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SH-wave propagation in a heterogeneous layer over an inhomogeneous isotropic elastic half-space

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Abstract. The present paper is devoted to study SH-wave propagation in heterogeneous layer laying over an inhomogeneous isotropic elastic half-space. The dispersion relation for propagation of said waves is derived with Green's function method and Fourier transform. As a special case when the upper layer and lower half-space are homogeneous, our derived equation is in agreement with the general equation of Love wave. Numerically, it is observed that the velocity of SH-wave increases with the increase of inhomogeneity parameter.

Keywords: non-homogeneity; Fourier transformation; Green's function; Dirac-delta function; isotropic; SH-waves

1. Introduction

The propagation of SH-waves in a hetrogeneous elastic medium can help us to understand earth-quakes, as the earth is made up of different hetrogeneous layers crust, mantle and core. The studies of nature of different layers of Earth inspired authors to work on the propagation of SH-wave in a hetrogeneous elastic medium. The schematics of the problem is taken as heterogeneous upper layer placed over inhomogeneous half-space, the formulation in upper layer and lower half-space has taken with different nonhomogeneity parameters. The problem is solved for layers with a unit impulse force in space and time followed by Green's function technique. The unit impulse force is represented by Dirac delta function, so an idealized point source or impulse of SH-wave can be described by this function.

Extensive and laudable work on wave propagation in various medium by using Green's function technique has been carried out by many researchers. Watanabe and Payton (2002) discussed SH- waves in a cylindrically monoclinic material with Green's function. Chattopadhyay *et al*. (2010, 2012) used Green's function technique to study propagation of SH-waves and heterogeneity on the SH-waves in a viscoelastic layer over a viscoelastic half-space under the effect of point source. Selvamani and Ponnusamy (2013) discussed wave propagation in a generalized thermo-elastic circular plate immersed in fluid. Gupta and Gupta (2013) formulated wave motion in anisotropic initially stressed fiber reinforced thermoelastic medium. Kakar and

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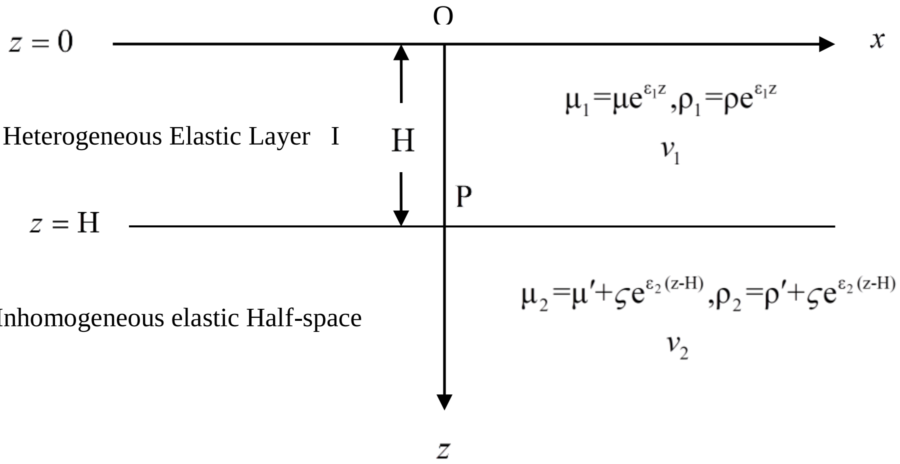

Fig. 1 Geometry of the problem

Kakar (2012) discussed propagation of Love waves in non-homogeneous elastic media. Delfim Soares Jr (2012) reviewed the iterative FEM-BEM coupling scheme for the investigation of wave propagation models. Chen *et al.* (2012) studied the medium coupling effect on propagation of guided waves in engineering structures and human bone phantoms. Kristel *et al.* (2013) analyzed wave propagation in unbounded elastic domains using the spectral element method. Kakar and Gupta (2014) investigated the existence of Love waves in an intermediate heterogeneous layer placed in between homogeneous and inhomogeneous half-spaces using Green's function technique. Xu *et al*. (2014) studied wave propagation in a 3D fully nonlinear NWT based on MTF coupled with DZ method for the downstream boundary.

In the present investigation, an attempt has been made to study the behaviour of SH-wave propagating in a heterogeneous elastic layer placed over inhomogeneous elastic half-space due to a point source. The heterogeneity is caused by consideration of exponential variation in rigidity and density in the upper elastic layer. Green's function technique is used to find the displacement in the elastic layer. Standard frequency equation of SH-waves is obtained in the special cases in closed form. The effects of inhomogeneity on the dimensionless phase velocity of SH-waves are also shown through figures with MATLAB software.

2. Formulation of the problem

Let H be the thickness of the upper layer with exponential variation of rigidity and density placed over inhomogeneous half-space. We consider x-axis along the direction of wave propagation and z-axis vertically downwards. We choose the source of disturbance P at the line of intersection of the interface and z-axis as shown (Fig. 1).

The variations of heterogeneous rigidity and density in the upper and lower layer are taken as

$$\left.\begin{aligned} \mu_1 &= \mu e^{\varepsilon_1 z} \\ \rho_1 &= \rho e^{\varepsilon_1 z} \end{aligned}\right\} \tag{1}$$

and

$$\left.\begin{aligned}\mu_2 &= \mu' + \varsigma e^{\varepsilon_2(z-H)} \\ \rho_2 &= \rho' + \varsigma e^{\varepsilon_2(z-H)}\end{aligned}\right\} \tag{2}$$

where ε_1 and ε_2 are inhomogeneous parameters of upper layer and lower half-space respectively, and ς is a small positive real constant such that $O(\varsigma^2)\rightarrow 0$.

3. Boundary conditions

The displacement components and stress components are continuous at $z=H$, and at $z=0$ there is no stress, therefore the geometry of the problem leads to the following conditions

$$\frac{\partial v_1}{\partial z} = 0 \tag{3a}$$

$$v_1 = v_2 \tag{3b}$$

$$\mu e^{\varepsilon_1 H} \frac{\partial v_1}{\partial z} = \mu' \frac{\partial v_2}{\partial z} \tag{3c}$$

4. Solution of the problem

The equation of motion for point source can be written as

$$\tau_{ij,j} + F_i = \rho \ddot{u}_i \tag{4}$$

where τ_{ij} are the stress components, ρ is the density of the medium and F_i are body forces.

For SH-wave propagation along the x-axis, we have

$$u = 0, \qquad w = 0, \qquad v = v(x, z, t) \tag{5}$$

Assuming the source is time harmonic and taking the time dependence $e^{i\omega t}$ to be understood throughout, such that the equation of motion for upper inhomogeneous isotropic medium is

$$\frac{\partial}{\partial x}\left(\mu_1 \frac{\partial v_1}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu_1 \frac{\partial v_1}{\partial z}\right) - \rho_1 \frac{\partial^2 v_1}{\partial t^2} = 4\pi\sigma_1(r,t) \tag{6}$$

Here 'r' is the distance from the origin, where the force is applied to a point of coordinates, '$\sigma_1(r,t)$' is the disturbances produced by the impulsive force at P and t is time.

As per our assumption $v_2(x,z,t) = \overline{v}_2(x,z)e^{i\omega t}$ and $\sigma_1(r,t) = \sigma(r)\,\mathrm{e}^{i\omega t}$, Eq. (6) reduces to

$$\frac{\partial}{\partial x}\left(\mu_1 \frac{\partial \overline{v}_1}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu_1 \frac{\partial \overline{v}_1}{\partial z}\right) + \rho_1 \omega^2 \overline{v}_1 = 4\pi\sigma(r) \tag{7}$$

The disturbances caused by the impulsive force '$\sigma(r)$' can be written in terms of Dirac-delta function at the source point as

$$\sigma(r) = \delta(x)\delta(z - H) \tag{8}$$

Therefore, Eq. (7) reduces to

$$\frac{\partial}{\partial x}\left(\mu_1 \frac{\partial \overline{v}_1}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu_1 \frac{\partial \overline{v}_1}{\partial z}\right) + \rho_1 \omega^2 \overline{v}_1 = 4\pi\delta(x)\delta(z - H) \tag{9}$$

Put Eq. (1) in Eq. (9), then we get

$$\frac{\partial^2 \overline{v}_1}{\partial x^2} + \frac{\partial^2 \overline{v}_1}{\partial z^2} + \varepsilon_1 \frac{\partial \overline{v}_1}{\partial z} + \frac{\rho}{\mu}\omega^2 \overline{v}_1 = \frac{4\pi}{\mu} e^{\varepsilon_1 z}\delta(x)\delta(z - H) \tag{10}$$

Again substituting $\overline{v}_1(x, z) = \overline{V}_1(x, z)e^{-\frac{\varepsilon_1}{2}z}$ in Eq. (10), we get

$$\frac{\partial^2 \overline{V}_1}{\partial x^2} + \frac{\partial^2 \overline{V}_1}{\partial z^2} + \left(\frac{\rho}{\mu}\omega^2 - \frac{\varepsilon_1^2}{4}\right)\overline{V}_1 = \frac{4\pi}{\mu} e^{-\frac{\varepsilon_1}{2}z}\delta(x)\delta(z - H) \tag{11}$$

To solve Eq. (11), the following Fourier transforms are taken

$$\left.\begin{aligned} V_r(\xi, z) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{v}_r(x, z)e^{i\xi x}dx \\ \overline{v}_r(x, z) &= \int_{-\infty}^{+\infty} V_r(\xi, z)e^{-i\xi x}d\xi \quad (\text{where, } r = 1, 2, 3) \end{aligned}\right\} \tag{12}$$

In terms of Fourier transforms, Eq. (11) can be written as

$$\left.\begin{aligned} \frac{d^2 V_1}{dz^2} - \alpha_1^2 V_1 &= \frac{2}{\mu} e^{-\frac{\varepsilon_1}{2}z}\delta(z - H) \\ \frac{d^2 V_1}{dz^2} - \alpha_1^2 V_1 &= \sigma_1(z) \end{aligned}\right\} \tag{13}$$

where,

$$\left.\begin{aligned} \alpha^2 &= \xi^2 - k_1^2 \\ \sigma_1(z) &= \frac{2}{\mu} e^{-\frac{\varepsilon_1}{2}z}\delta(z - H) \end{aligned}\right\} \tag{14}$$

and $k_1^2 = \left(\frac{\rho}{\mu}\omega^2 - \frac{\varepsilon_1^2}{4}\right)$, $\omega = kc$ is the angular frequency, k the wave number and c is the phase velocity.

The SH-waves are excited in the layer due to the presence of a point source at the interface of the layer and the lower half-space. Therefore, the equation of motion for lower semi infinite half-space by using Eq. (12) is

$$\frac{d^2V_2}{dz^2} - \beta^2 V_2 = 4\pi\sigma_2(z) \tag{15}$$

where,

$$\left.\begin{aligned} &\beta^2 = \xi^2 - k_2^2 \\ &4\pi\sigma_2(z) = -\frac{\varsigma}{\mu'}\left\{ e^{\varepsilon_2(z-H)}\frac{d^2V_2}{dz^2} + \varepsilon_2 e^{\varepsilon_2(z-H)}\frac{dV_2}{dz} + \left(\omega^2 e^{\varepsilon_2(z-H)} - e^{\varepsilon_2(z-H)}\xi^2\right)V_2 \right\} \end{aligned}\right\} \tag{16}$$

and $k_2^2 = \dfrac{\rho'}{\mu'}\omega^2$, $\omega = kc$ is the angular frequency, k the wave number and c is the phase velocity.

The displacement in lower half-space will determine from the Eq. (15), after assuming homogeneous to lower half-space, isotropic having source density distribution $\sigma^2(z)$.

Eq. (13) and Eq. (15) are solved by Green's Function technique under the prescribed boundary conditions in Eqs. (3a), (3b) and (3d). First of all we take the upper layer and it is solved with the help of Green's function $G_1(z/z_0)$.

The Eq. (13) will satisfy $G_1(z/z_0)$ as

$$\left.\begin{aligned} &\frac{d^2G_1(z/z_0)}{dz^2} - \alpha^2 G_1(z/z_0) = \delta(z - z_0) \\ &\text{together with the homogeneous boundary conditions} \\ &\frac{dG_1(z/z_0)}{dz} = 0 \quad \text{at } z = 0,\ \text{H} \end{aligned}\right\} \tag{17}$$

Here z_0 is arbitrary point in the upper medium. Multiplying Eq. (17) by $G_1(z/z_0)$, Eq. (13) by $V_2(\xi,z)$, subtracting and integrating over $0 \le z \le H$, we get

$$G_1(H/z_0)\left[\frac{dV_1}{dz}\right]_{z=H} - G_1(0/z_0)\left[\frac{dV_1}{dz}\right]_{z=0} = \left(\frac{2}{\mu}\right)e^{-\frac{\varepsilon_1 H}{2}} G_1(H/z_0) - V_1(z_0) \tag{18}$$

We know, from the properties of Green's function $G_1(H/z_0)=G_1(z/H)$ and replacing z_0 by z.

Thus we get the value of $V_1(z)$ at any point z in the upper medium from Eq. (18) as

$$V_1(z) = G_1(z/0)\left[\frac{dV_2}{dz}\right]_{z=0} - G_1(z/H)\left[\frac{dV_2}{dz}\right]_{z=H} + \left(\frac{2}{\mu}\right)e^{-\frac{\varepsilon_1 H}{2}} G_1(H/z) \tag{19}$$

Similarly, if $G_2(z/z_0)$ are Green's functions corresponding to lower homogeneous media, then Eq. (15) will satisfy as

$$\left.\begin{array}{l}\dfrac{d^2G_2(z/z_0)}{dz^2}-\beta^2G_1(z/z_0)=\delta(z-z_0)\\ \text{together with the homogeneous boundary conditions}\\ \dfrac{dG_2(z/z_0)}{dz}=0 \quad \text{at } z=H; \quad \dfrac{dG_2(z/z_0)}{dz}\to 0 \text{ as } z\to-\infty\end{array}\right\} \tag{20}$$

Multiplying Eq. (15) by $G_2(z/z_0)$, Eq. (20) by $V_2(\xi,z)$, subtracting and integrating with respect to z, $z=H$ to $z=\infty$, we get

$$-G_2(H/z_0)\left[\frac{dV_2}{dz}\right]_{z=H}=\int_H^\infty 4\pi\sigma_2(z)G_2(z/z_0)dz_0-V_2(z_0) \tag{21}$$

Replacing z by z_0 and using symmetry of Green's function, Eq. (21) become

$$V_2(z)=\int_H^\infty 4\pi\sigma_2(z)G_2(z/z_0)dz_0+G_2(z/H)\left[\frac{dV_2}{dz}\right]_{z=H} \tag{22}$$

Using boundary condition (3b) (v_1=v_2) in Eq. (19), we get

$$\frac{2}{\mu}e^{-\frac{\varepsilon_1}{2}H}G_1(H/H)-G_1(H/H)\left[\frac{dV_1}{dz}\right]_{z=H}+G_1(H/0)\left[\frac{dV_2}{dz}\right]_{z=0}=G_2(H/H)\left[\frac{dV_2}{dz}\right]_{z=H}+\int_H^\infty 4\pi\sigma_2(z_0)G_2(H/z_0)dz_0 \tag{23}$$

Similarly, using boundary condition (3b) $\left(\mu e^{\varepsilon_1 H}\dfrac{\partial v_1}{\partial z}=\mu'\dfrac{\partial v_2}{\partial z}\right)$ in Eq. (23), we get

$$\left[\frac{dV_1}{dz}\right]_{z=H}=\frac{1}{\mathrm{K}}\left\{\left(-\frac{2}{\mu}\right)e^{-\frac{\varepsilon_1}{2}H}G_1^2(H/0)+\left(\frac{2}{\mu}\right)e^{-\frac{\varepsilon_1}{2}H}AG_1(H/H)-A\int_H^\infty 4\pi\sigma_2(z_0)G_2(H/z_0)dz_0\right\} \tag{24}$$

where,

$$\mathrm{K}=AB-G_1^2(H/0),\ A=G_1(0/0),\ B=G_1(H/H)+\frac{\mu}{\mu'}e^{\varepsilon_1 H}G_2(H/\mathrm{H}). \tag{25}$$

Similarly, we have

$$\left[\frac{dV_2}{dz}\right]_{z=H}=\frac{\mu}{\mu'}e^{\varepsilon_1 H}\frac{A}{\mathrm{K}}\left(\left(-\frac{2}{\mu}\right)\frac{e^{-\frac{\varepsilon_1}{2}H}G_1^2(H/0)}{A}+\left(\frac{2}{\mu}\right)e^{-\frac{\varepsilon_1}{2}H}G_1(H/H)-\int_H^\infty 4\pi\sigma_2(z_0)G_2(H/z_0)dz_0\right) \tag{26}$$

Using Eq. (24) and Eq. (16) in Eq. (19) and using the property of delta function, we get

$$\begin{aligned}V_1(z)=&\left(\frac{2}{\mu\mathrm{K}}\right)e^{\frac{\varepsilon_1}{2}H}[B-G_1(H/H)][AG_1(z/H)-G_1(H/0)G_1(z/0)]\\&+\left(\frac{1}{\mathrm{K}}\right)[G_1(z/H)A-G_1(H/0)G_1(z/0)]\\&\times\int_H^\infty[e^{\varepsilon_2(z-H)}\frac{d^2V_2}{dz_0^2}+\varepsilon_2 e^{\varepsilon_2(z-H)}\frac{dV_2}{dz_0}+\left(\omega^2e^{\varepsilon_2(z-H)}-e^{\varepsilon_2(z-H)}\xi^2\right)V_2]G_2(H/z_0)dz_0\end{aligned} \tag{27}$$

In the similar manner, using Eq. (24) and Eq. (16) in Eq. (22) and using the property of delta function, we get

$$
\begin{aligned}
V_2(z) = {} & \frac{2e^{\frac{\varepsilon_1}{2}H} G_2(z/H)[AG_2(H/H) - G_2^2(H/0)]}{\mathrm{K}\mu'} + \frac{\varsigma\left(\mu e^{\varepsilon_1 H}\right) G_2(z/H) A}{\mu' \mathrm{K}} \\
& \times \int_H^\infty [e^{\varepsilon_2(z-H)} \frac{d^2 V_2}{dz_0^2} + \varepsilon_2 e^{\varepsilon_2(z-H)} \frac{dV_2}{dz_0} + \left(\omega^2 e^{\varepsilon_2(z-H)} - e^{\varepsilon_2(z-H)}\xi^2\right) V_2] G_2(H/z_0)\, dz_0 \\
& - \frac{\varsigma}{\mu'} \int_H^\infty [e^{\varepsilon_2(z-H)} \frac{d^2 V_2}{dz_0^2} + \varepsilon_2 e^{\varepsilon_2(z-H)} \frac{dV_2}{dz_0} + \left(\omega^2 e^{\varepsilon_2(z-H)} - e^{\varepsilon_2(z-H)}\xi^2\right) V_2] G_2(H/z_0)\, dz_0
\end{aligned}
\tag{28}
$$

Eq. (28) is an integral equation and $V_2(z)$ can be found from this equation by using successive approximations. The value of $V_2(z)$ obtained from Eq. (28), when substitute in Eq. (27) gives the value of $V_1(z)$. We are interested to find the value of $V_1(z)$, which will give the displacement in upper layer, and neglecting the higher order of ς, we take the first approximation as

$$
V_2(z) = \frac{2e^{\frac{\varepsilon_1}{2}H} G_2(z/H)[AG_1(H/H) - G_1^2(H/0)]}{\mu' \mathrm{K}} \tag{29}
$$

Eq. (29) represents the displacement at any point in the lower half-space. Putting this value of $V_2(z)$ in Eq. (27), we get

$$
\begin{aligned}
V_1(z) = {} & \frac{2e^{\frac{\varepsilon_1}{2}H} G_2(H/H)[AG_1(z/H) - G_1(H/0)G_1(z/0)]}{\mu' \mathrm{K}} - \frac{2\varsigma e^{\frac{\varepsilon_1}{2}H}[AG_1(z/H) - G_1(H/0)G_1(z/0)][AG_1(H/H) - G_1^2(H/0)]}{\mu'^2 \mathrm{K}^2} \\
& \times \int_H^\infty [e^{\varepsilon_2(z-H)} \frac{d^2 G_2(z_0/H)}{dz_0^2} + \varepsilon_2 e^{\varepsilon_2(z-H)} \frac{dG_2(z_0/H)}{dz_0} + \left(\omega^2 e^{\varepsilon_2(z-H)} - e^{\varepsilon_2(z-H)}\xi^2\right) G_2(z_0/H)] G_2(H/z_0)\, dz_0
\end{aligned}
\tag{30}
$$

We note that Eq. (30) completely represents the elastic displacements. These elastic displacements are due to a unit impulsive force in space and time. Also, the solution of Eq. (30) is incomplete because G_1 and G_2 are not known. We adopt the following method to find the unknown Green's function, Stakgold (1979).

We have considered $G_1(z/z_0)$ as a solution of Eq. (17).

A solution of Eq. (17) can also be found as

$$
\frac{d^2 L}{dz^2} - \alpha^2 L = 0 \tag{31}
$$

The two independent solutions of Eq. (31) will vanish at $z = -\infty$ and $z = \infty$ are $L_1(z) = e^{\alpha z}$ and $L_2(z) = e^{-\alpha z}$

Hence, the solution of Eq. (31) for an infinite medium is

$$
\left.
\begin{aligned}
& \frac{L_1(z) L_2(z_0)}{M} \quad \text{for} \quad z < z_0 \\
& \frac{L_1(z_0) L_2(z)}{M} \quad \text{for} \quad z > z_0
\end{aligned}
\right\} \tag{32}
$$

where, $M = L_1(z)L'_2(z) - L_2(z)L'_1(z) = -2\alpha$.

So we can write the solution of Eq. (17) as

$$-\frac{e^{-\alpha|z-z_0|}}{2\alpha} \tag{33}$$

Since $G_1(z/z_0)$ is to satisfy the homogeneous condition

$$\frac{dG_1(z/z_0)}{dz} = 0 \quad \text{at } z = 0; \quad \frac{dG_1(z/z_0)}{dz} \to 0 \text{ as } z \to -\infty \tag{34}$$

Therefore, we assume that

$$G_1(z/z_0) = -\frac{e^{-\alpha|z-z_0|}}{2\alpha} + Ce^{\alpha z} + De^{-\alpha z} \tag{35}$$

The conditions as mentioned in Eq. (33), we have

$$G_1(z/z_0) = -\frac{1}{2\alpha}\left[e^{-\alpha|z-z_0|} + e^{\alpha z}\left\{\frac{e^{-\alpha(z_0+H)} + e^{-\alpha(H-z_0)}}{e^{\alpha H} - e^{-\alpha H}}\right\} + e^{-\alpha z}\left\{\frac{e^{\alpha(H-z_0)} + e^{-\alpha(H-z_0)}}{e^{\alpha H} - e^{-\alpha H}}\right\}\right] \tag{36}$$

Therefore Eq. (36) takes the form as

$$G_1(z/H) = -\frac{1}{\alpha}\left[\frac{e^{-\alpha z} + e^{\alpha z}}{e^{\alpha H} - e^{-\alpha H}}\right] \tag{37}$$

$$G_1(H/H) = -\frac{1}{\alpha}\left[\frac{e^{-\alpha H} + e^{\alpha H}}{e^{\alpha H} - e^{-\alpha H}}\right] \tag{38}$$

Green's function $G_2(z/z_0)$ can be obtained in the similar manner as above by using the boundary conditions Eq. (3a) and Eq. (3b).

$$G_2(z/z_0) = -\frac{1}{2\beta}\left[e^{-\beta|z-z_0|} + e^{-\beta(z+z_0-2H)}\right] \tag{39}$$

Therefore Eq. (39) takes the form as

$$G_2(H/z_0) = -\frac{e^{-\beta|z_0-H|}}{\beta} \tag{40}$$

$$G_2(H/H) = -\frac{1}{\beta} \tag{41}$$

Substitute the value of Eqs. (36)-(41) in Eq. (30), simplifying, we get

$$V_1(z) = \frac{-2e^{\frac{\varepsilon_1}{2}H}\left\{\left[e^{\beta z} + e^{-\beta z}\right]\right\}}{P_0 + Q_0} \times \left[1 - \frac{\varsigma\left\{\left[e^{\alpha H} + e^{-\alpha H}\right]\right\}}{P_0 + Q_0}\left(\frac{2\beta\omega^2 - \varepsilon_2 k_1^2 - 2\varepsilon_2\beta^2}{4\beta^2 - \varepsilon_2}\right)\right] \tag{38}$$

where,

$$P_0 = \left(\alpha\mu e^{\varepsilon_1 H}\right)\left[e^{\alpha H} - e^{-\alpha H}\right], \qquad Q_0 = \left(\mu'\beta\right)\left[e^{\alpha H} + e^{-\alpha H}\right] \tag{39}$$

Neglecting the higher powers of ς then the Eq. (38) will become as

$$V_1(z) = \frac{-2e^{\frac{\varepsilon_1}{2}H}\left[e^{\alpha z} + e^{-\alpha z}\right]}{\left(P_0 + Q_0\right) + \varsigma\left[e^{\alpha H} + e^{-\alpha H}\right]\left(\dfrac{2\beta\omega^2 - \varepsilon_2 k_1^2 - 2\varepsilon_2\beta^2}{4\beta^2 - \varepsilon_2}\right)} \tag{40}$$

Taking the inverse Fourier transform of Eq. (40), the displacement in the upper layer may be obtained as

$$\bar{V}_1(z) = -2\int_{-\infty}^{\infty} \frac{e^{\frac{\varepsilon_1}{2}H}\left[e^{\alpha z} + e^{-\alpha z}\right]}{\left(P_0 + Q_0\right) + \varsigma\left[e^{\alpha H} + e^{-\alpha H}\right]\left(\dfrac{2\beta\omega^2 - \varepsilon_2 k_1^2 - 2\varepsilon_2\beta^2}{4\beta^2 - \varepsilon_2}\right)} \mathrm{e}^{-i\xi x}\,\mathrm{d}\xi \tag{41}$$

Using Eq. (12) into Eq. (41), we get

$$\bar{v}_1(z) = -2\int_{-\infty}^{\infty} \frac{e^{\frac{\varepsilon_1}{2}(z-H)}\left[e^{\alpha z} + e^{-\alpha z}\right]}{\left(P_0 + Q_0\right) + \varsigma\left[e^{\alpha H} + e^{-\alpha H}\right]\left(\dfrac{2\beta\omega^2 - \varepsilon_2 k_1^2 - 2\varepsilon_2\beta^2}{4\beta^2 - \varepsilon_2}\right)} \mathrm{e}^{-i\xi x}\,\mathrm{d}\xi \tag{42}$$

The dispersion equation of SH-wave can now be obtained by putting the denominator of Eq. (42) equal to zero, after putting the values of P_0 and Q_0, we have

$$\left.\begin{aligned}
&\left(\alpha\mu e^{\varepsilon_1 H}\right)\left[e^{\alpha H} - e^{-\alpha H}\right] + \left(\mu'\beta\right)\left[e^{\alpha H} + e^{-\alpha H}\right] + \varsigma\left[e^{\alpha H} + e^{-\alpha H}\right]\left(\frac{2\beta\omega^2 - \varepsilon_2 k_1^2 - 2\varepsilon_2\beta^2}{4\beta^2 - \varepsilon_2}\right) = 0\\
&\left(\alpha\mu e^{\varepsilon_1 H}\right)\left[e^{\alpha H} - e^{-\alpha H}\right] = -\left\{\left(\mu'\beta\right) + \varsigma\left(\frac{2\beta\omega^2 - \varepsilon_2 k_1^2 - 2\varepsilon_2\beta^2}{4\beta^2 - \varepsilon_2}\right)\right\}\left[e^{\alpha H} + e^{-\alpha H}\right]
\end{aligned}\right\} \tag{43}$$

Rearranging Eq. (43), we get

$$\tanh\left(\alpha H\right) = \frac{\left\{\left(\mu'\beta\right) - \varsigma\left(\dfrac{2\beta\omega^2 - \varepsilon_2 k_1^2 - 2\varepsilon_2\beta^2}{4\beta^2 - \varepsilon_2}\right)\right\}}{\alpha\mu e^{\varepsilon_1 H}} \tag{44}$$

Replacing α by ik, we get

$$\tan\left(kH\right) = \frac{\left\{\left(\mu'\beta\right) - \varsigma\left(\dfrac{2\beta\omega^2 - \varepsilon_2 k_1^2 - 2\varepsilon_2\beta^2}{4\beta^2 - \varepsilon_2}\right)\right\}}{k\mu e^{\varepsilon_1 H}} \tag{45}$$

We make following substitutions in Eq. (44) $\alpha = ik\left[\frac{c^2}{c_1^2} - \frac{\varepsilon_1^2}{4k^2} - 1\right]^{\frac{1}{2}}$ and $\beta = k\left[1 - \frac{c^2}{c_2^2}\right]^{\frac{1}{2}}$

where, $c_1 = \sqrt{\frac{\mu}{\rho}}$ and $c_2 = \sqrt{\frac{\mu'}{\rho'}}$ respectively.

We have

$$\tan\left\{kH\sqrt{\frac{c^2}{c_1^2} - \frac{\varepsilon_1^2}{4k^2} - 1}\right\} = \frac{\mu'\sqrt{1 - \frac{c^2}{c_2^2}}}{\mu e^{\varepsilon_1 H}\sqrt{\frac{c^2}{c_1^2} - \frac{\varepsilon_1^2}{4k^2} - 1}} + \frac{\varsigma}{2\mu e^{\varepsilon_1 H}\sqrt{\frac{c^2}{c_1^2} - \frac{\varepsilon_1^2}{4k^2} - 1}} \times \left[\frac{\frac{c^2}{c_1^2}\left(\frac{\mu}{\rho}\right)\sqrt{1 - \frac{c^2}{c_2^2} - \frac{\varepsilon_2}{k}\left(1 - \frac{c^2}{c_2^2}\right) - \frac{\varepsilon_2}{2k}\left(\frac{c^2}{c_2^2}\right)}}{\left(1 - \frac{c^2}{c_1^2} - \frac{\varepsilon_2^2}{4k^2}\right)}\right] \tag{46}$$

Eq. (46) is the dispersion equation of SH-waves in inhomogeneous upper layer placed in between two hetrogeneous half-spaces.

Case-1

In the absence of inhomogeneous parameters ε_1 and ε_2 of upper layer and lower half-space i.e., ε_1=0 and ε_2=0, Eq. (46) reduces to

$$\tan\left\{kH\sqrt{\frac{c^2}{c_1^2} - 1}\right\} = \frac{\mu'\sqrt{1 - \frac{c^2}{c_2^2}}}{\mu\sqrt{\frac{c^2}{c_1^2} - 1}} + \frac{\varsigma}{\mu\sqrt{\frac{c^2}{c_1^2} - 1}} \times \left[\frac{\frac{c^2}{c_1^2}\left(\frac{\mu}{\rho}\right)\sqrt{1 - \frac{c^2}{c_2^2}}}{\left(1 - \frac{c^2}{c_1^2}\right)}\right] \tag{47}$$

Case-2

When ς=0, ε_1=0 and ε_2=0, Eq. (46) reduces to

$$\tan\left\{kH\sqrt{\frac{c^2}{c_1^2} - 1}\right\} = \frac{\mu'\sqrt{1 - \frac{c^2}{c_2^2}}}{\mu\sqrt{\frac{c^2}{c_1^2} - 1}} \tag{48}$$

Eq. (48) is the classical dispersion equation of SH-waves given by Love (1911) and Ewing *et al.* (1957).

Table 1 Material parameters

Layer	Rigidity	Density
I	$\mu = 6.34\times10^{10} N/m^2$	$\rho = 3364 Kg/m^3$
II	$\mu' = 11.77\times10^{10} N/m^2$	$\rho' = 4148 Kg/m^3$

5. Numerical analysis

To show the effect of inhomogeneity parameters $l=\frac{\varepsilon_1}{k}$, $m=\frac{\varepsilon_2}{k}$ and $n=\frac{\varsigma}{2\mu}$ on nature of wave motion we have plotted non-dimensional phase velocity $\frac{c^2}{c_1^2}$ against dimensionless wave number *kH* on the propagation of SH-wave in hetrogeneous upper layer by using MATLAB software. Figs. 2-8 are plotted for Eq. (46) by taking parameters in Table 1, Gubbins (1990).

Fig. 2 shows the variation of dimensionless phase velocity $\frac{c^2}{c_1^2}$ v/s dimensionless wave number *kH* for the different values of heterogeneity parameter $\frac{\varepsilon_1}{k}=0.1, 0.3, 0.5, 0.7$ and inhomogeneity parameter $\frac{\varepsilon_2}{k}=0.1, 0.3, 0.5, 0.7$ at constant value of $\frac{\varsigma}{2\mu}=0.2$. It has been observed that phase velocity of SH-waves in upper layer increases with the increases of heterogeneity parameter $\frac{\varepsilon_1}{k}$ and inhomogeneity parameter $\frac{\varepsilon_2}{k}$ in presence of $\frac{\varsigma}{2\mu}$. The dimensionless phase velocity $\frac{c^2}{c_1^2}$ v/s dimensionless wave number *kH* for the different values of heterogeneity parameter $\frac{\varepsilon_1}{k}=0.1, 0.3, 0.5, 0.7$ and inhomogeneity parameter $\frac{\varepsilon_2}{k}=0.1, 0.3, 0.5, 0.7$ is plotted in Fig. 3 at different values of $\frac{\varsigma}{2\mu}=0.1, 0.2, 0.3, 0.4$. From this curve, it can be realized that phase velocity increases with the increases of heterogeneity parameter $\frac{\varepsilon_1}{k}$, inhomogeneity parameters $\frac{\varepsilon_2}{k}$ and $\frac{\varsigma}{2\mu}=0.2$. Fig. 4 represents the variation of dimensionless phase velocity $\frac{c^2}{c_1^2}$ v/s dimensionless wave number *kH* for the different values of heterogeneity parameter $\frac{\varepsilon_1}{k}=0.1, 0.3, 0.5, 0.7$ in the absence of $\frac{\varsigma}{2\mu}$ and inhomogeneity parameter $\frac{\varepsilon_2}{k}$. It is important to mention here, when we put $\frac{\varsigma}{2\mu}$ equal to zero, the SH-waves will only affected by the inhomogeneity parameter $\frac{\varepsilon_1}{k}$. It is observed in this curve that inhomogeneity parameters remarkably affect the SH-waves propagating in upper

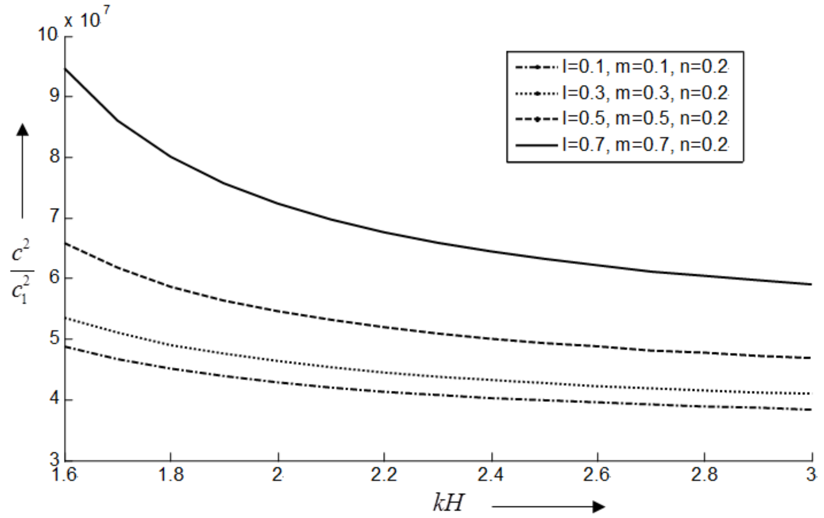

Fig. 2 Variation of dimensionless phase velocity $\frac{c^2}{c_1^2}$ v/s dimensionless wave number kH for the different values of heterogeneity parameter $\frac{\varepsilon_1}{k}$ and inhomogeneity parameter $\frac{\varepsilon_2}{k}$ at constant value of $\frac{\varsigma}{2\mu} = 0.2$

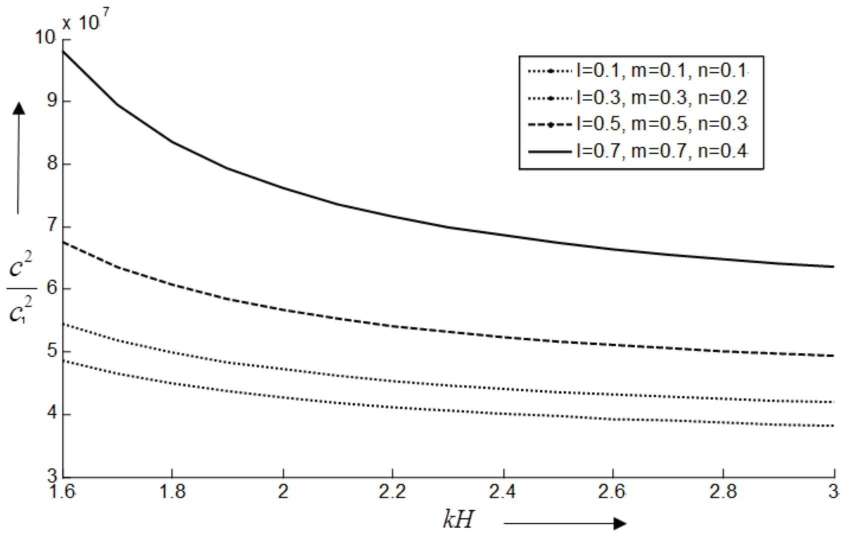

Fig. 3 Variation of dimensionless phase velocity $\frac{c^2}{c_1^2}$ v/s dimensionless wave number kH for the different values of heterogeneity parameter $\frac{\varepsilon_1}{k}$ and inhomogeneity parameter $\frac{\varepsilon_2}{k}$ at different value of $\frac{\varsigma}{2\mu}$

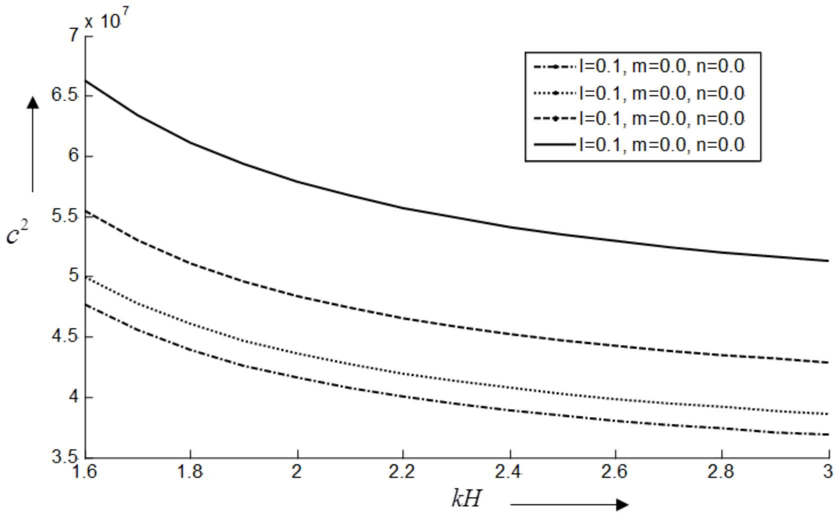

Fig. 4 Variation of dimensionless phase velocity $\frac{c^2}{c_1^2}$ v/s dimensionless wave number kH for the different values of heterogeneity parameter $\frac{\varepsilon_1}{k}$ in the absence of inhomogeneity parameter $\frac{\varepsilon_2}{k}$ and $\frac{\varsigma}{2\mu}$

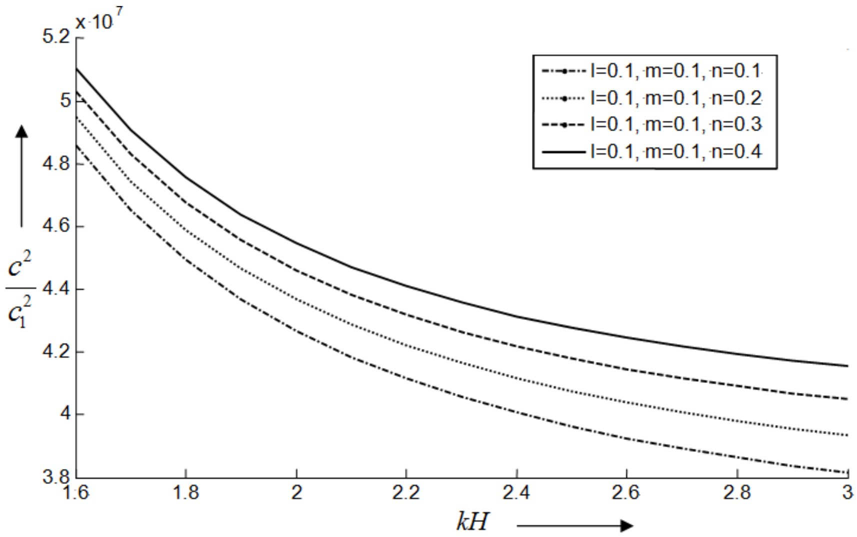

Fig. 5 Variation of dimensionless phase velocity $\frac{c^2}{c_1^2}$ v/s dimensionless wave number kH for the different values of $\frac{\varsigma}{2\mu}$ at constant values of heterogeneity parameter $\frac{\varepsilon_1}{k}=0.1$ and inhomogeneity parameter $\frac{\varepsilon_2}{k}=0.1$

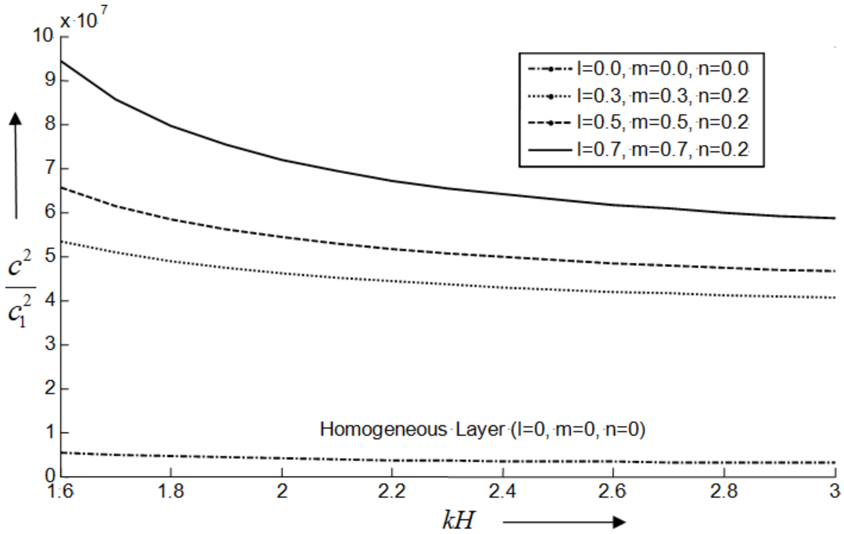

Fig. 6 Variation of dimensionless phase velocity $\frac{c^2}{c_1^2}$ v/s dimensionless wave number kH for the different values of heterogeneity parameter $\frac{\varepsilon_1}{k}$ and inhomogeneity parameter $\frac{\varepsilon_2}{k}$ at constant value of $\frac{\varsigma}{2\mu}=0$ and $\frac{\varsigma}{2\mu}=0.2$

layer in the absence of parameter $\frac{\varsigma}{2\mu}$. From this curve, it can be realized that phase velocity increases with the increases of parameter $\frac{\varepsilon_1}{k}$ in the absence of $\frac{\varsigma}{2\mu}$. Fig. 5 represents the variation of dimensionless phase velocity $\frac{c^2}{c_1^2}$ v/s dimensionless wave number kH for the different values of $\frac{\varsigma}{2\mu}$ at constant values of heterogeneity parameter $\frac{\varepsilon_1}{k}=0.1$ and inhomogeneity parameter $\frac{\varepsilon_2}{k}=0.1$. Fig. 6 is plotted to show the concept of homogeneity and non-homogeneity at constant value of $\frac{\varsigma}{2\mu}=0$ and $\frac{\varsigma}{2\mu}=0.2$.

6. Conclusions

In this problem we have taken two layers; hetrogeneous upper layer and inhomogeneous lower with exponential variation in rigidity and density. We have employed Green's function method to find the frequency equation due to a point source. Displacement in the upper layer is derived in closed form and the dispersion curves are drawn for various values of inhomogeneity parameters. In a particular case, the dispersion equation coincides with the well-known classical equation of

Love wave when the upper layer and lower half-space are homogeneous.

From above numerical analysis, it may be conclude that:

a. Phase velocity $\frac{c^2}{c_1^2}$ (non-dimensional) of SH-waves decreases with increase of wave number kH (non-dimensional).

b. The dimension less phase velocity of SH-wave shows remarkable change with heterogeneity parameter $\frac{\varepsilon_1}{k}$. It has been observed that the phase velocity increases with increase of heterogeneity parameters.

c. The dimension less phase velocity of SH-wave increases with increase of inhomogeneity parameter $\frac{\varepsilon_2}{k}$.

d. Phase velocity $\frac{c^2}{c_1^2}$ (non-dimensional) of SH-waves also increases with the increase of inhomogeneity parameter $\frac{\varsigma}{2\mu}$.

e. However, the dimensionless phase velocity of SH-waves increases with the increase of heterogeneity parameters $\frac{\varepsilon_1}{k}$ in upper layer keeping inhomogeneity parameter $\frac{\varepsilon_2}{k}$ constant, in absence of parameter $\frac{\varsigma}{2\mu}$ in lower half-space.

f. Also, the dimensionless phase velocity of SH-waves increases with the increase of parameters $\frac{\varepsilon_2}{k}$ in inhomogeneity upper layer keeping heterogeneity parameter $\frac{\varepsilon_1}{k}$ constant, in absence of parameter $\frac{\varsigma}{2\mu}$ in lower half-space.

The above results may be used to study surface wave propagation during earthquakes and artificial explosions.

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