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Stress state around cylindrical cavities in transversally isotropic rock mass

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Abstract. The present paper is dealing with the investigation of the stress field around the infinitely long cylindrical cavity, of a circular cross section, contained in the transversally isotropic elastic continuum. Investigation is based upon the determination of the stress function that satisfies the biharmonic equation, for the given boundary conditions and for rotationaly symmetric loading. The solution of the partial differential equation of the problem is given in the form of infinite series of Bessel's functions. Determination of the stress-strain field around cavities is a common requirement for estimation of safety of underground rock excavations.

Keywords: cylindrical cavity; stress state; rotationally symmetric loading; functions of loading; transversally isotropic medium

1. Introduction

An investigation of the influence of heterogeneities (inclusion, cavities, cracks...) on the effective properties of materials is of a great interest in applied mathematics and computational mechanics and a vast amount of literature covers this subject. Cavities are good approximations for modeling voids existing in natural materials, such as geo-materials. Particular practical importance of such investigations is related to determination of stress-strain fields around unsupported or supported cavities in a solid rock mass created by excavations for underground structures. These investigations enable a better understanding of interaction between underground structure and rock medium in three-dimensional conditions. The practical consequences that may be derived from the evaluation of disturbances of stresses and strains in vicinity of the cavities formed by underground excavations in solid rocks are related to the essential requirements for the safety of the tunneling works, particularly in cases where three-dimensional geometry of the cavities is playing significant role in the stress-strain changes (Lukić *et al*. 2010, Tomanovic 2012). Determination of the stress-strain state around a spherical, cylindrical and elliptical cavity (oblong ellipsoid) situated in elastic continuum, with unsupported internal boundary, has been recently considered in the work (Lukić *et al*. 2010). Also, in this paper a brief historical review of

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investigation of the mechanical properties of solids containing spherical, cylindrical and elliptical cavities is given. The researches particularly emphasized are: Eshelby (1959), and the authors of recent studies, (Chen 2004, Chen *et al*. 2003, Chen and Lee 2002, Dong *et al*. 2003, Duan *et al*. 2005, Lukić *et al*. 2009, Markenscoff 1998a, 1998b, Ou *et al*. 2008, 2009, Rahman 2002).

An initial determination of stress state around a cylindrical cavity is given in the work, Kirsch (1898), related to 2D problem. The solution in Schiff (1883) is based on a special transcendental equation that contains Bessel`s functions, and Fourier`s approach for division of variables. The solution of the derived equation for isotropic solid body is given in Prokopov (1949). The case with arbitrary loading perpendicular to the internal cavity surface was considered in Lur'e (1970). The solution of stresses around cylindrical cavity was given in Podil'chuk (1984) using infinite series that contain Bessel's functions. The more recent work in this field, presented in Chen (2004), dealing with a cylindrical bar that contains a spherical cavity or rigid inclusion, is based on the eigenfunction expansion variational method. Also, the work (Jabbari *et al*. 2008), even though related to a different topic, is using the generalized Bessel's function and direct method to solve the Navier's equations using the complex Fourier's series. The Improved Element-Free Galerkin method for solution the three-dimensional problems in linear elasticity is presented in Zhang and Liew (2010).

Relatively large interest exists in investigation of various aspects of cylindrical and spherical cavity expansion problems which are, to some extent, close to considered analysis. For instance, the paper (Zhang *et al*. 2009) analyses an elastoplastic cylindrical cavity expansion due to the anisotropic initial stress, (Hao *et al*. 2010) analyses the cylindrical cavity expansion with linear softening behavior, (Zou *et al*. 2010) considers the unified elasto plastic solution for cylindrical cavity expansion considering large strain and drainage conditions, while (Wang *et al*. 2007) considers the similar problem of expansion in elasto plastic brittle materials. The paper (Xue *et al*. 2009) considers the influence of the initial radius on expansion of cylindrical cavities, while (Qi *et al*. 2009) presents an unified analytical solutions for cylindrical cavity expansion in saturated soil,

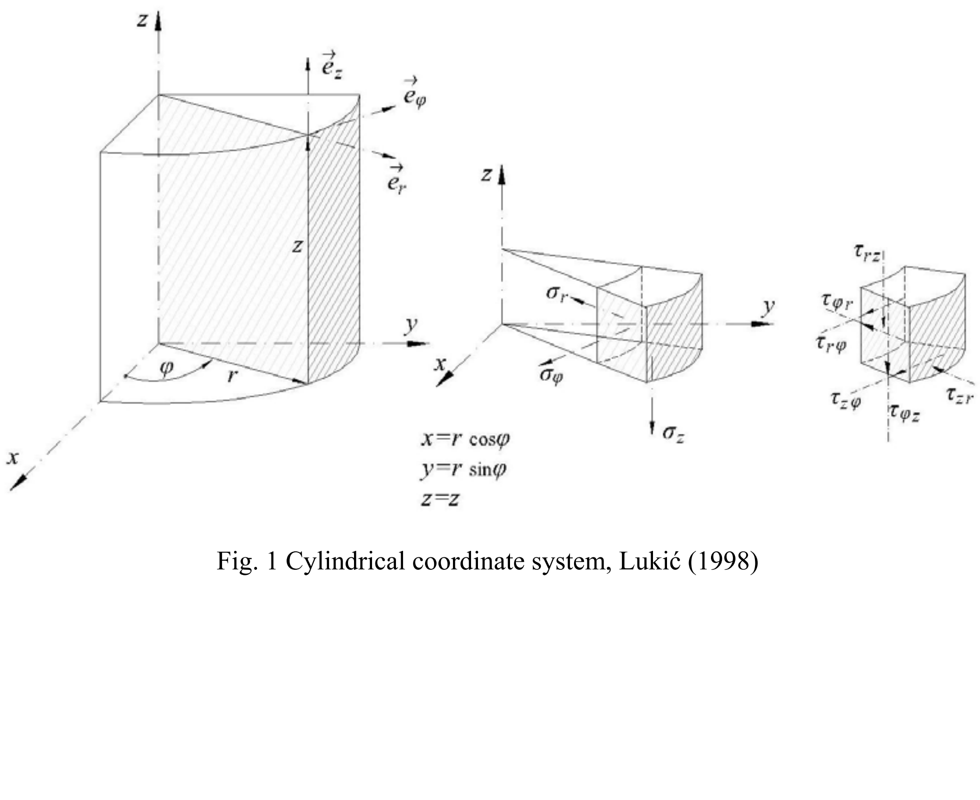

Fig. 1 Cylindrical coordinate system, Lukić (1998)

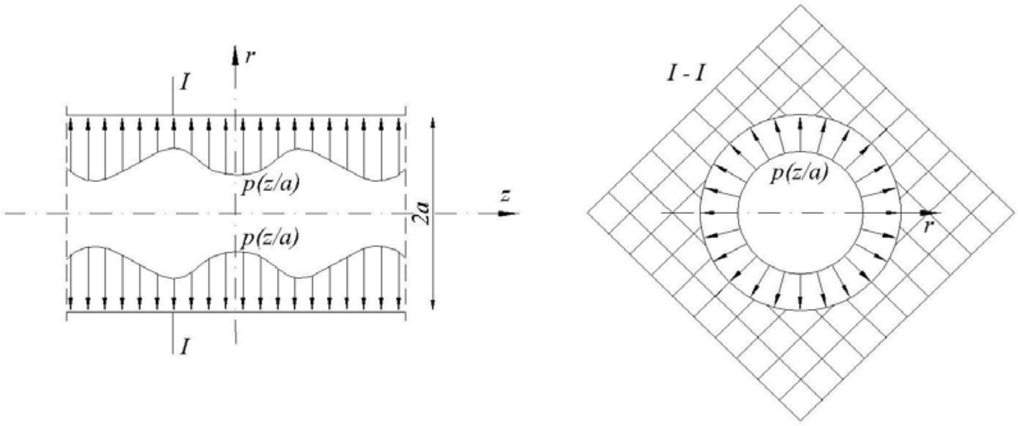

Fig. 2 Rotationally symmetric stress of a cylinder with infinite length in a transvresally isotropic medium

large deformations and undrained conditions. Finally, Zhao (2011) generalizes the conventional theory of cavity expansion, both cylindrical and spherical, to account for the effect of microstructure. Silvestri and Abou-Samra (2012) presented a method which may be employed whenever the soil can be modelled using the modified Cam clay.

The work (Karinski *et al*. 2009) is treating the stress analysis around an underground opening with sharp corners due to non-symmetric surface loads. The paper is analyzing the stress distribution and particularly the stress concentrations developed at sharp corners, using the BIE method. The present paper is an extension of the authors' previous work, (Lukić *et al*. 2010).

2. Axially symmetric stresses for a transversally isotropic medium

In the cylindrical coordinates (r, φ, z) (see Fig. 1), the operator ∇^2 is defined by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{\partial z^2} \tag{1}$$

A problem of partial, arbitrary, rotationally symmetrical stress of a cylinder with infinite length in a transversally isotropic medium, see Fig. 2, could be presented by homogenous partial differential equation of the fourth order ($\nabla^2\nabla^2\Phi = 0$), like the one used for homogenous and isotropic medium, in which the stress function Φ (r,z) is used.

2.1 Differential equation

If cylindrical coordinate system is used, with coordinates (r, φ, z) (see Fig. 1), where z is an axis of symmetry, the generalized Hooke's law for the case of symmetric stresses and for homogenous transversally isotropic medium could be given as follows

$$\varepsilon_r = \frac{\sigma_r}{E_r} - \nu_r \frac{\sigma_\varphi}{E_r} - \nu_z \frac{\sigma_z}{E_z}$$
$$\varepsilon_\varphi = \frac{\sigma_\varphi}{E_\varphi} - \nu_r \frac{\sigma_r}{E_r} - \nu_z \frac{\sigma_z}{E_z} \,, \quad \gamma_{rz} = \frac{1}{G}\tau_{rz} \tag{2}$$
$$\varepsilon_z = \frac{\sigma_z}{E_z} - \nu_r \frac{\sigma_r}{E_r} - \nu_r \frac{\sigma_\varphi}{E_r}$$

while the other componental deformations $\gamma_{r\varphi}$ and $\gamma_{\varphi z}$ are zero due to symmetry. Besides the obvious notation for strains and stresses in Eqs. (2), E_r is the elasticity modulus in radial direction, E_z is the elasticity modulus in axial direction, and ν_r and ν_z are the corresponding Poisson's coefficients, while G is the shear modulus.

In the generalized Hooke's law, given by Eq. (2), five constants are present. They are not mutually independent. Using the Maxwell-Betti's relation one obtains the next correlation

$$\nu_r E_z = \nu_z E_r \tag{3}$$

Thus, the number of mutually independent constants is four. However, in the following text it will be assumed that the Poisson's coefficients ν_r and ν_z are equal to zero. Under such assumption relations (2) are somewhat simplified and easier to handle, but also, obtained results under such assumption are closer to realistic conditions and results obtained by in situ meassurements made by the radial press. Now, assuming that ν_r and ν_z equal to zero, the generalized Hooke's law might be written as follows

$$\varepsilon_r = \frac{\sigma_r}{E_r} \quad \varepsilon_\varphi = \frac{\sigma_\varphi}{E_r} \quad \varepsilon_z = \frac{\sigma_z}{E_z} \quad \gamma_{rz} = \frac{1}{G}\tau_{rz} \tag{4}$$

Componental deformations for the case of axially symmetric stresses, for the cylindrical coordinate system, could be written as follows (see, for example, Klindukhov 2009)

$$\varepsilon_r = \frac{\partial u}{\partial r} \quad \varepsilon_\varphi = \frac{u}{r} \quad \varepsilon_z = \frac{\partial w}{\partial z} \quad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \tag{5}$$

The equilibrium conditions are given as

$$\begin{aligned} &\frac{\partial}{\partial r}(r\,\sigma_r) + \frac{\partial}{\partial z}(r\,\tau_{rz}) - \sigma_\varphi = 0 \\ &\frac{\partial}{\partial r}(r\,\tau_{rz}) + \frac{\partial}{\partial z}(r\,\sigma_z) = 0 \end{aligned} \tag{6}$$

If equilibrium equations are expressed through componental dilatations, by using Eqs. (4) and (5), the following equations are obtained.

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{G}{E_r}\frac{\partial^2 u}{\partial z^2} - \frac{u}{r^2} + \frac{G}{E_r}\frac{\partial^2 w}{\partial r\,\partial z} = 0 \\ &\frac{\partial^2 u}{\partial r\,\partial z} + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{E_z}{G}\frac{\partial^2 w}{\partial z^2} = 0 \end{aligned}\right\} \tag{7}$$

Eqs. (7) may be written in a form more convenient for further considerations

$$\left.\begin{aligned} & D_1^2 u + \frac{G}{E_r}\frac{\partial}{\partial r}e - \left(1 - \frac{G}{E_r}\right)\frac{u}{r^2} = 0 \\ & D_2^2 w + \frac{\partial}{\partial z}e = 0 \end{aligned}\right\} \tag{8}$$

where D_1^2 and D_2^2 are the differential operators given as

$$D_1^2 = \left(1 - \frac{G}{E_r}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{G}{E_r}\frac{\partial^2}{\partial z^2} \tag{9}$$

$$D_2^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \left(\frac{E_z}{G} - 1\right)\frac{\partial^2}{\partial z^2} \tag{10}$$

Also, e is the cubic dilatation expressed through componental displacements

$$e = \varepsilon_r + \varepsilon_\varphi + \varepsilon_z = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} \tag{11}$$

If the componental displacements are expressed using the two new unknown functions $\zeta\,(r, z)$ and $\psi\,(r, z)$ as follows

$$\left.\begin{aligned} & u = \left(1 - \frac{G}{E_r}\right)\frac{\partial \zeta}{\partial r} + 2\frac{\partial \psi}{\partial z} \\ & w = \frac{G}{E_r}\frac{\partial \zeta}{\partial z} - \frac{2}{r}\frac{\partial (r\psi)}{\partial r} \end{aligned}\right\} \tag{12}$$

then, using the expression (12) and notation (9), the cubic dilatation might be written as

$$e = D_1^2 \zeta \tag{13}$$

Inserting expressions (12) into Eq. (8), the first equilibrium equation may be expressed in terms of functions ζ and ψ

$$\left[D_1^2 - (1 - \frac{G}{E_r})\right]\left[\frac{\partial \zeta}{\partial r} + 2\frac{\partial \psi}{\partial z}\right] = 0 \tag{14}$$

The functions ζ and ψ could be expressed by the new function $\Phi\,(r, z)$, so that the first equilibrium equation becomes identically satisfied

$$\zeta = 2\frac{\partial \Phi}{\partial z} \qquad \psi = -\frac{\partial \Phi}{\partial r} \tag{15}$$

Inserting (12) and (15) into the second equilibrium Eq. (8), one obtains the differential equation of the problem

$$D_2^2\left(\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{G}{E_r}\frac{\partial^2\Phi}{\partial z^2}\right)+\frac{\partial^2}{\partial z^2}D_1^2\Phi=0 \tag{16}$$

If the following notation is used

$$D_3^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{G}{E_r}\frac{\partial^2}{\partial z^2} \tag{17}$$

then the differential Eq. (16) obtains a simple form

$$\left[D_2^2D_3^2+\frac{\partial^2}{\partial z^2}D_1^2\right]\Phi=0 \tag{18}$$

Inserting $E_r = E_z = E$ and $G = E/2$, and since from the beginning there is an assumption that Poisson's coefficients are zero, a known differential equation of axially symmetric stress for isotropic medium is obtained

$$D^2D^2\Psi=0 \tag{19}$$

where D stands for differential operator of the form

$$D=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2} \tag{20}$$

Differential Eq. (18) could be significantly simplified if the connection is established between modules of elasticity E_r and E_z and the shear modulus G

$$G=\frac{E_rE_z}{E_r+E_z} \tag{21}$$

Relation (21) is obtained from the St-Venant's formula for transverse isotropic elasticity assuming that the Poisson's coefficients are equal to zero. Such assumption leads to simplification of equations for transversally isotropic rock mass and also obtaining the results for the specific problem of the radial press. In doing so the authors had in mind that for the isotropic continuum this problem has the exact solution.

Inserting relation (21) into Eq. (18) differential equation for this special case could be easily obtained as

$$D^2D_4^2\Phi=0 \tag{22}$$

or in the expanded form

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+k^2\frac{\partial^2}{\partial z^2}\right)\Phi=0 \tag{23}$$

where

$$k^2=\frac{E_z}{E_r}\quad\text{and}\quad D_4^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+k^2\frac{\partial^2}{\partial z^2} \tag{24}$$

when the shear modulus G is an independent parameter, componental displacements and stresses displayed by function Φ are given as

$$u = -\frac{2G}{E_r}\frac{\partial}{\partial z}\frac{\partial \Phi}{\partial r} \quad w = 2\left[D^2\Phi + (\frac{G}{E_r} - 1)\frac{\partial^2 \Phi}{\partial z^2}\right] \tag{25}$$

$$\left.\begin{aligned}
&\sigma_r = -2G\frac{\partial}{\partial z}\frac{\partial^2\Phi}{\partial r^2}; \quad \sigma_\varphi = -2G\frac{\partial}{\partial z}\frac{1}{r}\frac{\partial \Phi}{\partial r}\\
&\sigma_z = 2E_z\frac{\partial}{\partial z}\left[D^2\Phi + (\frac{G}{E_r} - 1)\frac{\partial^2}{\partial z^2}\Phi\right]\\
&\tau = 2G\frac{\partial}{\partial r}\left[D^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right]
\end{aligned}\right\} \tag{26}$$

Componental displacements and stresses for the special case given by Eq. (21) are

$$\left.\begin{aligned}
&u = -2\frac{E_z}{E_r + E_z}\frac{\partial}{\partial z}\frac{\partial \Phi}{\partial r}\\
&w = 2\left[D^2\Phi + \frac{E_r}{E_r + E_z}\frac{\partial^2\Phi}{\partial z^2}\right]
\end{aligned}\right\} \tag{27}$$

$$\left.\begin{aligned}
&\sigma_r = -2\frac{E_r E_z}{E_r + E_z}\frac{\partial}{\partial z}\frac{\partial^2\Phi}{\partial r^2}\\
&\sigma_\varphi = -2\frac{E_r E_z}{E_r + E_z}\frac{\partial}{\partial z}\frac{1}{r}\frac{\partial\Phi}{\partial r}\\
&\sigma_z = 2\frac{E_r E_z}{E_r + E_z}\frac{\partial}{\partial z}\left[(1+k^2)D^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right]\\
&\tau = 2\frac{E_r E_z}{E_r + E_z}\frac{\partial}{\partial r}\left[D^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right]
\end{aligned}\right\} \tag{28}$$

In the further text the solution of differential Eq. (22) is shown, which is obtained by using the connection (21).

2.2 The solution of differential equation

When the solution is assumed in the form of trigonometric series

$$\Phi = \sum_{\lambda=1}^{\infty} f(r)\sin\lambda z \tag{29}$$

the partial differential Eq. (23) becomes an ordinary differential equation

$$E_1^2 E_2^2\ f = 0 \tag{30}$$

where E_1^2 and E_2^2 are differential operators given as

$$E_1^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \lambda^2 k^2 \tag{31}$$

$$E_2^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \lambda^2 \tag{32}$$

Apart from Eq. (23), i.e., Eq. (30), the function Φ must also satisfy the boundary conditions

$$\sigma_r = \sigma_\varphi = \sigma_z = \tau = 0 \tag{33}$$

for $r \to \infty$ and

$$\sigma_r = p\left(\frac{z}{a}\right) \quad \text{and} \quad \tau = 0 \tag{34}$$

for $r = a$. Introducing the notation.

$$E_2^2 f(r) = R(r) \tag{35}$$

and inserting it into (30), Eq. (30) will be certainly satisfied if

$$E_1^2 R = 0 \quad \text{or} \quad \frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \lambda^2 k^2 R = 0 \tag{36}$$

This is the well known Bessel's equation, and its general solution is

$$R = C_1 K_0(k\lambda r) + C_2 I_0(k\lambda r) \tag{37}$$

where $K_0(r)$ and $I_0(r)$ are the Bessel's functions of the order zero and of a complex argument, while C_1, C_2 are constants.

Inserting expression for R from Eq. (37) into Eq. (35), one obtains

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \lambda^2 f = C_1 K_0(k\lambda r) + C_2 I_0(k\lambda r) \tag{38}$$

If introducing the notation given by Eq. (39)

$$\gamma\left(k\lambda r\right) = C_1 K_0(k\lambda r) + C_2 I_0(k\lambda r) \tag{39}$$

then Eq. (38) may be written as follows

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \lambda^2 f = \gamma\left(k\lambda r\right) \tag{40}$$

Therefore, the Bessel's equation is obtained again, but it is the non-homogeneous now. The general solution for the homogeneous equation is known (L_1 and L_2 are constants)

$$f^1 = L_1 I_0(\lambda r) + L_2 K_0(\lambda r) \tag{41}$$

The particular integral of the non-homogeneous equation may be easily found using the Cauchy's method. The method says that if L_1I_0 (λr) and L_2K_0 (λr) are linearly independent particular integrals of homogeneous differential Eq. (40) for a certain value $r = \alpha$, then the constants L_1 and L_2 can be found, if

$$f(\lambda r) = 0 \quad \text{and} \quad \frac{d}{dr} f(\lambda r) = 1 \tag{42}$$

for $r = a$ and

$$L_1 I_0(\lambda\alpha) + L_2 K_0(\lambda\alpha) = 0 \tag{43}$$

$$\lambda\, L_1 I_1(\lambda\alpha) - \lambda\, L_2 K_1(\lambda\alpha) = 1 \tag{44}$$

$$L_1 = \alpha\, K_0(\lambda\alpha) \qquad L_2 = -\alpha\, I_0(\lambda\alpha) \tag{45}$$

The particular integral than could be written in the form of a definite integral

$$f_p = I_0(\lambda r)\int_{\alpha}^{r} \alpha\, K_0(\lambda\alpha)\gamma(k\lambda\alpha)\,d\alpha - K_0(\lambda r)\int_{\alpha}^{r} \alpha\, I_0(\lambda\alpha)\gamma(k\lambda\alpha)\,d\alpha \tag{46}$$

In a developed form and considering first that $\gamma(k\lambda r) = C_1K_0(k\lambda r)$, one obtains

$$f_p^1 = C_1\left[I_0(\lambda r)\int_{\alpha}^{r} \alpha\, K_0(\lambda\alpha)K_0(k\lambda\alpha)\,d\alpha - K_0(\lambda r)\int_{\alpha}^{r} \alpha\, I_0(\lambda\alpha)K_0(k\lambda r)\,d\alpha \right] \tag{47}$$

than, taking $\gamma(k\lambda r) = C_2I_0(k\lambda r)$, one obtains

$$f_p^2 = C_2\left[I_0(\lambda r)\int_{\alpha}^{r} \alpha\, K_0(\lambda\alpha)I_0(k\lambda\alpha)\,d\alpha - K_0(\lambda r)\int_{\alpha}^{r} \alpha\, I_0(\lambda\alpha)I_0(k\lambda\alpha)\,d\alpha \right] \tag{48}$$

It could be seen that particular integrals become integrals of the Lommel`s type, Watson (1922), 'so the final integration is possible

$$f_p^1 = C_1 \frac{1}{\lambda^2\left(1-k^2\right)}\left[K_0(k\lambda r) - D_1 K_0(\lambda r) + D_2 I_0(\lambda r)\right] \tag{49}$$

$$f_p^2 = C_2 \frac{1}{\lambda^2 \left(1-k^2\right)} \left[I_0(k\lambda r) - D_3 I_0(\lambda r) + D_4 K_0(\lambda r) \right] \tag{50}$$

The general integral of Eq. (30) now can be obtained in the form

$$\begin{aligned} f^T = {} & C_1 \frac{1}{\lambda^2 \left(1-k^2\right)} \left[K_0(k\lambda r) - D_1 K_0(\lambda r) + D_2 I_0(\lambda r) \right] \\ & + C_2 \frac{1}{\lambda^2 \left(1-k^2\right)} \left[I_0(k\lambda r) - D_3 I_0(\lambda r) + D_4 K_0(\lambda r) \right] + C_3 I_0(\lambda r) + C_4 K_0(\lambda r) \end{aligned} \tag{51}$$

Constants $C_{1\text{-}4}$ are arbitrary, unlike constants $D_{1\text{-}4}$ which are not, and in our case have this values

$$D_1 = D_3 = 1, \quad D_2 = D_4 = 0 \tag{52}$$

Therefore, the general integral for our case can be written in the final form

$$\begin{aligned} f^T = {} & C_1 \frac{1}{\lambda^2 \left(1-k^2\right)} \left[K_0(k\lambda r) - K_0(\lambda r) \right] \\ & + C_2 \frac{1}{\lambda^2 \left(1-k^2\right)} \left[I_0(k\lambda r) - I_0(\lambda r) \right] + C_3 I_0(\lambda r) + C_4 K_0(\lambda r) \end{aligned} \tag{53}$$

If one examines the behavior of $K_0(r)$ and $I_0(r)$ when $r \to \infty$, by the asymptotic development

$$I_0(t) \approx (2\pi t)^{-\frac{1}{2}} e^t, \quad K_0(t) \approx (2\pi t)^{\frac{1}{2}} e^{-t} \tag{54}$$

one might conclude that the part of the solution in (37), namely $C_2 I_0(k\lambda r)$, can not satisfy the boundary condition for $r \to \infty$, i.e., the condition (33), since for $r \to \infty$ the functions $I_0(k\lambda r)$ obtain an infinite value, while the functions $K_0(k\lambda r)$ tend to zero. Therefore, in the solution (37) one could use only the particular solution $C_1 K_0(k\lambda r)$. After some mathematical derivations, and with condition that due to previous analysis one excludes the particular integrals containing $I_0(k\lambda r)$, one could write the solution of differential Eq. (23) which corresponds to imposed boundary conditions.

It can be easily seen that the integral

$$f_I^T = C_1 \frac{1}{\lambda^2 \left(1-k^2\right)} \left[K_0(k\lambda r) - K_0(\lambda r) \right] + C_4 K_0(\lambda r) \tag{55}$$

satisfies the above given conditions in infinity and with its two free constants represents the solution for the states of stresses and deformations of infinite mass, from which the cylinder of infinite length is removed, and is loaded with rotationally symmetric load along the axis of removed cylinder, and for transversally isotropic mediums.

Integral

$$f_{II}^{T} = C_2 \frac{1}{\lambda^2\left(1-k^2\right)}\left[I_0\left(k\lambda r\right) - I_0\left(\lambda r\right)\right] + C_3 I_0\left(\lambda r\right) \tag{56}$$

is the solution of the case of a full cylinder loaded by the rotationally symmetric loading distributed over its rim, for transversally isotropic medium. The sum of those integrals given by (53) with four free constants can be used as a solution for a tube of arbitrary width. From this general integrals one can easily obtain the solution for above mentioned cases, and for homogeneous isotropic medium we should assume that $k \to 1$, i.e., $E_r = E_z$. For integral given by Eq. (55) one obtains

$$f_I^1 = C_4 K_0\left(\lambda r\right) - \frac{C_1}{2\lambda} r\, K_1\left(\lambda r\right) \tag{57}$$

Also, for the integral given by Eq. (56) one obtains

$$f_{II}^1 = C_3 I_0\left(\lambda r\right) + \frac{C_2}{2\lambda} r\, I_1\left(\lambda r\right) \tag{58}$$

so the sum of integrals (57) and (58) is given by

$$f^1 = \frac{r}{2\lambda}\left[C_2 I_1\left(\lambda r\right) - C_1 K_1\left(\lambda r\right)\right] + C_3 I_0\left(\lambda r\right) + C_4 K_0\left(\lambda r\right) \tag{59}$$

According to (29), the solution of Eq. (23) could be written in a form which corresponds to the assumed boundary conditions

$$\Phi = \sum_{\lambda=0}^{\infty}\left\{C_4 K_0\left(\lambda r\right) + C_1 \frac{1}{\lambda^2\left(1-k^2\right)}\left[K_0\left(k\lambda r\right) - K_0\left(\lambda r\right)\right]\right\} \sin\lambda z \tag{60}$$

Inserting the function Φ given by (60) into (35) one obtains

$$\left.\begin{aligned}
\sigma_r &= \frac{2E_r E_z}{E_r + E_z}\sum_{\lambda=0}^{\infty} -\lambda^2 \cos\lambda z \left\{ C_4\left[K_1(\lambda r)\frac{1}{r} + \lambda K_0(\lambda r)\right] + \right. \\
&\quad \left. + C_1 \frac{1}{\lambda(1-k^2)}\left[\frac{k}{\lambda r}K_1(k\lambda r) + k^2 K_0(k\lambda r) - K_1(\lambda r)\frac{1}{\lambda r} - K_0(\lambda r)\right]\right\} \\
\sigma_\varphi &= \frac{2E_r E_z}{E_r + E_z}\sum_{\lambda=0}^{\infty}\lambda^2 \cos\lambda z \left\{ C_4 \frac{K_1(\lambda r)}{r} + C_1 \frac{1}{\lambda(1-k^2)}\left[\frac{k}{\lambda r}K_1(k\lambda r) - \frac{K_1(\lambda r)}{\lambda r}\right]\right\} \\
\sigma_z &= \frac{2E_r E_z}{E_r + E_z}\sum_{\lambda=0}^{\infty}\lambda^2 \cos\lambda z \left\{ \frac{C_1}{\lambda(1-k^2)}\left[k^4 K_0(k\lambda r) - K_0(\lambda r)\right] + \lambda C_4 K_0(\lambda r)\right\} \\
\tau &= \frac{2E_r E_z}{E_r + E_z}\sum_{\lambda=0}^{\infty} -\lambda^2 \sin\lambda z \left\{ \frac{C_1}{\lambda(1-k^2)}\left[k^3 K_1(k\lambda r) - K_1(\lambda r)\right] + \lambda C_4 K_1(\lambda r)\right\}
\end{aligned}\right\} \tag{61}$$

From Eq. (61) one could see that the condition is fulfilled at infinity, i.e., all stresses become zero when $r \to \infty$, because each function tends to zero when $r \to \infty$.

To fulfill the conditions on boundary we have two free constants C_4 and C_1. The connection between constants is obtained from $\tau = 0$ for $r = a$

$$C_4 = \frac{C_1}{\lambda^2(1-k^2)}\left[1 - k^3\frac{K_1(k\lambda r)}{K_1(\lambda r)}\right] \tag{62}$$

If we use the notation $\lambda = \alpha/a$ and $r = a$, one obtains

$$C_4(\alpha) = \frac{C_1(\alpha)\,a^2}{\alpha^2(1-k^2)}\left[1 - k^3\frac{K_1(k\alpha)}{K_1(\alpha)}\right] \tag{63}$$

Using the condition that radial stress for $r = a$ equals to the given external load, we obtain

$$\sigma_r = \frac{2E_rE_z}{E_r+E_z}C_1(\alpha)\sum_{\alpha=0}^{\infty}\frac{k}{a\,(1-k^2)}\cos\frac{\alpha}{a}z \times \left\{K_1(k\alpha)\left[1-k^2\left(1+\alpha\frac{K_0(\alpha)}{K_1(\alpha)}\right)\right]+\alpha\,kK_0(k\alpha)\right\} \tag{64}$$

Loading can always be given sufficiently exactly by the Furrier's integral. In addition, if we consider the load as symmetric with respect to origin of the coordinate system, that is, as the even function, we obtain

$$p\left(\frac{z}{a}\right) = \frac{2}{\pi\,a}\int_0^{\infty}\cos\frac{\alpha\,z}{a}d\alpha\int_0^{\infty}f\left(\frac{z}{a}\right)\cos\frac{\alpha\,z}{a}dz \tag{65}$$

In expression (64) the sign of $\sum\limits_{\alpha=0}^{\infty}$ can be changed with definite integral $\int\limits_0^{\infty}$ considering C_1 as a function of α. By comparison of expressions (64) and (65) we explicitly obtain the constant C_1 as a function of α and a function $f\,(z/a)$

$$C_1 = \frac{E_r+E_z}{E_rE_z}\frac{1}{\pi}\int_0^{\infty}\frac{1-k^2}{k}\times\frac{1}{K_1(k\alpha)\left[1-k^2\left(1+\alpha\frac{K_0(\alpha)}{K_1(\alpha)}\right)\right]+\alpha\,kK_0(k\alpha)}\times f\left(\frac{z}{a}\right)\cos\frac{\alpha\,z}{a}dz \tag{66}$$

Using expression (63), which gives the connection between constants, and expression (66), we obtain the constant C_4 in the explicit form

$$C_4 = \frac{E_r+E_z}{E_rE_z}\frac{1}{\pi}\int_0^{\infty}\frac{a^2}{k\alpha^2}\times\frac{\left(1-k^3\frac{K_1(k\alpha)}{K_1(\alpha)}\right)}{K_1(k\alpha)\left[1-k^2\left(1+\alpha\frac{K_0(\alpha)}{K_1(\alpha)}\right)\right]+\alpha\,kK_0(k\alpha)}\times f\left(\frac{z}{a}\right)\cos\frac{\alpha\,z}{a}dz \tag{67}$$

After inserting the values for constants into Eqs. (47) and (48), after a short calculaton one obtain

$$\sigma_r = -\frac{2}{\pi}\int_0^\infty \cos\frac{\alpha}{a}z \times \frac{\left(-k^2\frac{K_1(k\alpha)}{K_1(\alpha)}\right)\left[K_1\left(\frac{\alpha r}{a}\right)\frac{1}{r}+\frac{\alpha}{a}K_0\left(\frac{\alpha}{a}r\right)\right] + \frac{1}{r}K_1\left(k\alpha\frac{r}{a}\right)+k\frac{\alpha}{a}K_0\left(k\alpha\frac{r}{a}\right)}{K_1(k\alpha)\left[1-k^2\left(1+\alpha\frac{K_0(\alpha)}{K_1(\alpha)}\right)\right]+\alpha\, kK_0(k\alpha)}$$

$$\times\int_0^\infty f\left(\frac{z}{a}\right)\cos\frac{\alpha\, z}{a}dz\, d\alpha$$

$$\sigma_\varphi = \frac{2}{\pi}\int_0^\infty \cos\frac{\alpha}{a}z \times \frac{\frac{1}{r}\left[-k^2\frac{K_1(k\alpha)}{K_1(\alpha)}K_1\left(\frac{\alpha}{a}r\right)+K_1\left(k\alpha\frac{r}{a}\right)\right]}{K_1(k\alpha)\left[1-k^2\left(1+\alpha\frac{K_0(\alpha)}{K_1(\alpha)}\right)\right]+\alpha\, kK_0(k\alpha)}\times\int_0^\infty f\left(\frac{z}{a}\right)\cos\frac{\alpha\, z}{a}dz\, d\alpha \tag{68}$$

$$\sigma_z = \frac{2}{\pi}\int_0^\infty \cos\frac{\alpha}{a}z \times \frac{k^2\left[-\frac{K_1(k\alpha)}{K_1(\alpha)}K_0\left(\frac{\alpha}{a}r\right)+kK_0\left(k\alpha\frac{r}{a}\right)\right]}{K_1(k\alpha)\left[1-k^2\left(1+\alpha\frac{K_0(\alpha)}{K_1(\alpha)}\right)\right]+\alpha\, kK_0(k\alpha)}\times\int_0^\infty f\left(\frac{z}{a}\right)\cos\frac{\alpha\, z}{a}dz\, d\alpha$$

$$\tau = \frac{2}{\pi}\int_0^\infty \sin\frac{\alpha}{a}z \times k^2\frac{K_1\left(k\alpha\frac{r}{a}\right)-\frac{K_1(k\alpha)}{K_1(\alpha)}K_1\left(\frac{\alpha}{a}r\right)}{K_1(k\alpha)\left[1-k^2\left(1+\alpha\frac{K_0(\alpha)}{K_1(\alpha)}\right)\right]+\alpha\, kK_0(k\alpha)}\times\int_0^\infty f\left(\frac{z}{a}\right)\cos\frac{\alpha\, z}{a}dz\, d\alpha$$

The pure deformation, that is dilatations and shears can be easily obtained from the known expressions

$$\varepsilon_r = \frac{\sigma_r}{E_r}\ ;\quad \varepsilon_\varphi = \frac{\sigma_\varphi}{E_r}\ ;\quad \varepsilon_z = \frac{\sigma_z}{E_z}\ ;\quad \gamma_{rz} = \frac{E_r+E_z}{E_rE_z}\tau \tag{69}$$

Displacements of points are given by the following expressions

$$\varepsilon_r = \frac{\partial u}{\partial r}\ ;\quad \varepsilon_\varphi = \frac{u}{r}\ ;\quad \varepsilon_z = \frac{\partial w}{\partial z}\ ;\quad \gamma_{zr} = \frac{\partial u}{\partial z}+\frac{\partial w}{\partial r} \tag{70}$$

The states of stresses and deformations are defined in this way, for every point of the homogenous transversally isotropic medium of infinite mass from which the cylinder of an infinite length, loaded on the circumference by an arbitrary rotationally symmetric loading, is extracted.

2.3 A case of a partial uniformly distributed load along the z axis

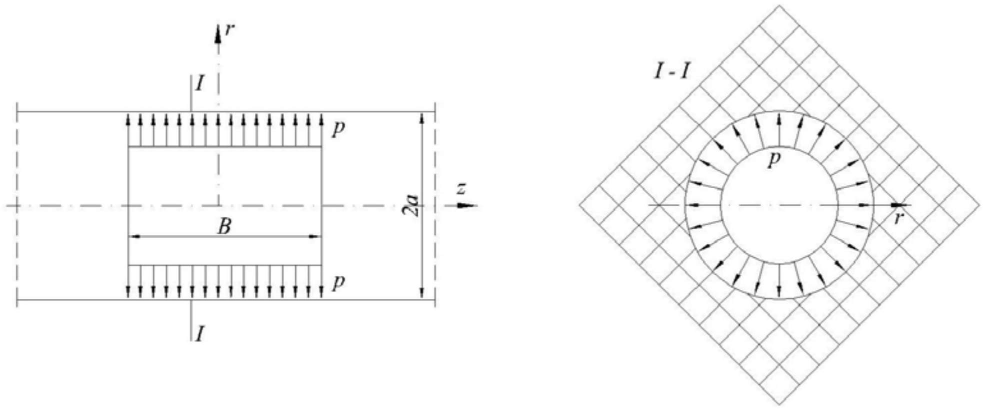

Fig. 3 Axisymmetric partial loading

The states of stresses and deformations for the case of loading shown in Fig. 3, can be easily derived from equations presented in the former section. The expression (65) applied to this case of loading given in Fig. 3 is given as

$$p\left(\frac{z}{a}\right)=\frac{2}{\pi a}\int_0^\infty \cos\frac{\alpha z}{a}d\alpha\times\int_0^{B/2} p\cos\frac{\alpha z}{a}dz \tag{71}$$

since $f(z/a)=p$. Now, Eq. (71) becomes

$$p\left(\frac{z}{a}\right)=\frac{2p}{\pi a}\int_0^\infty \frac{a}{\alpha}\sin\frac{\alpha B}{2a}\cos\frac{\alpha z}{a}d\alpha=\frac{2p}{\pi}\int_0^\infty \frac{1}{\alpha}\sin\frac{\alpha B}{2a}\cos\frac{\alpha z}{a}d\alpha \tag{72}$$

Constant C_1 for this case of loading is obtained as

$$C_1=\frac{E_r+E_z}{E_rE_z}\frac{pa}{\pi}\frac{1}{\alpha}\frac{1-k^2}{k}\sin\frac{\alpha B}{2a}\times\frac{1}{K_1(k\alpha)\left[1-k^2\left(1+\alpha\frac{K_0(\alpha)}{K_1(\alpha)}\right)\right]+\alpha kK_0(k\alpha)} \tag{73}$$

Using (63) and (73) expressions for stresses are easily obtained

$$\sigma_r=-\frac{2p}{\pi}\int_0^\infty \cos\frac{\alpha z}{a}\sin\frac{\alpha B}{2a}\times\frac{\left(-k^2\frac{K_1(k\alpha)}{K_1(\alpha)}\right)\left[K_1\left(\frac{\alpha r}{a}\right)\frac{a}{r}+\alpha K_0\left(\frac{\alpha}{a}r\right)\right]+\frac{a}{r}K_1\left(k\alpha\frac{r}{a}\right)+k\alpha K_0\left(k\alpha\frac{r}{a}\right)}{\alpha\left\{K_1(k\alpha)\left[1-k^2\left(1+\alpha\frac{K_0(\alpha)}{K_1(\alpha)}\right)\right]+\alpha kK_0(k\alpha)\right\}}d\alpha \tag{74}$$

$$\sigma_\varphi = \frac{2p}{\pi}\int_0^\infty \cos\frac{\alpha z}{a}\sin\frac{\alpha B}{2a}\times\frac{\frac{a}{r}\left[-k^2\frac{K_1(k\alpha)}{K_1(\alpha)}K_1\left(\frac{\alpha}{a}r\right)+K_1\left(k\alpha\frac{r}{a}\right)\right]}{\alpha\left\{K_1(k\alpha)\left[1-k^2\left(1+\alpha\frac{K_0(\alpha)}{K_1(\alpha)}\right)\right]+\alpha\, kK_0(k\alpha)\right\}}d\alpha$$

$$\sigma_z = \frac{2p}{\pi}\int_0^\infty \cos\frac{\alpha z}{a}\sin\frac{\alpha B}{2a}\times\frac{k^2\left[-\frac{K_1(k\alpha)}{K_1(\alpha)}K_0\left(\frac{\alpha}{a}r\right)+kK_0\left(k\alpha\frac{r}{a}\right)\right]}{K_1(k\alpha)\left[1-k^2\left(1+\alpha\frac{K_0(\alpha)}{K_1(\alpha)}\right)\right]+\alpha\, kK_0(k\alpha)}d\alpha \tag{74}$$

$$\tau = -\frac{2p}{\pi}\int_0^\infty \sin\frac{\alpha z}{a}\sin\frac{\alpha B}{2a}\times\frac{k^2\left\{K_1\left(k\alpha\frac{r}{a}\right)-\frac{K_1(k\alpha)}{K_1(\alpha)}K_1\left(\frac{\alpha}{a}r\right)\right\}}{K_1(k\alpha)\left[1-k^2\left(1+\alpha\frac{K_0(\alpha)}{K_1(\alpha)}\right)\right]+\alpha\, kK_0(k\alpha)}d\alpha$$

Upon analysis of convergency of integrals (74) one might conclude that all stresses become zero when $r \to \infty$ (integrals are convergent), i.e. the effect of cavity upon the state of stress ceases. Therefore, the boundary condition (33) for stresses when $r \to \infty$ is fulfilled. In convergency analysis of integrals (74) the tabular values of Bessel's functions up to $\alpha = 16$ were used, Watson (1922). For $\alpha > 16$ the values of functions $K_0(\alpha)$ and $K_1(\alpha)$ were determined from the asymptotic development of these functions

$$K_n(\alpha)=\left(\frac{\pi}{2\alpha}\right)^{\frac{1}{2}}e^{-\alpha}\left[1+\frac{\left(4n^2-1^2\right)}{1(8\alpha)}\left(1+\frac{\left(4n^2-3^2\right)}{2(8\alpha)}\left(1+\frac{4n^2-5^2}{3(8\alpha)}(1+...)\right)\right)\right] \tag{75}$$

The convergency of integrals for the shear stress (74) at the boundary $r = a$ was also analyzed. Due to asymptotic development of Bessel's functions ($K_0(\alpha)$ and $K_1(\alpha)$) and the theorems for convergency of the improper integrals it was shown that this condition is also fulfilled.

Radial displacement "u" expressed by stress function is given by (27). The same result will be obtained if the presumption is

$$\sigma_\varphi = E_r\varepsilon_\varphi \ , \ \ \varepsilon_\varphi = \frac{u}{r} \tag{76}$$

Radial displacement than becomes

$$u = r\frac{\sigma_\varphi}{E_r} \tag{77}$$

If the stresses from Eq. (74) are inserted into (77), displacement "u" obtains the final form

$$u = \frac{2pa}{E_r\pi}\int_0^\infty \cos\frac{\alpha z}{a}\sin\frac{\alpha B}{2a}\times\frac{\left(-k^2\frac{K_1(k\alpha)}{K_1(\alpha)}\right)\left[K_1\left(\frac{\alpha r}{a}\right)\right]+K_1\left(k\alpha\frac{r}{a}\right)}{\alpha\left\{K_1(k\alpha)\left[1-k^2\left(1+\alpha\frac{K_0(\alpha)}{K_1(\alpha)}\right)\right]+\alpha\, kK_0(k\alpha)\right\}}d\alpha \tag{78}$$

For the edge $r = a$ on the circumference of the cylinder displacement "u" is obtained directly from Eq. (78) inserting simply $r = a$

$$u = \frac{2pa}{E_r \pi}\int_0^\infty \cos\frac{\alpha z}{a}\sin\frac{\alpha B}{2a}\times\frac{K_1(k\alpha)(1-k^2)}{\alpha\left\{K_1(k\alpha)\left[1-k^2\left(1+\alpha\frac{K_0(\alpha)}{K_1(\alpha)}\right)\right]+\alpha\, kK_0(k\alpha)\right\}}d\alpha \tag{79}$$

That is

$$u = \frac{2pa}{E_r \pi}\int_0^\infty \cos\frac{\alpha z}{a}\sin\frac{\alpha B}{2a}\times\frac{1-k^2}{\alpha\left\{\left[1-k^2\left(1+\alpha\frac{K_0(\alpha)}{K_1(\alpha)}\right)\right]+\alpha\, k\frac{K_0(k\alpha)}{K_1(k\alpha)}\right\}}d\alpha \tag{80}$$

Determination of componental stresses and componental displacements, as shown in Eqs. (74) and (80), is reduced to calculating a definite integral within the bounds of 0 to ∞. Integration in the closed form is not possible. The approximate value of integral that can be acceptable for the practical purposes, can be obtained by numerical integration methods. In this case it is even more possible because the functions under integral actually represent a damped oscillation whose values for higher α are irrelevant.

Inserting the notations: $\eta = B/2a$; $\xi = z/a$ into Eq. (80), one obtains for "u"

$$u_r = \frac{2pa}{E_r \pi}\int_0^\infty \cos\alpha\xi\sin\alpha\eta\times\frac{1-k^2}{\alpha\left\{\left[1-k^2\left(1+\alpha\frac{K_0(\alpha)}{K_1(\alpha)}\right)\right]+\alpha\, k\frac{K_0(k\alpha)}{K_1(k\alpha)}\right\}}d\alpha \tag{81}$$

It is not hard to see that the integrand becomes undefined expression of the form 0/0, when $\alpha \to 0$. This limit value can be easily obtained by applying the L`Hospital`s rule onto the function under integral

$$\lim_{\alpha\to 0}\frac{(1-k^2)\cos\alpha\xi\sin\alpha\eta}{\alpha\left\{\left[1-k^2\left(1+\alpha\frac{K_0(\alpha)}{K_1(\alpha)}\right)\right]+\alpha\, k\frac{K_0(k\alpha)}{K_1(k\alpha)}\right\}} = \eta \tag{82}$$

As can be seen, the limit value of the function under integral does not depend on parameter k for $\alpha = 0$. Calculated and presented in a diagram in Fig. 4, there is a change of coefficient ψ, which in this case, apart from depending on the diameter of the cylinder "a" and the width of the loading B, depends also on parameter k, that is on the ratio of modules of elasticity. The coefficient ψ in this case is

$$\psi = \frac{2}{\pi}\int_0^\infty \cos\alpha\xi\sin\alpha\eta\times\frac{1-k^2}{\alpha\left\{\left[1-k^2\left(1+\alpha\frac{K_0(\alpha)}{K_1(\alpha)}\right)\right]+\alpha\, k\frac{K_0(k\alpha)}{K_1(k\alpha)}\right\}}d\alpha \tag{83}$$

Displacement can be also written as

$$u_{r=a} = \psi\frac{pa}{E_r} \tag{84}$$

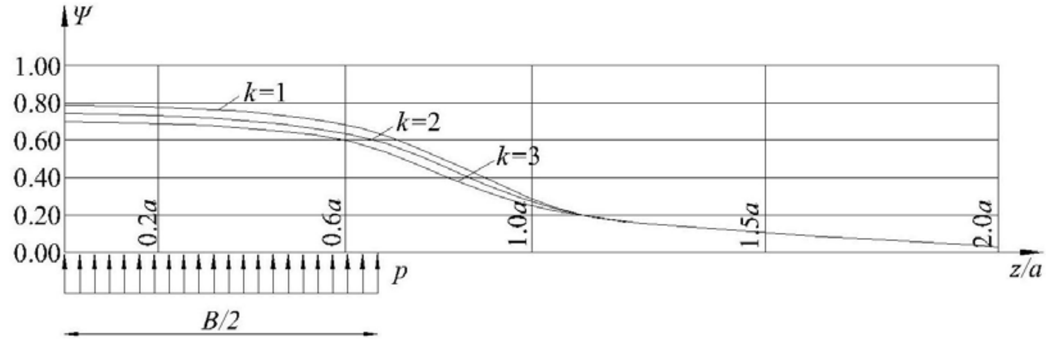

Fig. 4 Change of the coefficient ψ with respect to the ratio of modules of elasticity k
(k = 1- isotropic; k = 2, 3 - transversally isotropic)

The coefficient ψ is calculated for $\eta = B/2a = 0{,}79285$ ratio, where $2a$ is diameter of the cylinder and B is the width of loading. Given decimal ratio is taken because it corresponds to the actual construction of a radial press. The coefficient ψ is calculated for k = 2,3 since k = 1 corresponds to isotropic medium. However, the values of the coefficient ψ for k = 1 are also shown in Fig. 4, so the influence of anisotropy could be comparatively seen. Expression (84) can be used for practical purposes, that is for determination of the modulus of elasticity in radial direction E_r if parameter $k = \sqrt{E_r/E_z}$ and deformation measured on the spot are known. In this case from Eq. (84) we get

$$E_r = \psi \frac{pa}{u_0}, \quad u_0 = u_{r=a} \tag{85}$$

Similarly, the next problem could also be solved. Namely, if the known quantities are: radial displacement "u", actual deformations measured on the spot by radial press, and modulus of elasticity E_r, then the parameter k is determined via the coefficient ψ

Verification of obtained coefficients ψ is given by comparison of the calculated values of ψ with the results of measurements conducted in the tunnel of the Hydro Power Plant Mavrovo (H.E. Mavrovo), in former Yugoslavia, now FYROM (Former Yugoslav Republic of Macedonia). Measurements were made due to the partial uniform loading distributed along the tunnel axis, applied by the radial press, as presented by Lazarević and Kujundžić (1954). Together with the theoretical curves (k = 1 for the homogeneous isotropic and k = 2, 3 for the transversally isotropic medium), the curve obtained from measurements in the tunnel of the H.E. Mavrovo is also given in Fig. 5. As may be seen, measured values of ψ generally comply quite well with the theoretical ones. Values of measured ψ fall between the theoretical curves for k = 2 and k = 3, in the area where the partial loading is applied, indicating the corresponding ratio of modulii of elasticity and transverse isotropy at the particular tunnel location. Measured values of ψ in the area away from the loading, further along the axis of the tunnel, are somewhat bellow the theoretical curves.Obtained differencies in ψ away from the applied loading might be within the measurement error of the instruments, due to various reasons, e.g., temperature effects, and are of no practical consequence since the deformations at unloaded area are relatively small. Also, there are no

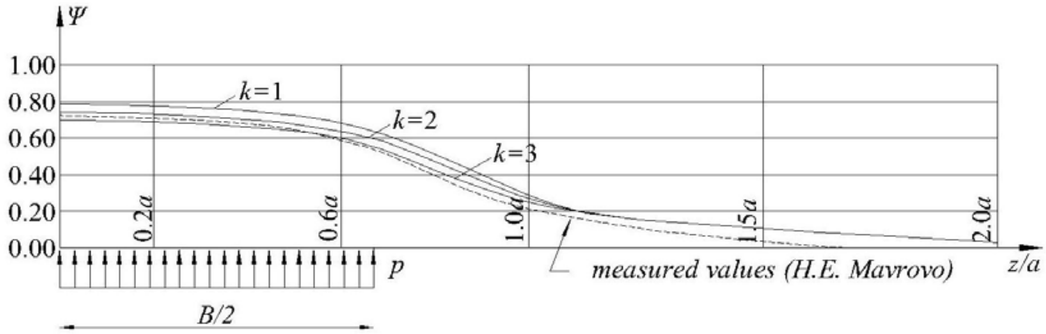

Fig. 5 Comparative analysis of theoretical solutions and experimental measurements
(k = 1 – isotropic; k = 2, 3 - transversally isotropic)

Table 1 Numerical data

Partial uniform load intensity	p = 300 kPa
Isotropic rock mass data	γ = 28 kN/m^3, ν = 0.3, E = 20 GPa
Transversally isotropic rock mass data (radial)	γ = 28 kN/m^3, ν_r = 0.30, E_r = 20 GPa
Transversally isotropic rock mass data (axial)	ν_z = 0.33, E_z = 18 GPa
Geometric characteristics	H = 100 m, r_0 = 2.0 m, r = 0.2 m

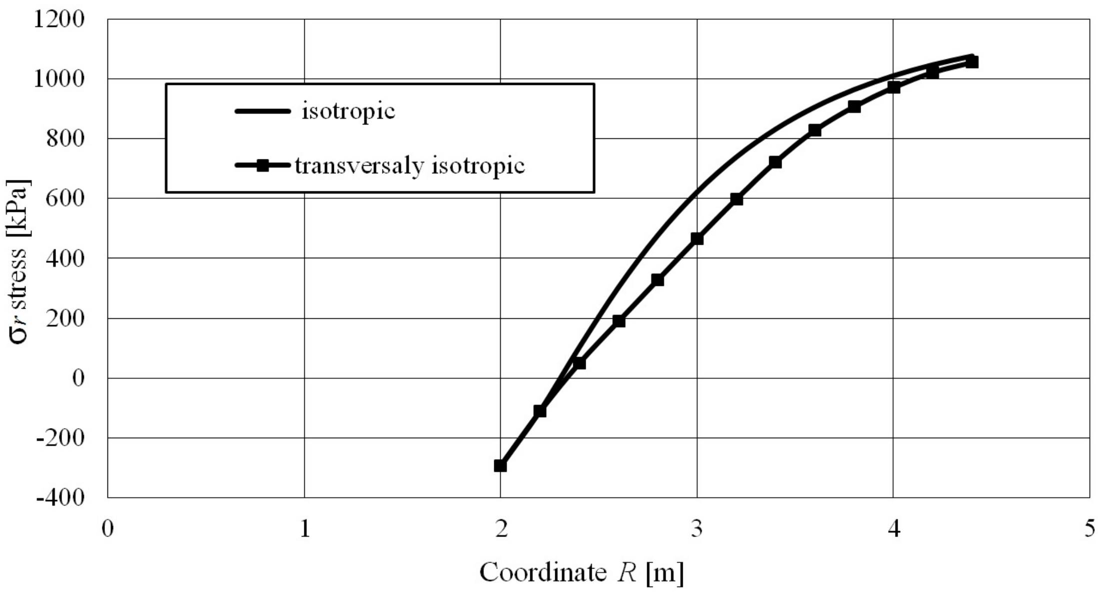

Fig. 6 Partially loaded contour - radial stresses

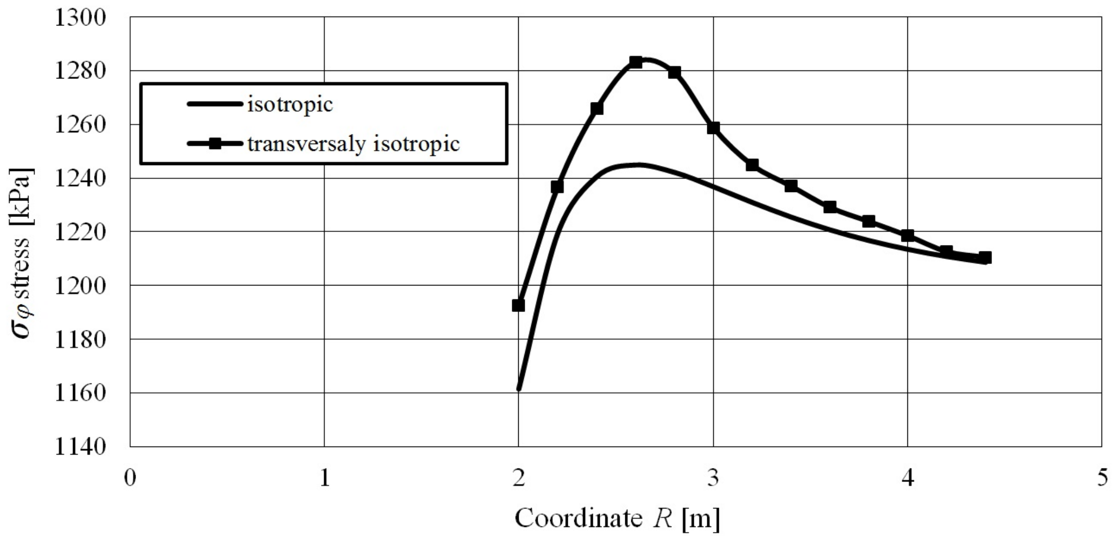

Fig. 7 Partially loaded contour - tangential stresses

measured data at distances of approximatelly 1.7*a* and more from the loading (2*a* is the tunnel diameter).

Besides the given deformation analysis and comparison with measured data, the comparative analysis of the radial and tangential stresses (σ_r, σ_φ) for isotropic (Lukić *et al.* 2010) and for transversally isotropic medium, considered in this work, is also given, see Figs. 6 and 7.

The radial and tangential stresses have been computed for the cylindrical cavity using the following data, see Table 1.

3. Conclusions

This paper presents an analysis of the aspects of cylindrical cavities in rock mass and its potential applications. Analyzing the results shown in diagram (Fig. 4), for the ratio of modules of elasticity k = 1,2 and 3, it could be seen that for $k > 1$ the coefficient ψ has a lower value when compared to $k = 1$ which represents the change of the coefficient ψ for homogenous and isotropic medium. It is logical, because the case $k > 1$ means that $E_r > E_z$, so the lower value of deformation is to be expected in radial direction. In other words, because of the higher modulus of elasticity in radial direction compared to the modulus of elasticity in axial direction, mass resists to deformation more, that is, the capacity for deformation of the mass is lower. Apart from the previous, the family of curves for $k > 1$ tend to 0 more slowly than those for $k < 1$. This is because for $k > 1$, $E_z < E_r$ and the other way around.

Comparison of theoretical predictions of radial displacements with available experimental measurements in the tunnel of the hydro power plant Mavrovo in Macedonia is performed. Measurement data are, practically, in a complete agreement with predicted values. Also, comparative calculations of the radial and tangential stresses around the partially loaded cylindrical cavity surrounded by the isotropic and transversally isotropic mediums are given and presented in Figs. 6 and 7.

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