Earthquakes and Structures, *Vol. 13, No. 2 (2017) 201-209*
DOI: https://doi.org/10.12989/eas.2017.13.2.201

Thermoviscoelastic orthotropic solid cylinder with variable thermal conductivity subjected to temperature pulse heating

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(Received March 8,2017, Revised September 3,2017 , Accepted September 7,2017)

Abstract. This work aims to analyze the thermo-viscoelastic interaction in an orthotropic solid cylinder. The medium is considered to be variable thermal conductivity and subjected to temperature pulse. Analytical solution based on dual-phase-lags model with Voigt-type for behavior of viscoelastic material has been effectively proposed. All variables are deduced using method of Laplace transforms. Numerical results for different distribution fields, such as temperature, displacement and stress components are graphically presented. Results are discussed to illustrate the effect of variability thermal conductivity parameter as well as phase-lags and viscoelasticity on the field quantities. Results are obtained when the viscosity is ignored with and without considering variability of thermal conductivity. A comparison study is made and all results are investigated.

Keywords: thermoviscelasticity; orthotropic medium; solid cylinder; temperature pulse; variable thermal conductivity; dual-phase-lags

1. Introduction

One of the experimental probes of microphysical processes is the viscoelasticity which is of interest in different applications. Mechanical analysis of a viscoelastic solid is sensitive to variation in environmental factors such as temperature, humidity and presence of diffusion. Viscoelastic solid plays important roles in engineering applications. Solutions may be investigated for viscoelastic wave equations and velocities of seismic wave propagating. The attenuation of seismic waves in viscoelastic media are very important for geophysical prospecting technology. Governing equations of viscoelastic solids maybe constructed according to Boltzmann superposition principle in most cases of linear viscoelasticity. It is interested to extend linear theory of viscoelasticity to most famous theory of thermo-viscoelasticity at finite strains. This process may be done after taken into consideration several requirements. The constitutive theory of finite thermoelasticity can be reduced during sufficiently deformation process as a second requirement.

Different investigations are dealt with generalized or coupled thermoviscoelastic problems for many applications (Abd-Alla *et al.* 2004, Othman 2005, Tian and Shen 2005, Sarkar and Lahiri 2013, Ezzat *et al.* 2014, Abd-Alla *et al.* 2017). Kovalenko and Karnaukhov (1972) discussed the influences of the heat effect via a generalized linearized theory of thermoviscoelasticity. Drozdov (1999) studied the non-isothermal viscoelastic behavior of polymers and derived the constitutive relations at finite strains. Kundu and Mukhopadhyay (2005) discussed variable distributions in viscoelastic solid with spherical cavity due to theory of generalized thermoelasticity with relaxation time effect. Baksi *et al.* (2006) studied an infinite rotating magneto-thermo-visco-elastic media subjected to heat source with one relaxation parameter to derive and solve its fundamental equations. Baksi *et al.* (2008) presented a thermoviscoelastic problem in an infinite isotropic medium subjected to point heat source in two dimensions. Kanoria and Mallik (2010) studied the thermoviscoelastic interaction in an infinite viscoelastic medium due to periodically varying heat sources taken into consideration Kelvin-Voigt-type. Kumar and Partap (2011) studied the micropolar thermoelastic interactions in an infinite viscoelastic thermally conducting plate employing the coupled dynamic thermoelasticity and generalized theories of thermoelasticity. Ezzat *et al.* (2013) presented 1-D problem in the frame of thermoviscoelasticity with heat sources to deal with the coupled fractional relaxation equations due to the fractional calculus.

Zenkour and his colleagues (Zenkour *et al.* 2013, Abbas and Zenkour 2014, Abouelregal and Zenkour 2014, Zenkour 2015, Zenkour and Abouelregal 2015, Zenkour 2016, Zenkour *et al.* 2015) have investigated the effect of dual-phase-lags (DPLs) on thermoelastic structures subjected to different heating sources. The present DPLs model developed by Tzou (1995, 1996) is considered as an extension to the well-known generalized thermoelasticity theory (Lord and Shulman 1967, Green and Lindsay 1971, Green and Naghdi 1993). In this article, thermoelastic interactions in the present body in context of a generalized thermoelasticity with DPLs are investigated. Conducting orthotropic medium with variability thermal conductivity

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ISSN: 2092-7614 (Print), 2092-7622 (Online)

including a solid cylinder is initially presented. The cylindrical boundaries are subjected to a temperature pulse and its surface is traction free (Zenkour and Abouelregal 2014). Numerical results for all variables of the thermoviscoelastic body are graphically presented. A comparison has been made in two cases of the presence and absence of viscosity field and temperature-dependent thermal conductivity.

2. Basic equations

The Kelvin-Voigt approach is one of the macroscopic mechanical approaches that used to describe the viscoelastic response of a medium. It represents the delayed elastic response subjected to stress when the deformation is time dependent but recoverable. Here, we consider a viscoelastic orthotropic solid cylinder at environment temperature T_0. The outer surface of cylinder is traction-free and subject to temperature pulse. The mentioned linear viscoelasticity Kelvin-Voigt approach maybe employed to deal with viscoelastic nature of present cylinder. The cylindrical coordinates system (r, ξ, z) is chosen to address this problem in which z-axis is lying along axis of cylinder.

For the present axially symmetric problem, the displacement field is reduced to

$$u_r = u(r,t), \quad u_\xi(r,t) = u_z(r,t) = 0. \tag{1}$$

The Cauchy relations will be

$$\varepsilon_{rr} = \frac{\partial u}{\partial r}, \quad \varepsilon_{\xi\xi} = \frac{u}{r}. \tag{2}$$

For a Kelvin-Voigt type, generalized Hooke's law of the cylinder takes the form (Eringen 1967)

$$\begin{Bmatrix}\sigma_{rr}\\ \sigma_{\xi\xi}\\ \sigma_{zz}\end{Bmatrix} = \tau_m \begin{bmatrix} c_{11} & c_{12}\\ c_{12} & c_{22}\\ c_{13} & c_{23}\end{bmatrix}\begin{Bmatrix}\frac{\partial u}{\partial r}\\ \frac{u}{r}\end{Bmatrix} - \begin{Bmatrix}\beta_{11}\\ \beta_{22}\\ \beta_{33}\end{Bmatrix}\theta, \tag{3}$$

where $\tau_m = 1 + t_0 \frac{\partial}{\partial t}$.

After neglecting body forces, one can obtain dynamic equation of cylindrical cavity as

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\xi\xi}}{r} = \rho \frac{\partial^2 u}{\partial t^2}. \tag{4}$$

Substituting Eq. (3) into Eq. (4) yields

$$\begin{aligned} c_{11}\tau_m\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) - c_{22}\tau_m \frac{u}{r^2} \\ = \rho \frac{\partial^2 u}{\partial t^2} + \beta_{11}\frac{\partial \theta}{\partial r} + (\beta_{11} - \beta_{22})\frac{\theta}{r}. \end{aligned} \tag{5}$$

The modified Fourier's law may be presented as

$$\left(1 + \tau_q \frac{\partial}{\partial t}\right)\vec{q} = -K_r\left(1 + \tau_\theta \frac{\partial}{\partial t}\right)\nabla\theta. \tag{6}$$

The energy conservation equation can be expressed as

$$-\nabla\cdot\vec{q} = \rho C_E \frac{\partial \theta}{\partial t} + T_0 \frac{\partial}{\partial t}\left(\beta_{11}\frac{\partial u}{\partial r} + \beta_{22}\frac{u}{r}\right). \tag{7}$$

By using Eqs. (6) and (7) to eliminate $\vec{q}$, one can obtain heat conduction equation with DPLs (without heat source) in the form

$$\begin{aligned} \left(1 + \tau_\theta \frac{\partial}{\partial t}\right)\left(K_r \theta_{,r}\right)_{,r} \\ = \left(1 + \tau_q \frac{\partial}{\partial t}\right)\Big[\rho C_E \frac{\partial \theta}{\partial t} \\ + T_0 \frac{\partial}{\partial t}\left(\beta_{11}\frac{\partial u}{\partial r} + \beta_{22}\frac{u}{r}\right)\Big]. \end{aligned} \tag{8}$$

Different field equations in context of generalized thermoelasticity with the first relaxation time can be obtained from Eqs. (1)-(8) by setting mechanical PLs parameters $\tau_\theta = 0$ and $\tau_\theta = \tau_0$ (τ_0 is first relaxation time). Putting thermal PLs $\tau_\theta = \tau_q = 0$, one obtains the different field equations for coupled theory of thermoelasticity. Also on putting thermal PLs $\tau_\theta = \tau_q = 0$, and the thermomechanical coupling parameters $\beta_{11} = \beta_{22} = 0$ gives the uncoupled thermoelasticity governing equations.

The thermal material properties in thermosensitivity medium may be temperature-dependent and give nonlinear heat conduction problem. To get exact solution one can assume simply nonlinear properties of the material, in which thermal conductivity K_r and specific heat C_E are linearly temperature-dependent (Noda 1986, Zenkour and Abouelregal 2016), but the thermal diffusivity k ($k = K_r/\rho C_E$) is considered to be constant. That is

$$K_r = K_r(\theta) = k_0(1 + k_1\theta). \tag{9}$$

So, new variable ψ maybe assumed to represent heat conduction in Kirchhoff transformation (Noda 1986) in the form

$$\psi = \frac{1}{k_0}\int_0^\theta K_r(\theta)\mathrm{d}\theta. \tag{10}$$

The substitution of Eq. (9) into Eq. (10) gives

$$\psi = \theta\left(1 + \tfrac{1}{2}k_1\theta\right). \tag{11}$$

From Eq. (11), it follows that

$$\nabla\psi = \frac{K_r(\theta)}{k_0}\nabla\theta, \quad \frac{\partial \psi}{\partial t} = \frac{K_r(\theta)}{k}\frac{\partial \theta}{\partial t}. \tag{12}$$

Finally, the substitution of Eq. (12) into Eq. (8) gives general heat equation considering variable thermal conductivity as

$$\begin{aligned} \left(1 + \tau_\theta \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r}\right) \\ = \left(1 + \tau_q \frac{\partial}{\partial t}\right)\Big[\rho C_E \frac{\partial \psi}{\partial t} \\ + \frac{T_0}{k_0}\frac{\partial}{\partial t}\left(\beta_{11}\frac{\partial u}{\partial r} + \beta_{22}\frac{u}{r}\right)\Big]. \end{aligned} \tag{13}$$

From Eqs. (11), the equation of motion will be

$$\begin{aligned} c_{11}\tau_m\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\right) - c_{22}\tau_m \frac{u}{r^2} \\ = \rho \frac{\partial^2 u}{\partial t^2} + \frac{\beta_{11}}{1 + 2k_1\theta}\frac{\partial \psi}{\partial r} \\ + (\beta_{11} - \beta_{22})\left(\frac{\sqrt{1 + 2k_1\psi} - 1}{k_1 r}\right). \end{aligned} \tag{14}$$

In the linear form, since $\theta = T - T_0$ such that and $|\theta/T_0| \ll 1$, then the governing equations are reduced to

$$c_{11}\tau_m\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)-c_{22}\tau_m\frac{u}{r^2} = \rho\frac{\partial^2 u}{\partial t^2}+\beta_{11}\frac{\partial\psi}{\partial r}+(\beta_{11}-\beta_{22})\frac{\psi}{r}, \quad (15)$$

$$\begin{Bmatrix}\sigma_{rr}\\ \sigma_{\xi\xi}\\ \sigma_{zz}\end{Bmatrix}=\begin{bmatrix}\tau_m c_{11} & \tau_m c_{12} & -\beta_{11}\\ \tau_m c_{12} & \tau_m c_{22} & -\beta_{22}\\ \tau_m c_{13} & \tau_m c_{23} & -\beta_{33}\end{bmatrix}\begin{Bmatrix}\frac{\partial u}{\partial r}\\ \frac{u}{r}\\ \psi\end{Bmatrix}. \quad (16)$$

The following dimensionless variables maybe considered here

$$\{u',r',R_i'\}=\frac{c_0}{k}\{u,r,R_i\},\quad \{t',t_0',\tau_q',\tau_\theta'\}=\frac{c_0^2}{k}\{t,t_0,\tau_q,\tau_\theta\},$$
$$\sigma_{ij}'=\frac{\sigma_{ij}}{c_{11}},\quad k_1'=T_0k_1,\quad \psi'=\frac{\psi}{T_0},\quad c_0^2=\frac{c_{11}}{\rho}. \quad (17)$$

Using the quantities (17) in the governing Eqs. (16) and suppressing dashes, we obtain

$$\tau_m\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)-c_2\tau_m\frac{u}{r^2}=\frac{\partial^2 u}{\partial t^2}+\varepsilon_1\frac{\partial\psi}{\partial r}+\varepsilon_3\frac{\psi}{r}, \quad (18)$$

$$\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\nabla^2\psi=\left(1+\tau_q\frac{\partial}{\partial t}\right)\left[\frac{\partial\psi}{\partial t}+\frac{\partial}{\partial t}\left(\varepsilon_4\frac{\partial u}{\partial r}+\varepsilon_5\frac{u}{r}\right)\right], \quad (19)$$

$$\begin{Bmatrix}\sigma_{rr}\\ \sigma_{\xi\xi}\\ \sigma_{zz}\end{Bmatrix}=\begin{bmatrix}\tau_m & \tau_m c_1 & -\varepsilon_1\\ \tau_m c_1 & \tau_m c_2 & -\varepsilon_2\\ \tau_m c_3 & \tau_m c_4 & -\varepsilon_6\end{bmatrix}\begin{Bmatrix}\frac{\partial u}{\partial r}\\ \frac{u}{r}\\ \psi\end{Bmatrix}, \quad (20)$$

where

$$c_1=\frac{c_{12}}{c_{11}},\quad c_2=\frac{c_{22}}{c_{11}},\quad c_4=\frac{c_{23}}{c_{11}},\quad \varepsilon_1=\frac{\beta_{11}T_0}{c_{11}},\quad \varepsilon_2=\frac{\beta_{22}T_0}{c_{11}},$$
$$\varepsilon_3=\frac{(\beta_{11}-\beta_{22})T_0}{c_{11}},\quad \varepsilon_4=\frac{\beta_{11}}{\rho C_E},\quad \varepsilon_5=\frac{\beta_{22}}{\rho C_E},\quad \varepsilon_6=\frac{\beta_{33}T_0}{c_{11}}. \quad (21)$$

To solve the present problem one can consider some initial and boundary conditions. The initial conditions may be given by as

$$u(r,t)|_{t=0}=\left.\frac{\partial u(r,t)}{\partial t}\right|_{t=0}=0,\quad \theta(r,t)|_{t=0}=\left.\frac{\partial\theta(r,t)}{\partial t}\right|_{t=0}=0,$$
$$\psi(r,t)|_{t=0}=\left.\frac{\partial\psi(r,t)}{\partial t}\right|_{t=0}=0. \quad (22)$$

Assuming small value of disturbance and confined to neighborhood of interface $r = R$ and hence vanish as $r \to 0$. Then, the regularity conditions are

$$u(r,t)=\psi(r,t)=\theta(r,t)=0 \quad \text{and} \quad r\to\infty. \quad (23)$$

3. Solution of the problem

Firstly, we assume that the surface $r = R$ of cylinder is subjected to temperature pulse in the form

$$\theta(R,t)=\begin{cases}\theta_0\sin(\omega t), & 0\le t\le\frac{\pi}{\omega},\\ 0, & t>\frac{\pi}{\omega},\end{cases} \quad (24)$$

where θ_0 is the amplitude. Secondly, the boundary plane surface $r = R$ of cylinder is traction free. That is

$$\sigma_{rr}(R,t)=0. \quad (25)$$

Using Eq. (11), then one gets

$$\psi(R,t)=\theta_0\sin(\omega t)+\tfrac{1}{2}k_1\theta_0^2\sin^2(\omega t). \quad (26)$$

Applying the Laplace transform to Eqs. (18)-(20) with the aid of Eq. (22) and let us take a material with $\beta_{11} = \beta_{22}$ (i.e., $\varepsilon_4 = \varepsilon_5 = \varepsilon$) and $c_{11} = c_{22}$, one obtains

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{1}{r^2}-\frac{s^2}{1+t_0s}\right)\bar{u}=\frac{\varepsilon_1}{1+t_0s}\frac{\mathrm{d}\bar{\psi}}{\mathrm{d}r}, \quad (27)$$

$$\nabla^2\bar{\psi}=\frac{s(1+\tau_q s)}{1+\tau_\theta s}\left[\bar{\psi}+\varepsilon\left(\frac{\mathrm{d}\bar{u}}{\mathrm{d}r}+\frac{\bar{u}}{r}\right)\right], \quad (28)$$

$$\begin{Bmatrix}\bar{\sigma}_{rr}\\ \bar{\sigma}_{\xi\xi}\\ \bar{\sigma}_{zz}\end{Bmatrix}=\begin{bmatrix}1 & c_1 & -\varepsilon_1\\ c_1 & 1 & -\varepsilon_2\\ c_3 & c_4 & -\varepsilon_6\end{bmatrix}\begin{Bmatrix}(1+t_0s)\frac{\mathrm{d}\bar{u}}{\mathrm{d}r}\\ (1+t_0s)\frac{\bar{u}}{r}\\ \bar{\psi}\end{Bmatrix}. \quad (29)$$

Here, an over bar represents Laplace transform of corresponding function and s is Laplace variable. Eqs. (27) and (28) maybe given as

$$\left(DD_1-\frac{s^2}{1+t_0s}\right)\bar{u}=\frac{\varepsilon_1}{1+t_0s}D\bar{\psi}, \quad (30)$$

$$\varepsilon qD_1\bar{u}=(D_1D-q)\bar{\psi}, \quad (31)$$

where

$$D=\frac{\mathrm{d}}{\mathrm{d}r},\quad D_1=\frac{\mathrm{d}}{\mathrm{d}r}+\frac{1}{r},\quad q=\frac{s(1+\tau_q s)}{1+\tau_\theta s}. \quad (32)$$

Assuming that the radial displacement u is represented as first derivative of new thermoelastic potential unknown ϕ. That is

$$u=\frac{\mathrm{d}\phi}{\mathrm{d}r}, \quad (33)$$

then, Eqs. (30) and (31) are given by

$$\left(D_1D-\frac{s^2}{1+t_0s}\right)\bar{\phi}=\frac{\varepsilon_1}{1+t_0s}\bar{\psi}, \quad (34)$$

$$\varepsilon qD_1D\bar{\phi}=(D_1D-q)\bar{\psi}. \quad (35)$$

Eliminating $\bar{\psi}$ from Eqs. (34) and (35), one gets

$$\left\{\nabla^4-\left[\frac{s^2}{1+t_0s}+q\left(\frac{\varepsilon_1\varepsilon}{1+t_0s}+1\right)\right]\nabla^2+\frac{qs^2}{1+t_0s}\right\}\bar{\phi}=0, \quad (36)$$

which can be rewritten as:

$$(\nabla^2-m_1^2)(\nabla^2-m_2^2)\bar{\phi}=0, \quad (37)$$

where m_1^2 and m_2^2 are the roots of the equation

$$m^4-\left[\frac{s^2}{1+t_0s}+q\left(\frac{\varepsilon_1\varepsilon}{1+t_0s}+1\right)\right]m^2+\frac{qs^2}{1+t_0s}\bar{\phi}=0. \quad (38)$$

These roots are given by

$$m_1^2, m_2^2 = \tfrac{1}{2}\left(2A \pm \sqrt{A^2 - 4B}\right), \tag{39}$$

where

$$A = \tfrac{s^2 + q\varepsilon_1\varepsilon}{1+t_0 s} + q, \quad B = \tfrac{qs^2}{1+t_0 s}. \tag{40}$$

Eq. (37) tends to the following modified Bessel's equation of zero order

$$\left(\tfrac{\mathrm{d}^2}{\mathrm{d}r^2} + \tfrac{1}{r}\tfrac{\mathrm{d}}{\mathrm{d}r} - m_1^2\right)\left(\tfrac{\mathrm{d}^2}{\mathrm{d}r^2} + \tfrac{1}{r}\tfrac{\mathrm{d}}{\mathrm{d}r} - m_2^2\right)\bar{\phi} = 0. \tag{41}$$

It is easy to solve the above equation under regularity conditions that u, θ, $\psi \to 0$ as $r \to 0$. This solution is expressed as

$$\bar{\phi} = \textstyle\sum_{i=1}^{2} A_i I_0(m_i r), \tag{42}$$

where A_i, $i = 1{,}2$ represent two parameters depending on s of Laplace transform. The substitution of Eq. (42) into Eq. (34) gives

$$\tfrac{\varepsilon_1}{1+t_0 s}\bar{\psi} = \textstyle\sum_{i=1}^{2}(m_i^2 - s^2)A_i I_0(m_i r). \tag{43}$$

Substituting from Eq. (42) into Laplace transform of Eq. (33), one obtains

$$\bar{u} = \textstyle\sum_{i=1}^{2} m_i A_i I_1(m_i r). \tag{44}$$

The corresponding stress components maybe obtained in the form

$$\bar{\sigma}_{rr} = -\frac{1+t_0 s}{2r}\sum_{i=1}^{2}\{r(m_i^2 - 2s^2)I_0(m_i r) + m_i[2c_1 I_1(m_i r) + r m_i I_2(m_i r)]\}A_i, \tag{45}$$

$$\bar{\sigma}_{\xi\xi} = \frac{1+t_0 s}{2r}\sum_{i=1}^{2}\{r[m_i^2(c_1 - 2) + 2s^2]I_0(m_i r) + m_i[2I_1(m_i r) + r c_1 m_i I_2(m_i r)]\}A_i, \tag{46}$$

$$\bar{\sigma}_{zz} = \frac{1+t_0 s}{2}\sum_{i=1}^{2}\left\{\left[c_3 m_i^2 - \frac{2\varepsilon_6}{\varepsilon_1}(m_i^2 - s^2)\right]I_0(m_i r) + \frac{2m_i c_4 I_1(m_i r)}{r} + c_3 m_i^2 I_2(m_i r)\right\}A_i. \tag{47}$$

The boundary conditions appeared in Eqs. (25) and (26) in Laplace domain may be transfer to

$$\bar{\psi}(R,s) = \theta_0\omega\left(\tfrac{1}{s^2+\omega^2} + \tfrac{k_1\theta_0\omega}{s^2+4s\omega^2}\right) = \bar{G}(s), \tag{48}$$

$$\bar{\sigma}_{rr}(R,s) = 0. \tag{49}$$

Using Eqs. (43) and (45) into Eqs. (48) and (49) to get a system of two equations in A_i as

$$\sum_{i=1}^{2}(m_i^2 - s^2)A_i I_0(m_i r) = \frac{\varepsilon_1\bar{G}(s)}{1+t_0 s}, \tag{50}$$

$$\sum_{i=1}^{2}\{R(m_i^2 - 2s^2)I_0(m_i R) + m_i[2c_1 I_1(m_i R) + R m_i I_2(m_i R)]\}A_i = 0. \tag{51}$$

The solution maybe completed after getting A_i. Moreover, the temperature $\bar{\theta}$ can be easily obtained from Eq. (11) after applying Laplace transform as

$$\bar{\theta}(r,s) = \tfrac{\sqrt{1+2K_1\bar{\psi}}-1}{k_1}. \tag{52}$$

4. Numerical results and discussions

The obtained solution for temperature, radial displacement, and stresses is attempted in Laplace transform domain. In this section, we try to get the distributions of such variables in their inverted forms. Numerical inversion method based on a Fourier series expansion (Honig and Hirdes 1984) is adopted to invert Laplace transform in Eqs. (43)-(47). The variable quantity in Laplace domain maybe inverted to the time domain by using the expression

$$f(t) = \tfrac{\mathrm{e}^{ct}}{t}\left[\tfrac{\bar{f}(c)}{2} + \mathrm{Re}\left\{\textstyle\sum_{n=1}^{N}(-1)^n \bar{f}\left(c + \tfrac{in\pi}{t}\right)\right\}\right], \tag{53}$$

in which c is experimentally satisfies the relation $ct \approx 4.7$ (Honig and Hirdes 1984).

In order to observe the validity and efficiency of our system and also to get the distribution responses for different field variables like displacement u, temperature θ and stresses σ_{rr}, $\sigma_{\xi\xi}$ and σ_{zz} inside the medium we have done numerical computations with the help of computer programming. The results have been graphically presented for thermoviscoelastic (TVE) and thermoelastic (TE) cylinder. Results are calculated by choosing *Cobalt* as an orthotropic material with elastic properties (in SI units) at $T_0 = 298\ K$ (Misra *et al.* 1996) as

$$\begin{gathered} c_{11} = c_{22} = 3.071\times10^{11}\ \mathrm{N/m}, \quad c_{12} = 1.650\times10^{11}\ \mathrm{N/m}, \quad \rho = 8836\ \mathrm{kg/m^3}, \\ k_0 = 69\ \mathrm{W/(mKs)}, \quad C_E = 427\ \mathrm{J/(kg\,K)}, \quad \beta_{11} = \beta_{22} = 7.04\times10^6\mathrm{N/(m^2K)}, \\ \beta_{33} = 6.90\times10^6\mathrm{N/(m^2K)}. \end{gathered} \tag{54}$$

The outer radius of cylinder is taken as $R = 1$ and period of time is considered as $t = 0.12$. Results are illustrated in Figs. 1-15. The nature of variations of various fields observed in these figures indicates that the system of equations of viscoelastic orthotropic materials of efficiently compute the numerical solutions of the problem. Also the obtained solutions are in complete agreement with boundary conditions of the problem. From these figures, we find that the field quantities depend not only on state and space variables t and r, but also depend on variability thermal conductivity parameter and phase-lags parameters. It is to be noted that field quantities are plotted along radial direction (from right to left) with $r = 1$ as a starting point and $r = 0$ as an ending point. Three cases are discussed here as follows:

In the first case, three values of variability thermal conductivity parameter k_1 in the case of viscous solids are used. The values $k_1 = -1$ and $k_1 = -0.5$ are taken for variable thermal conductivity while $k_1 = 0$ when thermal conductivity is temperature-independent. The variations with spatial coordinate r has been observed in Figures 1-5 when τ_q and τ_θ remain constants ($\tau_q = 0.2$, $\tau_\theta = 0.1$). It is seen that the parameter k_1 has significant effects on all the fields. We also observed the following important facts:

Fig. 1 shows that the variations of temperature distributions change initially and decrease with the passage of time. It can also be seen that temperature decreases as parameter k_1 decreases.

It is seen from Fig. 2 that value of displacement increases with the increase of parameter k_1. It is also found that the effect of disturbance approaches zero at a distance far from the surface of the cylinder. Similar observations can be made from Figs. 3, 4, and 5 when thermal stresses are considered.

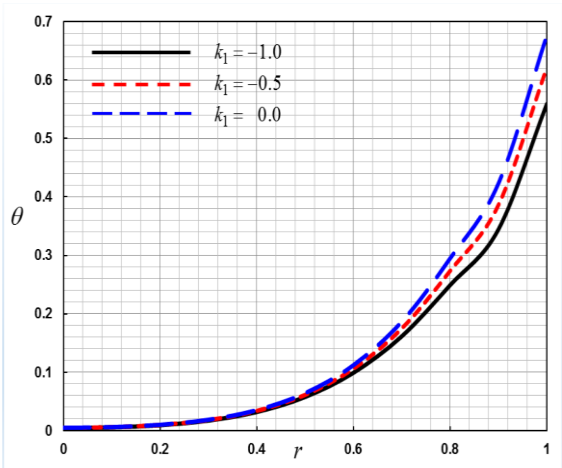

Fig. 1 Distribution of temperature θ for different values of variability thermal conductivity

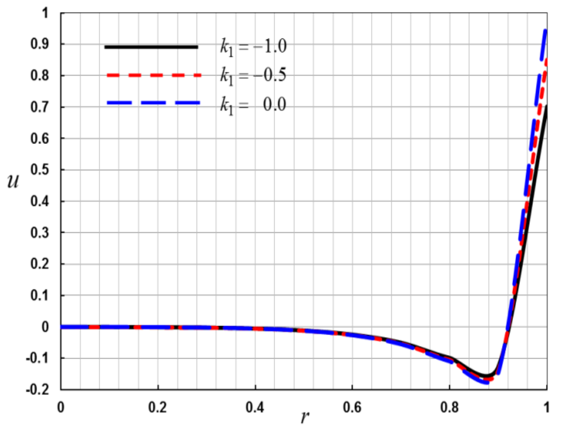

Fig. 2 Distribution of radial displacement u for different values of variability thermal conductivity

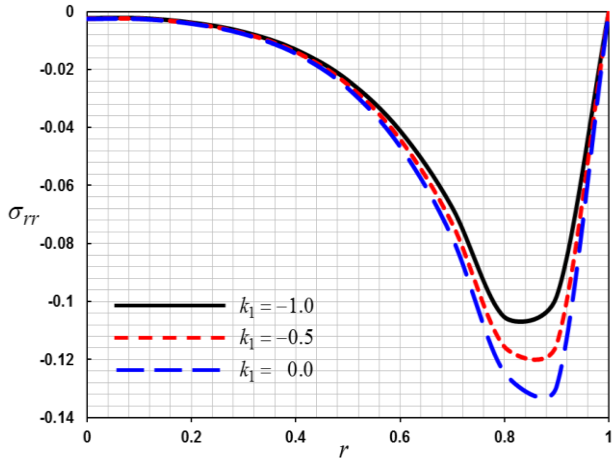

Fig. 3 Distribution of stress σ_{rr} for different values of variability thermal conductivity

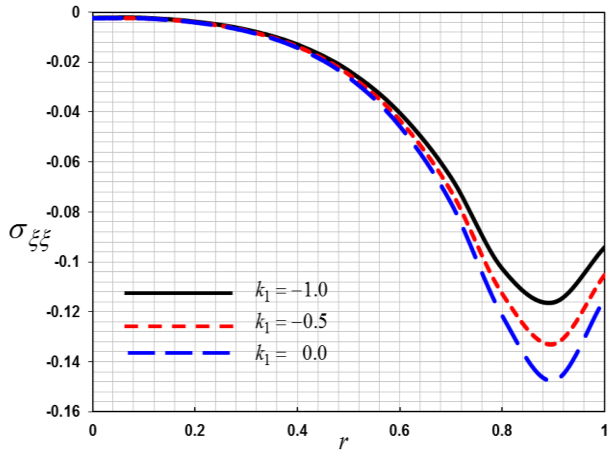

Fig. 4 Distribution of stress $\sigma_{\xi\xi}$ for different values of variability thermal conductivity

• It is observed in Fig. 3 that the variation of σ_{rr} ends with zero value at r=1 for all cases which agrees with boundary condition and it decreases continuously to attain its lowest value. It attains its highest negative at $r \approx 0.9$ and has decreasing behavior for the interval $1 \geq r \geq 0.9$ and increasing behavior for the interval $0.9 \geq r \geq 0$. It can also be seen from the plot that variability thermal conductivity parameter k_1 acts to increase the magnitude of stress σ_{rr}. It is found that the disturbance is prominent in the neighborhood of the surface of the cylinder and the disturbance gradually diminishes as the radial distance decreases. Similar observations can be made from Figs. 4 and 5 when the stresses $\sigma_{\xi\xi}$ and σ_{zz} are plotted against r.

• It is also apparent from the figure that $\sigma_{\xi\xi}$ and σ_{zz} increase as the parameter k_1 values decreases.

The second case is devoted to discuss effect of mechanical relaxation time due to viscosity t_0 on temperature, displacement and stresses when $\tau_q = 0.2$, $\tau_\theta = 0.1$ and $k_1 = -0.5$. Three values ($t_0 = 0.2$, 0.1 and 0) are considered in Figs. 6-10. The nature of variation of the fields clearly changes with viscosity parameter t_0 and a prominent effect of viscosity parameter upon all profiles under the DPL theory is indicated.

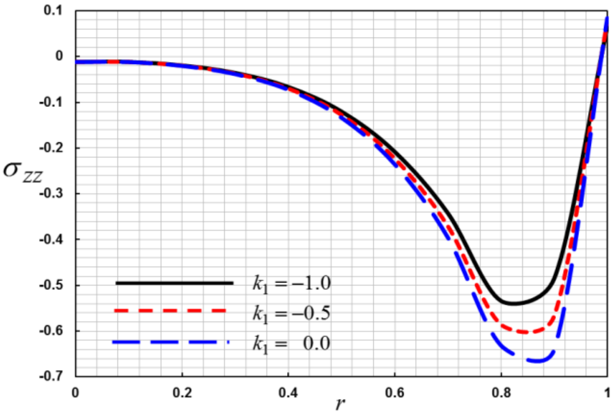

Fig. 5 Distribution of stress σ_{zz} for different values of variability thermal conductivity parameter

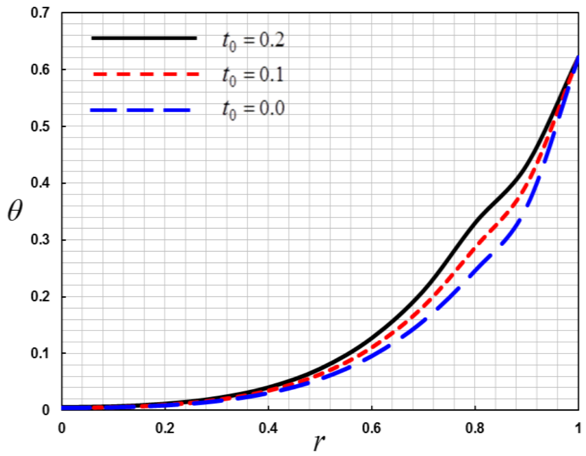

Fig. 6 Distribution of temperature θ for mechanical relaxation time due to viscosity t_0

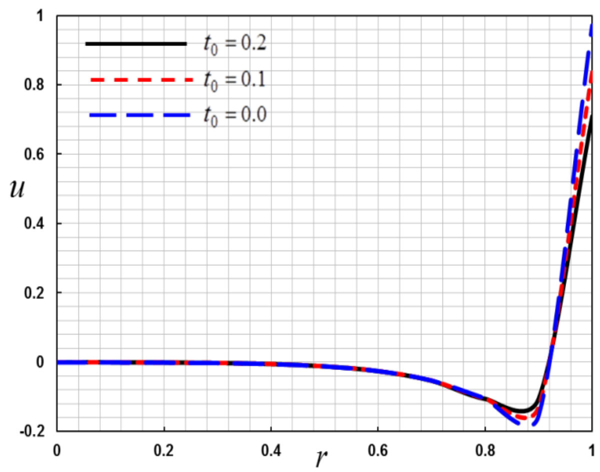

Fig. 7 Distribution of displacement u for mechanical relaxation time due to viscosity t_0

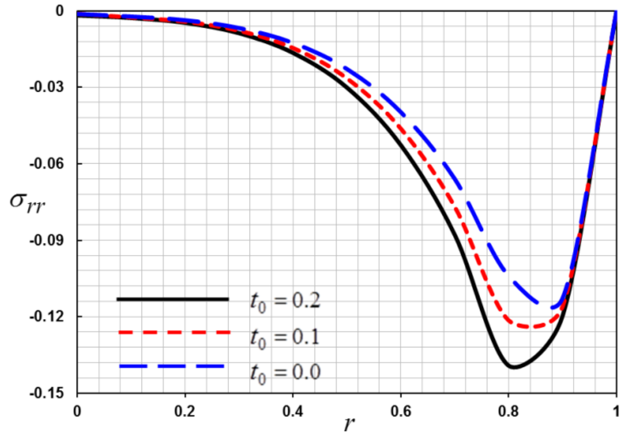

Fig. 8 Distribution of stress σ_{rr} for mechanical relaxation time due to viscosity t_0

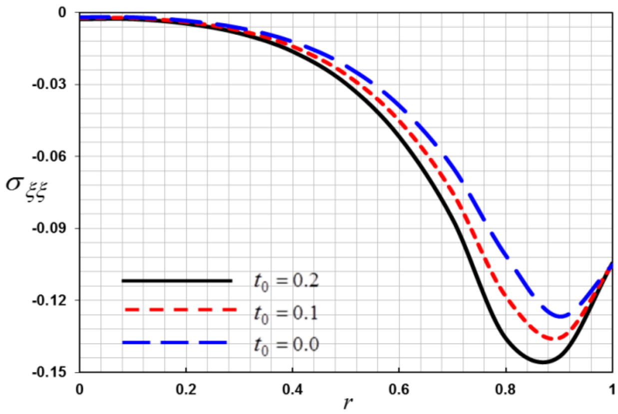

Fig. 9 Distribution of stress $\sigma_{\xi\xi}$ for mechanical relaxation time due to viscosity t_0

The comparison of dimensionless physical quantities is made for the two different cases: (i) thermoviscoelastic solid (TVE) when $t_0 = 0.2$ and $t_0 = 0.1$, and (ii) thermoelastic solid (TE) when $t_0 = 0$. It is also observed the following important notes:

- The influence of viscosity parameter is very pronounced on temperature and stresses.
- Fig. 6 shows that viscosity parameter acts to increase magnitude of temperature distribution. It is observed that θ in TEV theory is larger than its behavior as compared to TE theory. As the value of t_0 increases the absolute values of θ increases. The behavior of temperature for both theories (TEV and TE) is alike.
- From Fig. 7 we see that, when the viscosity increases, the absolute value of radial displacement u decreases.
- Values of u in TE theory are the largest in comparison with those in TEV theory. The radial displacement component is also having similar pattern in the discussed three theories.
- In Figs. 8 and 9, the absolute values of stresses σ_{rr} and σ_{zz} increase as t_0 increases.
- The difference in values of $\sigma_{\xi\xi}$ at a particular point for various values of viscosity parameter is illustrated in Fig. 10.

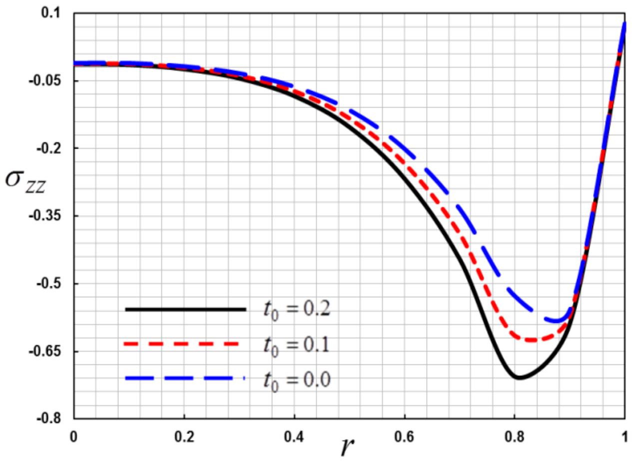

Fig. 10 Distribution of stress σ_{zz} for mechanical relaxation time due to viscosity t_0

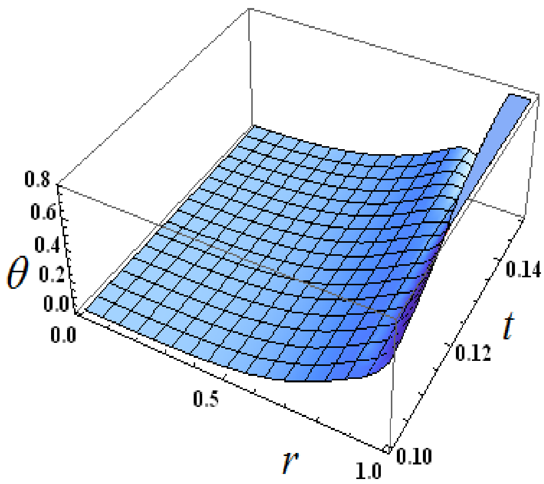

Fig. 11 Distribution of the temperature θ for different times t

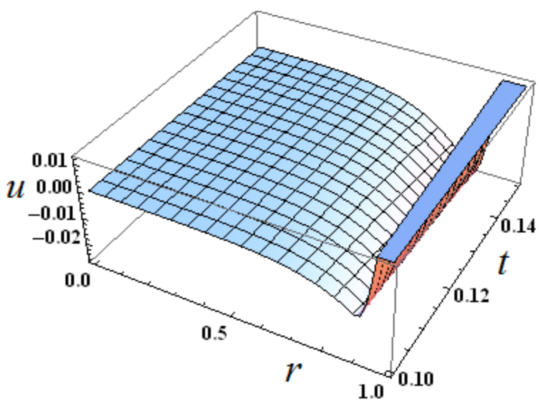

Fig. 12 Distribution of the displacement u for different times t

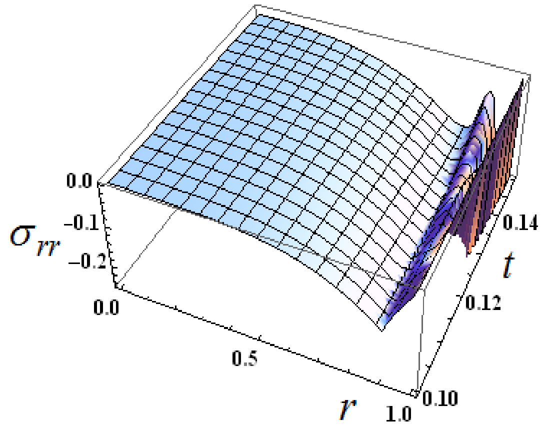

Fig. 13 Distribution of stress σ_{rr} for different times t

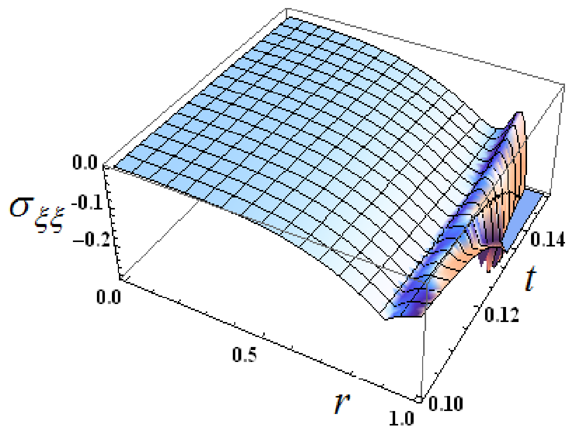

Fig. 14 Distribution of stress $\sigma_{\xi\xi}$ for different times t

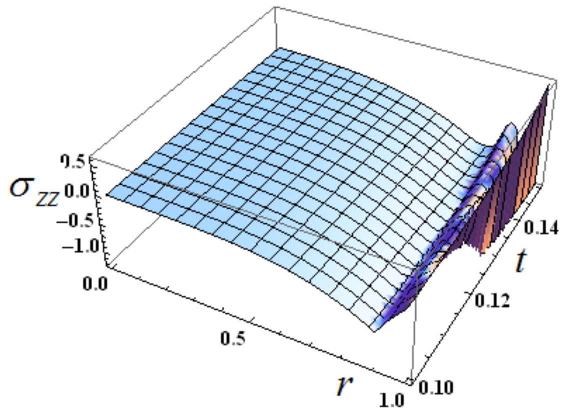

Fig. 15 Distribution of stress σ_{zz} for different times t

The third case is to investigate how temperature, displacement and stresses vary with t when phase-lags τ_q, τ_θ and viscosity parameter t_0 remain constants (Figs. 11-15). We can see the significant effect of the time t on all the studied fields. We found that, increasing in value of time causes increasing in values of temperature, displacement and stresses fields.

5. Conclusions

This article constructs the model of generalized thermoviscoelasticity for a homogeneous orthotropic infinite solid cylinder with a variable thermal conductivity based on DPL model. Outer surface is taken to be traction-free and under temperature pulse. The problem is numerically solved using Laplace transform technique. Numerical results for the displacement, temperature, stresses distributions are illustrated graphically. Comparisons are made of results due to various theories in cases of temperature dependent and independent modulus of elasticity. From the numerical results, it is concluded that:

1. The viscosity parameter plays an important role and is more pronounced in thermoviscoelasticity case.
2. The variability thermal conductivity parameter has significant effects on speed of wave propagation of all the studied fields.
3. The phase-lags have great effects on the field quantities.
4. The effects of the time parameter on all the studied fields are very significant.
5. The theories of coupled thermoelasticity, generalized thermoelasticity with one relaxation time can extracted as special cases.

Finally, the outputs of this article should prove useful to investigators in the development of continuum mechanics, as well as to investigators in neighbor branches. Also, the results presented here may provide interesting information for experimental scientists and researchers working on this subject.

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Nomenclature

C_E	specific heat at uniform strain
c_{ij}	isothermal elastic constants
$I_0(\cdot)$, $I_1(\cdot)$, $I_2(\cdot)$	modified Bessel's functions of second kinds of order zero, one and two
K_r	thermal conductivity
k_0	thermal conductivity at ambient temperature T_0
k_1	slope of thermal conductivity-temperature curve divided by k_0
T_0	environment temperature
t_0	mechanical relaxation time due to the viscosity
$\vec{q}$	heat flux vector
u_r	radial displacement
β_{ij}	thermal elastic coupling components
ε_{rr}, $\varepsilon_{\xi\xi}$	radial and circumferential strains
(r, ξ, z)	cylindrical coordinates system
ρ	material density
σ_{rr}, $\sigma_{\xi\xi}$, σ_{zz}	normal mechanical stress components
$\theta = T - T_0$	thermodynamical temperature
τ_θ	phase-lag of temperature gradient
τ_q	phase-lag of heat flux
ω	circular frequency of sinusoidal pulse