (Ji 2013)

$$\boldsymbol{n}_{k+1} = \frac{1}{\nabla g(\boldsymbol{n}_k)^T R \nabla g(\boldsymbol{n}_k)}\left[\nabla g(\boldsymbol{n}_k)^T \boldsymbol{n}_k - g(\boldsymbol{n}_k)\right]\boldsymbol{R}\nabla g(\boldsymbol{n}_k) \tag{4}$$

where $\boldsymbol{n}_k$ is the kth iteration point in n-space, $g(\boldsymbol{n}_k)$ and$\nabla g(\boldsymbol{n}_k)$ are the performance function and gradient vector of the performance function evaluated at $\mathbf{n}_k$, respectively, and $\boldsymbol{R}$ is the correlation matrix. Details of the derivation of Eq. (4) are given in Appendix. To obtain the $\nabla g(\boldsymbol{n}_k)$, the following relationship is useful

$$\frac{\partial g(\boldsymbol{n}_k)}{\partial n_{k,i}} = \sigma_k^N \frac{\partial g(\boldsymbol{x}_k)}{\partial x_{k,i}} \tag{5}$$

where $n_{k,i}$ is the ith component of the reduced vector $\boldsymbol{n}_k$, $x_{k,i}$ is the ith component of the original variable vector $\boldsymbol{x}_k$.

In general, the gradient vector is not constant when the performance function is nonlinear. As a result, iterative evaluation of Eq. (4) is needed to obtain the final iteration point (i.e., the design point $\boldsymbol{n}^*$). The algorithm is repeated until convergence, satisfying two criteria: $|\boldsymbol{n}_{k+1} - \boldsymbol{n}_k|$ and $|g(\boldsymbol{n}_{k+1}) \le \varepsilon_2|$; both ε_1 and ε_2 are small quantities. After obtaining $\boldsymbol{n}^*$, the reliability index β can be computed by Eq. (2).

2.3 Procedure of FORM with sensitivity analysis in x-space

For most engineering problems involving implicit performance function with correlated non-normals, Eq. (4) in combination with Eq. (5) can be directly used for the reliability analysis. Since the analytical differentiation of $g(\boldsymbol{x}_k)$ is not available, the sensitivity analysis (e.g., finite difference method) could be used to approximate the $\partial g(\boldsymbol{x}_k)/\partial x_{k,i}$. The procedure for implementing the recursive algorithm FORM with sensitivity analysis in x-space is summarized as follows:

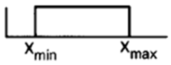
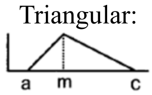
Type of $F(x)$	$x = F^{-1}[\Phi(n)]$
Normal: mean μ_x, std. dev. σ_x	$x = \mu_x + n\sigma_x$
LogNormal: mean μ_x , std. dev. σ_x	$x = \exp(\lambda + n\zeta),\ \ \zeta = \sqrt{\ln(1+(\sigma_x/\mu_x)^2}\ ,\ \ \lambda = \ln\mu_x - 0.5\zeta^2$
Extreme Value (Gumbel): mean μ_x, std. dev. σ_x	$x = u - \dfrac{\ln[-\ln(\Phi(n))]}{\alpha},\ \ \alpha = \dfrac{1}{\sqrt{6}}\left(\dfrac{\pi}{\sigma_x}\right),\ \ u = \mu_x - \dfrac{0.5722}{\alpha}$
Exponential: mean μ_x	$x = -\mu_x \ln[1-\Phi(n)]$
Uniform: 	$x = x_{\min} + (x_{\max} - x_{\min})\cdot\Phi(n)$
Triangular: 	$x = a + \sqrt{\Phi(n)\times(m-a)\times(c-a)},$ if $\Phi(n) \le (m-a)/(c-a)$, $x = c - \sqrt{[1-\Phi(n)]\times(c-a)\times(c-m)}$, otherwise
Weibull CDF: $1 - \exp[-(x/\lambda)^\alpha]$	$x = \lambda\cdot[-\ln(1-\Phi(n))]^{1/\alpha}$
Gamma with PDF: $f(x) = \dfrac{x^{\alpha-1}\exp(-x/\alpha)}{\lambda^\alpha\Gamma(\alpha)}$	Para 1 = α, Para 2 = λ, Mean: $\mu_x = \alpha + \lambda$, Newton method used to obtain x from n, starting from μ_x
Beta distribution	Para 1 = α, Para 2 = λ, Para 3 = min, Para 4 = max, Mean: μ_x = min + (max – min)α / ($\alpha + \lambda$), Newton method used to obtain x from n, starting from μ_x
PERT distribution	Para 1 = min, Para 2 = mode, Para 3 = max, Mean: μ_x = (min + 4 × mode + max) / 6, Newton method used to obtain x from n, starting from μ_x

Fig. 1 Function *x_i* for obtaining x_i from n_i (after Low and Tang 2007)

Step 1: first select an initial iteration point $\boldsymbol{n_k}$ in n-space, and find $\boldsymbol{x_k}$ using a *x_i* function as shown in Fig. 1. Since $\boldsymbol{n_k}$ is the vector of reduced equivalent standard normal variables, initial $\boldsymbol{n_k}$ can be at its mean value $\boldsymbol{n_0} = \mathbf{0}$;

Step 2: evaluate the performance function $g(\boldsymbol{x}_k)$. Use finite difference method to compute $\Delta g(\boldsymbol{x}_k)/\Delta x_{k,i}$, and obtain the gradient vector $\nabla g(\boldsymbol{x}_{k,i})$;

Step 3: use Eq. (5) to compute the gradient vector $\nabla g(\boldsymbol{n}_{k,i})$;

Step 4: compute the new vector $\boldsymbol{n}_{k+1}$ using Eq. (4), and compute a nominal index β_{k+1} using Eq. (2) where $\boldsymbol{n}^*$ is taken to be the new vector $\boldsymbol{n}_{k+1}$;

Step 5: using the new vector $\boldsymbol{n}_{k+1}$, repeat steps 1 to 4 until convergence of the vector $\boldsymbol{n}$ or β value and $g(\boldsymbol{n}_{k+1}) = 0$.

It is worth pointing out that for the recursive algorithm FORM with sensitivity analysis, one may encounter difficulties in achieving the convergence of vector $\boldsymbol{n}$ if ε_1 is set to be a very small quantity. This is because the sensitivity analysis always brings some errors in estimating the gradient vector of performance function. On the other hand, since our objective is to find the reliability index β, one can reduced the convergence criteria for ε_1 by using $\varepsilon_1 = |\beta_{k+1} - \beta_k|$ instead of $\varepsilon_1 = |\boldsymbol{n}_{k+1} - \boldsymbol{n}_k|$, where β_k is a nominal reliability index computed by Eq. (2) at the tentative design point $\boldsymbol{n}_k$. This is particularly useful when finite difference method is used for the sensitivity analysis.

3. Validatition of the proposed approach using simple examples

3.1 Application to correlated non-normal variables subjected to explicit limit state surface

The proposed sensitivity-based FORM procedure is first illustrated using a problem containing correlated non-normal variables which was solved by Low and Tang (2004, 2007) using the constraint optimization approach. The example problem illustrates a reliability analysis of a steel beam section. The fully plastic flexural capacity is given as YZ, where Y = the yield strength of steel and Z = section modulus of the section. Subjected to a bending moment M at that section, the performance function is defined as $g = YZ - M$. The statistical information of the three variables Y, Z and M is shown in Fig. 2. Note that the performance function of this example problem is available. As a result, accurate solution of the reliability index is found to be 2.6646.

A detailed reliability analysis of the above-mentioned example using the proposed FORM procedure is also shown in Fig. 2. Initially, the tentative point $\boldsymbol{n_0}$ is chosen to be [0, 0, 0]. Since the performance function is given in x-space other than directly in n-space, the one-to-one transformation from n-space to x-space is carried out using the user-defined function *x_i* in a spreadsheet in order to evaluate the performance function at the initial point. The gradient vector $\nabla g(\boldsymbol{x}_0)$ is directly obtained from the performance function, $(Y_0, Z_0, -1)$. Using the proposed 5-step procedure, the first tentative β value was found to be 2.8998. After four iterations, the β value was found converged to 2.6646, which is the same as the accurate solution.

3.2 Application to a slope reliability analysis

To investigate the applicability of the sensitivity-based FORM in slope reliability analysis, a homogeneous $1H{:}1V$ cohesive slope as shown in Ji *et al*. (2012)'s Fig. 12 is re-visited. The

Transformation of variables from n-space to x-space

ProbDist	Variable	Para1	Para2	n_k	x_k	Δg (x_k)
Lognorma	Y	40	5	-1.29	33.79	47.758
Lognorma	Z	50	2.5	-0.89	47.76	33.786
ExtValue	M	1000	200	2.29	1614	-1.000

Computation of the next iteration point based on recursive algorithm in n-space

[R]			g (n_k) = g (x_k)	Δg (n_k)	n_{k+1}	β
1	0.4	0	2E-05	200.9	-1.29	2.6646
0.4	1	0		80.63	-0.89	
0	0	1		-413.2	2.29	

Note:
(1) the performance function $g(x_K) = YZ - M$;
(2) the first component of n_k vector is associated with *Y* and the second associated with *Z, the third associated with M*; *Y* and *Z* are correlated with a correlation coefficient of 0.4;
(3) the gradient vector $\Delta g(x_k)$ is dirercly obtained from the closed-form $g(\mathbf{x}_K)$

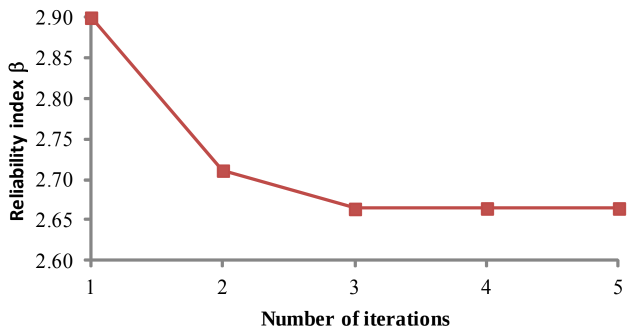


Fig. 2 Reliability analysis of an example problem by sensitivity-based FORM

Transformation of variables from n-space to x-space

ProbDist	Variable	μ	σ	n_k	x_k	Δg (x_k)
Normal	c'	15	4.5	-1.33	15.00	0.042
Normal	φ'	23	2.3	0.23	23.00	0.030

Computation of the next iteration point based on recursive algorithm in n-space

[R]		g (n_k) = g (x_k)	Δg (n_k)	n_{k+1}	β
1	-0.5	0.000	0.189	-1.33	1.428
-0.5	1		0.068	0.23	

Note:
(1) the performance function $g(x_k) = F_s - 1$;
(2) the factor of safety F_s is computed by Spencer's method in a spreadsheet;
(3) the first component of n_k vector is associated with *c* and the second associated with *φ*; *c* and *φ* are correlated with a correlation coefficient of −0.5;
(4) the gradient vector $\Delta g(x_k)$ is obtained by finite difference analysis for each variables

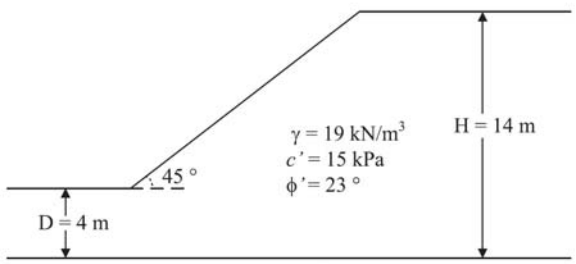


Fig. 3 Reliability analysis of a homogeneous earth slope by the recursive algorithm FORM

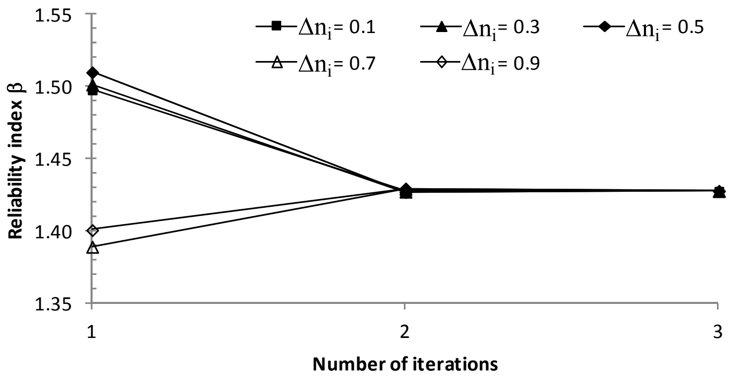


Fig. 4 Influence of perturbation Δn_i value on reliability index β

Spencer's method implemented in a spreadsheet was used to compute the factor of safety F_s. The optimization approach for FORM using EXCEL's SOLVER obtained a β value equal to 1.428. For comparison, Fig. 3 shows the reliability analysis by the sensitivity-based FORM in n-space. To compute the gradient vector $\nabla g(\boldsymbol{x}_0)$ or $\Delta g(\boldsymbol{x}_{0,j})/\Delta x_j$ (j = 1 to 2), the forward finite difference sensitivity analysis was performed on each component of $\boldsymbol{x}_\mathbf{0}$. Note that performance function is always evaluated in x-space other than directly in n-space. After 3 iterations, the β value has converged to 1.428, which is practically the same as that obtained by the constrained optimization approach. The advantage of the sensitivity-based FORM is that the integration of the reliability methods and deterministic stability model is not required, so that more complicated numerical methods for deterministic stability evaluation (e.g., FEM or FDM) can be used and extended into probabilistic analysis, as illustrated next.

3.3 *Effect of perturbation values in n-space*

It is noted that a constant perturbation value of Δn_i of 0.5 was used in the above sensitivity analysis for obtaining the gradient vector $\nabla g(\boldsymbol{n_k})$. Other perturbation values in the n-space can also be used for the sensitivity analysis. Since the reduced variable n_i has mean value of '0' and standard deviation of '1', a parametric study of the influence of Δn_i (varying from 0.1 to 0.9) on β was carried out. The results are briefly shown in Fig. 4. For the first iteration, the β value is much sensitive to the perturbation value used. When more iterations were carried out, the β value converges very fast. Thus, it may be concluded that the β value is not really sensitive to the value of Δn_i when an iterative procedure is used to improve the results.

4. Probabilistic slope stability analysis by strength-reduction FEM – Clarence Cannon Dam

4.1 *Problem background*

Clarence Cannon Dam is located in the Salt River in northeastern Missouri and forms Mark Twain Lake. The dam is part of a multi-purpose project which provides flood control, recreation, water supply, fish and wildlife conservation, and hydropower. Completed in 1983, the dam has a 305 m long earth embankment, a gated concrete spillway section and a concrete powerhouse adjacent to the spillway. The earth embankment volume is approximately 2.74 million m^3. The dam crest at elevation 200 m above mean sea level is about 35 m above the floodplain and 42 m above the stream bed.

The embankment section analyzed is located at survey station 12 + 75 and is representative of the dam near its maximum height. The embankment geometry at this location is shown in Fig. 5. The section includes a compacted clay foundation cutoff trench through an abandoned glacial channel. The cutoff trench and embankment base are constructed of Phase I fill materials and the remainder of the embankment is constructed of Phase II fill materials.

The stability of Cannon Dam has been previously studied by many others. For example, Wolff (1985) investigated four different cases of stability of the dam, such as when the dam is at the end of construction, with partial pool of reservoir, at steady seepage, and with sudden drawdown. Corps' force-equilibrium method was used to compute the factor of safety along a limited number of specified non-circular critical slip surfaces. Hassan and Wolff (1999) analyzed the downstream stability of the dam without considering the pore water pressure, and they employed simplified Bishop method of slices and Spencer method of slices with both circular and non-circular slip surfaces. It is noted that all previous studies found in the literature were limited to limit equilibrium methods of slices. For comparison, the Cannon Dam is re-studied in this study by strength-reduction finite element analysis using the numerical code PLAXIS.

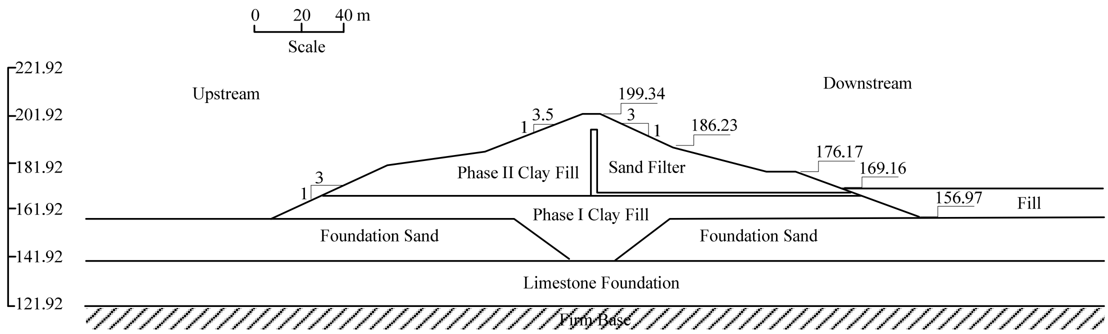


Fig. 5 Cannon Dam at section of station 12 + 75

Table 1 Soil properties of Cannon Dam for FEM stability analysis

Material	Young's modulus E (MPa)	Poisson ratio v	Cohesion c (kPa)	Friction angle φ (°)	Unit weight γ (kN/m^3)
Phase II clay fill	15	0.3	143.64	15	19
Phase I clay fill	10	0.3	117.79	8.5	19
Sand filter	30	0.3	0	30	16
Foundation sand	30	0.3	5	18	17.5
Fill	50	0.3	5	35	20
Limestone	20000	0.25	200	35	25

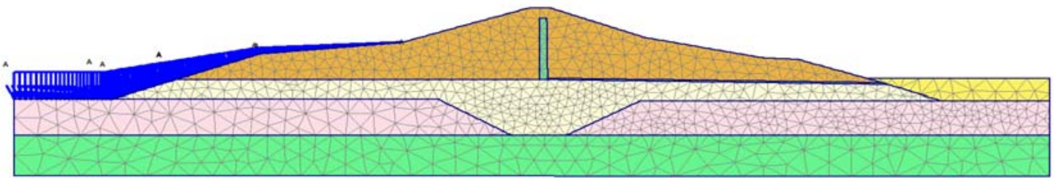


Fig. 6 Finite element model of the Cannon Dam

4.2 Strength-reduction FE stability analysis based on mean values

Soil properties of the Cannon Dam as reported by Hassan and Wolff (1999) and Slide's manual (http://www.rocscience.com/products/8/Slide) are used for the present study. It is noted that only the shear strength parameters of clay fills were reported in past studies. In this study, some missing values of soil properties which were not provided in the past studies are assumed in accordance to their engineering behavior. In addition, elastic modulus E and Poisson's ratio ν are also required in the strength-reduction FE stability analysis (based on elastic-perfectly plastic constitutive model and Mohr-Coulomb failure criterion). In fact, it is shown in the literature that neither E nor v influences the stability factor (e.g., Griffiths and Lane 1999). Thus the following study assumes nominal values of E and v which are typical to the relevant geo-materials. The input parameters are detailed in Table 1.

Fig. 6 shows the finite element model for Cannon Dam. 15-node triangular element is used to mesh the FE model. For the current study, a total of 1828 elements which include 14921 nodes are used. A relatively coarse mesh is applied to the limestone foundation (bottom layer) due to the fact that failure will not take place in this layer. Since the downstream (right side) stability of the dam is of particular interest, as investigated by others (Hassan and Wolff 1999), the mesh has been refined in this side. By trial-and-error, it is found that such a mesh scheme is adequate for the stability analysis.

Fig. 7 shows the displacements and shear strain increments at the state of failure, which is computed using the strength reduction technique of PLAXIS. An almost circular failure band is observed, and the failure band tends to cross the Phase I fill, Phase II fill and the foundation sand layers. It is noted that the shear strain increments is larger around the slope toe area, and becomes smaller towards the top surface along the failure band. This implies that the slope failure would first take place around the toe area, and progressively extends to the top surface of the dam.

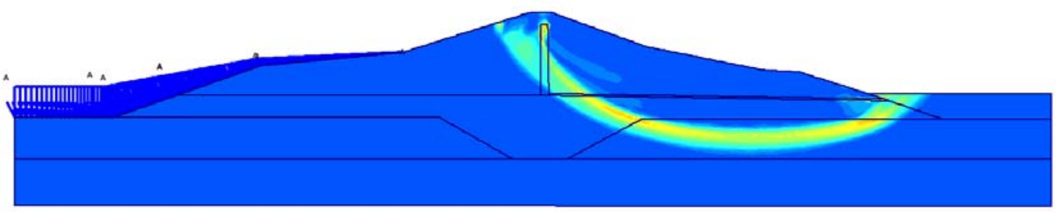


Fig. 7 Deterministic critical failure mechanism of the Cannon Dam by FEM (mean value point)

The critical factor of safety F_s by strength-reduction FE analysis was found to be 2.384. For the cross-validation purpose, the stability of Cannon Dam was also examined using limit equilibrium methods of slices. The results are compared with others in Table 2. In general, the factors of safety F_s computed by various methods are in close agreement with each other. For the limit equilibrium methods of slices, when circular slip surface is assumed, generalized limit equilibrium method of slices (GLE) obtains a F_s of 2.471, which is slightly higher than that by simplified Bishop method, 2.462. Since GLE is the most rigorous limit equilibrium method of slices, a non-circular slip surface analysis was also performed using GLE, which yields a lower value of F_s of 2.374 than does GLE with circular slip surface. In comparison, the F_s by strength-reduction FE stability analysis, 2.384, lies between the two values computed by GLE with circular and noncircular slip surfaces. Thus, it may be concluded that the strength-reduction FE stability analysis of Cannon Dam yields a reasonable F_s value compared with various limit equilibrium methods of slices. One advantage of strength-reduction FE stability analysis is that the failure mechanism is found automatically, and does not need assumptions of the location and shape of slip surfaces.

4.3 Probabilistic strength-reduction FE stability analysis

As reported by Wolff (1985) and Hassan and Wolff (1999), strength parameters of the Phase I clay fill and Phase II clay fill were considered as random variables. Based on UU test of record samples, statistical moments for the strength parameters of the two fill layers are summarized in Table 3. It is seen that the strength parameters are of very high variability; the coefficient of variation (c.o.v.) for φ_1 is as high as 1.0. In addition, cross correlation between cohesion c and friction angle φ of each layer is also reported: a positive cross correlation of the strength parameters in Phase I clay fill; but a negative cross correlation in Phase II clay fill. Based on the basic statistical information, a probabilistic investigation of the stability of Cannon Dam can be conducted. For example, Wolff (1985) used point-estimate method to estimate the probability of failure of the Cannon Dam along a specified non-circular critical slip surface (the deterministic critical one). Hassan and Wolff (1999) carried out a reliability analysis of the Cannon Dam using MVFOSM with search for the critical-β surface; both circular and non-circular slip surface

Table 2 Comparison between stability analyses of the Cannon Dam

Method of analyses	Sliding mode	Critical factor of safety F_s
Corps' force-equilibrium method (Wolff 1985)	Non-circular slip surface with 3-segment polyline	2.36
Simplified Bishop method (Hassan and Wolff 1999)	Circular slip surface	2.352
Simplified Bishop method (Hassan and Wolff 1999)	Non-circular slip surface with 3-segment polyline	1.980
Simplified Bishop method in this study	Circular slip surface	2.358
GLE in this study	Circular slip surface	2.471
GLE in this study	Non-circular slip surface	2.374
FEM analysis in this study	Almost circular failure band	2.384

Table 3 Statistical information of strength parameters of the Cannon Dam (after Hassan and Wolff 1999)

Soil description	Strength parameter	Mean value	Standard deviation	Coefficient of variation	Correlation coefficient
Phase II clay fill	c_2 (kPa)	143.64	79	0.55	− 0.55
	φ_2 (°)	15	9	0.6	
Phase I clay fill	c_1 (kPa)	117.79	58.89	0.5	0.10
	φ_1 (°)	8.5	8.5	1.0	

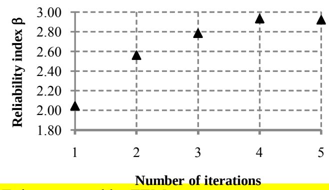
were considered. Bhattacharya *et al*. (2003) applied a new numerical procedure for locating the critical-β surface in their reliability study of the Cannon Dam.

For the present strength-reduction FE stability analysis of the Cannon Dam, it is also possible to extend into probabilistic analysis using the proposed procedure for implicit performance function with the aid of sensitivity analysis in x-space. It is worth pointing out that the sensitivity-based FORM requires the least runs of FE stability evaluation compared with RSM-based FORM (Xu and Low 2006). Note that past studies using point-estimate method and MVFOSM did not mention the statistical distribution of strength parameters but only assumed a lognormal distribution for the computed factor of safety F_s. In contrast, the FORM analysis of this study requires this additional statistical information. Considering the fact that high variability of the strength parameters were reported, Lognormal distribution is recommended in the next probabilistic study, since it helps avoid negative values of strength parameters which is meaningless in practice.

Details of the probabilistic FE evaluation are presented in Fig. 8, where forward finite difference scheme was used to approximate the gradient vector of performance function. It is shown that the reliability index β converges very fast, although there is still a very small fluctuation of β value after the forth iteration. This is due to the numerical instability of FE analysis when the F_s is close to unity as observed by the author. For convenience, the β value of

Transformation of variables from n-space to x-space

ProbDist	Variable	Para1	Para2	Para3	Para4	x*	Correlation matrix R				n	Varying index Δn_j
Lognormal	c_2	143.6	79.0			80.90	1	-0.55	0	0	-0.86	0.05
Lognormal	ϕ_2	15.0	9.0			12.18	-0.55	1	0	0	-0.1	0.2
Lognormal	c_1	117.8	58.9			33.67	0	0	1	0.1	-2.42	0.05
Lognormal	ϕ_1	8.5	8.5			1.79	0	0	0.1	1	-1.46	0.2



Procedure for Probabilistic Sensitivity-based FORM Δn_j = varying, $g = F_s - 1$, where F_s is computed by FEM

Step		Iteration 1			
Step 1	Initial $\mathbf{n}_i$	0	0	0	0
	$g(\mathbf{n}_i)$	1.203			
Step 2	$g(\mathbf{n}_{i,new})$	1.226	1.234	1.221	1.245
Step 3	$\Delta g(\mathbf{n}_i)$	0.023	0.031	0.018	0.042
Step 4	$\Delta g(\mathbf{n}_i)/\Delta n_j$	0.46	0.155	0.36	0.21
Step 5	$\mathbf{n}_{i+1}$	-1.303	0.341	-1.325	-0.855
	β	2.045			

Step		Iteration 2			
Step 1	Initial $\mathbf{n}_i$	-1.303	0.341	-1.325	-0.855
	$g(\mathbf{n}_i)$	0.382			
Step 2	$g(\mathbf{n}_{i,new})$	0.387	0.404	0.399	0.426
Step 3	$\Delta g(\mathbf{n}_i)$	0.005	0.022	0.017	0.044
Step 4	$\Delta g(\mathbf{n}_i)/\Delta n_j$	0.1	0.11	0.34	0.22
Step 5	$\mathbf{n}_{i+1}$	-0.233	-0.324	-2.133	-1.497
	β	2.561			

Step		Iteration 3			
Step 1	Initial $\mathbf{n}_i$	-0.233	-0.324	-2.133	-1.497
	$g(\mathbf{n}_i)$	0.116			
Step 2	$g(\mathbf{n}_{i,new})$	0.124	0.127	0.127	0.144
Step 3	$\Delta g(\mathbf{n}_i)$	0.008	0.011	0.011	0.028
Step 4	$\Delta g(\mathbf{n}_i)/\Delta n_j$	0.16	0.055	0.22	0.14
Step 5	$\mathbf{n}_{i+1}$	-1.184	0.301	-2.136	-1.479
	β	2.785			

Step		Iteration 4			
Step 1	Initial $\mathbf{n}_i$	-1.184	0.301	-2.136	-1.479
	$g(\mathbf{n}_i)$	0.057			
Step 2	$g(\mathbf{n}_{i,new})$	0.064	0.074	0.069	0.085
Step 3	$\Delta g(\mathbf{n}_i)$	0.007	0.017	0.012	0.028
Step 4	$\Delta g(\mathbf{n}_i)/\Delta n_j$	0.14	0.085	0.24	0.14
Step 5	$\mathbf{n}_{i+1}$	-0.875	-0.075	-2.385	-1.54
	β	2.934			

Step		Iteration 5			
Step 1	Initial $\mathbf{n}_i$	-0.875	-0.075	-2.385	-1.54
	$g(\mathbf{n}_i)$	-0.002			
Step 2	$g(\mathbf{n}_{i,new})$	0.002	0.008	0.005	0.013
Step 3	$\Delta g(\mathbf{n}_i)$	0.004	0.01	0.007	0.015
Step 4	$\Delta g(\mathbf{n}_i)/\Delta n_j$	0.08	0.05	0.14	0.075
Step 5	$\mathbf{n}_{i+1}$	-0.86	-0.098	-2.415	-1.457
	β	2.921			

*Note: A total of 25 FEM runs are performed to obtain the reliability index

Fig. 8 Probabilistic evaluation of the Cannon Dam using the sensitivity-based FORM

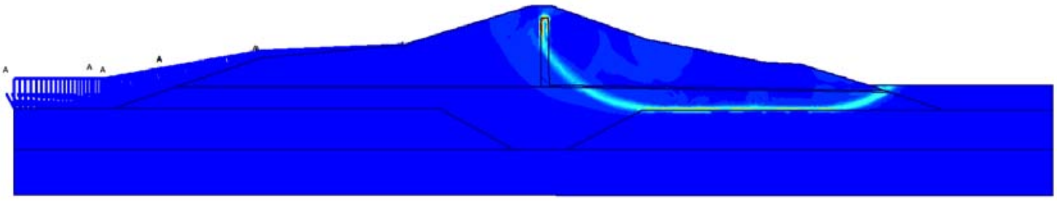


Fig. 9 Probabilistic critical failure mechanism of the Cannon Dam by FEM (design point)

2.921 at the fifth iteration is taken to be the final (critical) reliability index. From the final design point, it is shown that the normalized indices n_i for the strength parameters $\{c_2, \varphi_2\}$ and $\{c_1, \varphi_1\}$ are {-0.86, -0.10} and {-2.42, -1.46}, respectively. Obviously, the n_i's for $\{c_1, \varphi_1\}$ have greater influence on β than those for $\{c_2, \varphi_2\}$ do. The implication is that the strength parameters of Phase I clay fill will contribute much more to the stability of the Cannon Dam than does the strength parameters of Phase II clay fill.

Based on the final design point of the fifth iteration, the probabilistic critical failure mechanism of the Cannon Dam can be determined, as shown in Fig. 9. It is very interesting to compare the probabilistic critical failure mechanism with the deterministic critical failure mechanism of Fig. 7. Both are directly determined by strength-reduction FE stability analysis without making assumption with respect to the shape and location of slip surface. However, the two failure mechanisms differ very much: an almost circular (critical) failure band was observed from deterministic stability analysis; but a very non-circular (critical) failure band was observed from probabilistic analysis. Similar findings were previously reported by Cho (2009) where SRFDM were used for the stability analysis.

The results of this probabilistic strength-reduction FE stability analysis of Cannon Dam are compared with others using limit equilibrium methods (LEM), as shown in Table 4. Both Hassan and Wolf (1999) and Bhattacharya *et al*. (2003) used MVFOSM in combination with LEMs of slices. Thus reliability indices β reported by the two studies are in close agreement each other. But it is somewhat strange that the probabilistic critical slip surfaces from the two similar studies differ significantly. In contrast, a relatively higher value of β is found in this study. This is reasonable, since the mean value F_s by FEM is higher than by the other two LEM analyses, while the potential uncertainty of soil properties are the same. Another interesting finding is that even though the β

Table 4 Comparison between reliability analyses of the Cannon Dam

Method of analyses	Probabilistic critical failure mechanism	Reliability index β
MVFOSM and simplified Bishop method (Hassan and Wolff 1999)	Non-circular surface	2.664
MVFOSM and Spencer method of slices (Bhattacharya et al. 2003)	Non-circular surface	2.674
Probabilistic strength-reduction FE analysis in this study (FORM)	Non-circular band	2.921

value by FE differs from that by LEM, the probabilistic critical failure band from FE is very similar to the probabilistic critical failure surface from Bhattacharya *et al*. (2003)'s LEM.

5. Comparison between RSM-based and sensitivity-based reliability analyses

From the above illustration, it has been shown that both RSM-based FORM and the sensitivity-based FORM are able to deal with problems in the absence of explicit performance functions. The concept behind the use of RSM for reliability analysis is simple. The main efforts lie in the estimation of the actual performance function. However, considering computational efficiency and accuracy, the implementation of RSM-based reliability analysis could be impractical when the performance function is highly nonlinear and the number of basic random variables is very large (high dimensional reliability analysis). For example, for a problem involving n basic random variables $(n+1)$ experimental evaluation of the performance function are required when using a linear RSM; when using a simple second-order polynomial RSM, $2n+1$ experimental evaluations are required when cross terms are not considered, and $(n+1)(n+2)/2$ experimental evaluations are required when including the cross terms. Note that the number of experimental evaluations increases exponentially with increasing order of polynomial to be used. Other the other hand, a stochastic RSM method has been used to overcome the shortcomings of RSM to deal with high dimensional reliability analysis of geotechnical problems (Jiang *et al*. 2014a, b).

As an alternative to the RSM-based FORM, the sensitivity-based FORM does not require approximating the actual performance function, thus it is more suitable to use when no information about the actual performance function (whether linear or nonlinear) is available. Moreover, the number of experimental evaluations of the performance function is reduced significantly especially compared with nonlinear RSM. For example, for a problem involving n basic random variables, only $n+1$ (the same number as used in linear RSM) experimental evaluations of the actual performance function are required for each iteration when forward or backward difference scheme is used for sensitivity analysis, and $2n+1$ experimental evaluations are required for each iteration when central difference scheme is used. The use of sensitivity-based FORM for high dimensional reliability analysis is referred to Ji *et al*. (2013) and Ji (2013).

6. Conclusions

The stability of earth slope has long been an important topic in geotechnical engineering, and recent studies focus on the probabilistic assessment that incorporates the uncertainty of soil parameters. In practice, commercial numerical codes are often employed for the factor of safety analysis since most earth slopes are complex in the geometry and/or stratification. As such, it is interesting to extend the deterministic analysis by those stand-alone numerical codes to probabilistic analysis. In this regard, this paper presents a new procedure for the reliability analysis involving implicit limit state surface. It is shown that the approach is fast and efficient in the determination of design point and manipulable in the space of original variables. It helps extend the strength-reduction finite element stability analysis of slopes using stand-alone numerical codes (e.g., PLAXIS) into probabilistic assessment. The proposed approach was then applied to a well-known case history – the Clarence Cannon Dam.

The strength-reduction FE analysis of Clarence Cannon Dam showed that there is an almost circular failure band if deterministic (mean) values of the strength parameters were used. Moreover, the F_s by FE analysis was consistent with that by LEM with circular slip surface. When probabilistic assessment was conducted, it was shown that the probabilistic critical failure band could be very non-circular. This is very different from deterministic FE analysis.

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Appendix: derivation of the recursive algorithm FORM in the space of original random variables

Consider the basic random variables in x-space are subjected to a correlation matrix $\boldsymbol{R}$. Cholesky decomposition of $\boldsymbol{R}$ yields a lower triangular matrix $\boldsymbol{L}$ and an upper triangular matrix $\boldsymbol{U} = \boldsymbol{L}^T$ and the two relationships $\boldsymbol{u} = \boldsymbol{L}^{-1}\boldsymbol{n}$ and $\boldsymbol{n} = \boldsymbol{L}\boldsymbol{u}$ can be obtained, where $\boldsymbol{n}$ and $\boldsymbol{u}$ denote the random variable vector $\boldsymbol{x}$'s transformation into n-space and u-space, respectively. Since $\boldsymbol{L}$ is a lower triangular matrix, the derivative of the performance function has the chain rule of the form

$$\frac{\partial g(\cdot)}{\partial u_{\mathrm{k,i}}} = \frac{\partial g(\cdot)}{\partial n_{\mathrm{k,i}}} \cdot \frac{\partial n_{\mathrm{k,i}}}{\partial u_{\mathrm{k,i}}} + \frac{\partial g(\cdot)}{\partial n_{\mathrm{k,i+1}}} \cdot \frac{\partial n_{\mathrm{k,i+1}}}{\partial u_{\mathrm{k,i}}} + \cdots + \frac{\partial g(\cdot)}{\partial n_{\mathrm{k,i}}} \cdot \frac{\partial n_{\mathrm{k,m}}}{\partial u_{\mathrm{k,i}}} \tag{A1}$$

where $u_{k,i}$ and $n_{k,i}$ are respectively the ith component of the vector $\boldsymbol{u}_k$ and $\boldsymbol{n}_k$, m is the total number of variables.

The gradient vector of performance function can be translated from u-space to n-space, such that

$$\begin{Bmatrix} \dfrac{\partial g(\cdot)}{\partial u_{\mathrm{k,1}}} \\ \vdots \\ \dfrac{\partial g(\cdot)}{\partial u_{\mathrm{k,m}}} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial n_{\mathrm{k,1}}}{\partial u_{\mathrm{k,1}}} & \cdots & \dfrac{\partial n_{\mathrm{k,m}}}{\partial u_{\mathrm{k,1}}} \\ 0 & \ddots & \vdots \\ 0 & 0 & \dfrac{\partial n_{\mathrm{k,m}}}{\partial u_{\mathrm{k,m}}} \end{bmatrix} \cdot \begin{Bmatrix} \dfrac{\partial g(\cdot)}{\partial n_{\mathrm{k,1}}} \\ \vdots \\ \dfrac{\partial g(\cdot)}{\partial n_{\mathrm{k,m}}} \end{Bmatrix} = \boldsymbol{L}^T \cdot \begin{Bmatrix} \dfrac{\partial g(\cdot)}{\partial n_{\mathrm{k,1}}} \\ \vdots \\ \dfrac{\partial g(\cdot)}{\partial n_{\mathrm{k,m}}} \end{Bmatrix} \tag{A2}$$

or

$$\nabla g(\boldsymbol{u}_k) = \boldsymbol{L}^T \nabla g(\boldsymbol{u}_k) \tag{A3}$$

Multiplying $\boldsymbol{L}$ to both sides of Eq. (3), and using Eq. (A3), the recursive algorithm is reformulated to be

$$\boldsymbol{n}_{k+1} = \frac{1}{\nabla g(\boldsymbol{n}_k)^T \boldsymbol{R} \nabla g(\boldsymbol{n}_k)} \left[\nabla g(\boldsymbol{n}_k)^T \boldsymbol{n}_k - g(\boldsymbol{n}_k) \right] \boldsymbol{R} \nabla g(\boldsymbol{n}_k) \tag{A4}$$

where $\boldsymbol{n}_k$ is the kth iteration point in n-space, $g(\boldsymbol{n}_k)$ and $\nabla g(\boldsymbol{n}_k)$ are the performance function and gradient vector of the performance function evaluated at $\boldsymbol{n}_k$, respectively, and $\boldsymbol{R}$ is the correlation matrix.