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Energy based approach for solving conservative nonlinear systems

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Abstract. This paper concerns two new analytical approaches for solving high nonlinear vibration equations. Energy Balance method and Hamiltonian Approach are presented and successfully applied for nonlinear vibration equations. In these approaches, there is no need to use small parameters to solve and only with one iteration, high accurate results are reached. Numerical procedures are also presented to compare the results of analytical and numerical ones. It has been established that, the proposed approaches are in good agreement with numerical solutions.

Keywords: conservative systems; nonlinear vibration; Energy Balance method; Hamiltonian Approach

1. Introduction

To consider physical phenomena it has been needed to prepare a mathematical model. Usually these models are in nonlinear case. Nonlinear oscillators are wildly used form physical and engineering problems. It an easy task to have exact solutions for linear engineering problems, but in nonlinear problems its hard to prepare an exact solution and in some cases its impossible. There are many numerical methods for solving nonlinear engineering problems. Many researchers have been working on different analytical and semi- analytical solutions for nonlinear engineering problems in recent years. One of the traditional analytical methods is perturbation technique, which has many shortcomings. Perturbation technique is useful only for weak nonlinear problems not high nonlinear ones. The new proposed approaches have been studied recently to over come the shortcomings of traditional methods such as:

Harmonic balance method (Huseyin *et al*. 1991, Civalek 2013, 2006, Lau *et al*. 1983), Hamiltonian approach (He 2010, Xu 2010), Energy balance method (Jamshidi *et al*. 2010, Mehdipour 2010, He 2002), Homotopy perturbation method (Shaban et al. 2010), Hamiltonian approach (He 2010, Xu 2010), Energy balance method (Jamshidi *et al*. 2010, Mehdipour 2010, He 2002), Max-Min approach (Zeng *et al*. 2009), and the other analytical and numerical (Sedighi *et al*. 2016, 2015, Bayat *et al*. 2015a, b, 2016, Cai and Liu 2011, Wu 2011, Bayat *et al*. 2012, Cunedioglu and Beylergil 2014). Among of above methods, Energy Balance Method (EBM) and Hamiltonian approach (HA) have been considered in this paper to solve high nonlinear problems. The most benefit of analytical approaches are preparing a great understanding from the behavior of the system and considering the effects of different parameters on response of the problem.

The results of these two approaches are compared with the numerical solution using Runge-Kutta's algorithm.

The paper has been organized as follows:

Mathematical problem is studied and the governing equation of the problem is developed. Basic idea of Hamiltonian approach and Energy balance method have been described, then applications of Hamiltonian Approach and Energy Balance Method have been studied in detailed procedure to illustrate the applicability and accuracy of these methods. Some comparisons between analytical and numerical solutions have done and tabulated. Eventually, we have conclude the contains of the most significant findings of the paper.

2. Thin circular sector cylinder formulation

Swinging oscillation of thin circular sector cylinder In this condition a thin circular sector cylinder is considered as Shown in Fig. 1. As before thin circular sector cylinder rolls in an oscillatory motion back and for thon a flat stationary support, with no sliding effect. Governing equation of the oscillation is as follow (Shaban *et al*. 2010)

$$\begin{gathered}\left(R^2-R\bar{y}\cos(\theta)\right)\left(2\ddot{\theta}\right)+R\left(\bar{y}\sin(\theta)\right)\dot{\theta}^2+\left(g\bar{y}\right)\sin(\theta)=0\\ \theta(0)=A,\ \ \dot{\theta}(0)=0,\end{gathered}\tag{1}$$

Where the geometrical parameters are shown in Fig. 1. The height of mass center obtained as below

$$\bar{y}=\frac{R\sin(\alpha)}{\alpha}\tag{2}$$

Introducing the dimensionless time variable

$$\bar{t}=\sqrt{\frac{1}{\bar{y}}}t=\left(\sqrt{\frac{R\sin(\alpha)}{\alpha}}\right)^{-1}t.\tag{3}$$

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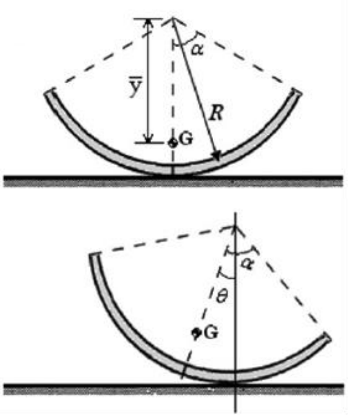


Fig. 1 Geometric parameters of the homogeneous thin circular sector cylinder

Eq. (57) becomes

$$\left(\frac{R}{\bar{y}}-\cos(\theta)\right)\left(2\ddot{\theta}\right)+\sin(\theta)\dot{\theta}^2+\frac{g\bar{y}}{R}\sin(\theta)=0$$
$$\theta(0)=A,\quad \dot{\theta}(0)=0. \tag{4}$$

And by introducing the dimensionless geometrical parameter

$$\lambda=\frac{\bar{y}}{R}=\frac{\sin(\alpha)}{\alpha} \tag{5}$$

Eq. (60) becomes

$$\left(\frac{1}{\lambda}-\cos(\theta)\right)\left(2\ddot{\theta}\right)+\sin(\theta)\dot{\theta}^2+g\lambda\sin(\theta)=0$$
$$\theta(0)=A,\quad \dot{\theta}(0)=0. \tag{6}$$

3. Basic idea of Hamiltonian Approach (HA)

Recently, He (2010) has proposed the Hamiltonian approach to overcome the shortcomings of the energy balance method. This approach is a kind of energy method with a vast application in conservative oscillatory systems. In order to clarify this approach, consider the following general oscillator

$$\ddot{\theta}+f\left(\theta,\dot{\theta},\ddot{\theta}\right)=0 \tag{7}$$

With initial conditions

$$\theta(0)=A,\quad \dot{\theta}(0)=0. \tag{8}$$

Oscillatory systems contain two important physical parameters, i.e., the frequency ω and the amplitude of oscillation A. It is easy to establish a variational principle for Eq. (1), which reads

$$J(\theta)=\int_0^{T/4}\left\{-\frac{1}{2}\dot{\theta}^2+F(\theta)\right\}dt \tag{9}$$

Where Tis period of the nonlinear oscillator, $\partial F/\partial\theta=f$.

In the Eq. (3), $\frac{1}{2}\dot{\theta}^2$ is kinetic energy and $F(\theta)$ potential energy, so the Eq. (3) is the least Lagrangian action, from which we can immediately obtain its Hamiltonian, which reads

$$H(\theta)=\frac{1}{2}\dot{\theta}^2+F(\theta)=\text{constant} \tag{10}$$

From Eq. (4), we have

$$\frac{\partial H}{\partial A}=0 \tag{11}$$

Introducing a new function, $\bar{H}(\theta)$, defined as

$$\bar{H}(u)=\int_0^{T/4}\left\{\frac{1}{2}\dot{\theta}^2+F(\theta)\right\}dt=\frac{1}{4}TH \tag{12}$$

Eq. (5) is, then, equivalent to the following one

$$\frac{\partial}{\partial A}\left(\frac{\partial\bar{H}}{\partial T}\right)=0 \tag{13}$$

or

$$\frac{\partial}{\partial A}\left(\frac{\partial\bar{H}}{\partial(1/\omega)}\right)=0 \tag{14}$$

From Eq. (8) we can obtain approximate frequency-amplitude relationship of a nonlinear oscillator.

4. Basic idea of Energy Balance Method (EBM)

In the present paper, we consider a general nonlinear oscillator in the Form (He 2008)

$$\ddot{\theta}+f\left(\theta(t)\right)=0 \tag{15}$$

In which θ and t are generalized dimensionless displacement and time variables, respectively. Its variational principle can be easily obtained

$$J(\theta)=\int_0^{t}\left(-\tfrac{1}{2}\dot{\theta}^2+F(\theta)\right)dt \tag{16}$$

Where $T=\frac{2\pi}{\omega}$ is period of the nonlinear oscillator, $F(\theta)=\int f(\theta)d\theta.$

Its Hamiltonian, therefore, can be written in the form

$$H=\tfrac{1}{2}\dot{\theta}^2+F(\theta)=F(A) \tag{17}$$

Or

$$\Re(t)=\tfrac{1}{2}\dot{\theta}^2+F(\theta)-F(A)=0 \tag{18}$$

Oscillatory systems contain two important physical parameters, i.e..

The frequency ω and the amplitude of oscillation *A*. So let us consider such initial conditions

$$\theta(0)=A, \quad \dot{\theta}(0)=0 \tag{19}$$

We use the following trial function to determine the angular frequency

$$\theta(t)=A\cos(\omega t) \tag{20}$$

Substituting (13) into θ term of (11), yield

$$\Re(t)=\frac{1}{2}\omega^2A^2\sin^2(\omega t)+F\left(A\cos(\omega t)\right)-F\left(A\right)=0 \tag{21}$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make $\Re$ zero for all values of *t* by appropriate choice of ω. Since Eq. (13) is only an approximation to the exact solution, $\Re$ cannot be made zero everywhere. Collocation at $\omega t=\pi/4$ gives

$$\omega=\sqrt{\frac{2\left(F\left(A\right)\right)-F\left(A\cos(\omega t)\right)}{A^2\sin^2(\omega t)}} \tag{22}$$

Its period can be written in the form

$$T=\frac{2\pi}{\sqrt{\frac{2\left(F\left(A\right)\right)-F\left(A\cos(\omega t)\right)}{A^2\sin^2(\omega t)}}} \tag{23}$$

5. Application

5.1 Solution using HA

By using the Taylor's series expansion for $\cos(\theta(t))$, $\sin(\theta(t))$ we can re-write Eq. (6) in the following form

$$\begin{aligned}&\left(\frac{1}{\lambda}-\left(1-\tfrac{1}{2}\theta^2+\tfrac{1}{24}\theta^4\right)\right)\left(2\ddot{\theta}\right)\\&\quad+\left(\theta-\tfrac{1}{6}\theta^3\right)\dot{\theta}^2+g\lambda\left(\theta-\tfrac{1}{6}\theta^3\right)=0\end{aligned} \tag{24}$$

The Hamiltonian of Eq. (22) is constructed as

$$H=-\frac{\dot{\theta}^2}{\lambda}+\dot{\theta}^2-\tfrac{1}{2}\dot{\theta}^2\theta^2+\tfrac{1}{24}\dot{\theta}^2\theta^4+\frac{g\lambda}{2}\theta^2-\frac{g\lambda}{24}\theta^4 \tag{25}$$

Integrating Eq. (21) with respect to t from 0 to *T*/4, we have

$$\bar{H}=\int_0^{T/4}\left(-\frac{\dot{\theta}^2}{\lambda}+\dot{\theta}^2-\tfrac{1}{2}\dot{\theta}^2\theta^2+\tfrac{1}{24}\dot{\theta}^2\theta^4+\frac{g\lambda}{2}\theta^2-\frac{g\lambda}{24}\theta^4\right)dt \tag{26}$$

Assume that the solution can be expressed as

$$\theta(t)=A\cos(\omega t) \tag{27}$$

Substituting Eq. (25) into Eq. (24), we obtain

$$\begin{aligned}\bar{H}&=\int_0^{T/4}\begin{pmatrix}-\dfrac{A^2\omega^2\sin^2(\omega t)}{\lambda}+A^2\omega^2\sin^2(\omega t)\\-\tfrac{1}{2}A^4\omega^2\sin^2(\omega t)\cos^2(\omega t)\\+\tfrac{1}{24}A^6\omega^2\sin^2(\omega t)\cos^4(\omega t)\\+\tfrac{g\lambda}{2}A^2\cos^2(\omega t)-\tfrac{g\lambda}{24}A^4\cos^4(\omega t)\end{pmatrix}dt\\&=\int_0^{\pi/2}\begin{pmatrix}-\dfrac{A^2\omega^2\sin^2 t}{\lambda}+A^2\omega^2\sin^2 t\\-\tfrac{1}{2}A^4\omega^2\sin^2 t\cos^2 t+\tfrac{1}{24}A^6\omega^2\sin^2 t\cos^4 t\\+\tfrac{g\lambda}{2}A^2\cos^2 t-\tfrac{g\lambda}{24}A^4\cos^4 t\end{pmatrix}dt\\&=-\frac{1}{4}\frac{\omega\pi}{\lambda}A^2+\frac{\omega\pi}{4}A^2-\frac{\omega\pi}{32}A^4\\&\quad+\frac{\omega\pi}{768}A^6-\frac{1}{128}\frac{g\lambda\pi}{\omega}A^4+\frac{1}{8}\frac{g\lambda\pi}{\omega}A^2\end{aligned} \tag{28}$$

Setting

$$\begin{aligned}\frac{\partial}{\partial A}\left(\frac{\partial\bar{H}}{\partial(1/\omega)}\right)&=-\frac{\pi\omega^2}{2\lambda}A+\frac{\pi\omega^2}{2}A-\frac{\omega^2\pi}{8}A^3\\&\quad+\frac{\omega^2\pi}{128}A^5-\frac{1}{32}\pi g\lambda A^3+\frac{1}{4}\pi g\lambda A\end{aligned} \tag{29}$$

Solving the above equation, an approximate frequency as a function of amplitude equal to

$$\omega_{HA}=\frac{2\lambda\sqrt{g\left(A^2-8\right)}}{\sqrt{\left(\lambda A^4-16\lambda A^2+64\lambda-64\right)}} \tag{30}$$

Hence, the approximate solution can be readily obtained

$$\theta(t)=A\cos\left(\frac{2\lambda\sqrt{g\left(A^2-8\right)}}{\sqrt{\left(\lambda A^4-16\lambda A^2+64\lambda-64\right)}}\,t\right) \tag{31}$$

5.2 Solution using EBM

By using the Taylor's series expansion for $\cos(\theta(t))$, $\sin(\theta(t))$ the variational formulation can be obtain from Eq. (6) as follows

$$J(\theta)=\int_0^t\left(-\frac{\dot{\theta}^2}{\lambda}+\dot{\theta}^2-\tfrac{1}{2}\dot{\theta}^2\theta^2+\tfrac{1}{24}\dot{\theta}^2\theta^4+\frac{g\lambda}{2}\theta^2-\frac{g\lambda}{24}\theta^4\right)dt. \tag{32}$$

Its Hamiltonian, therefore, can be written in the form

$$\begin{aligned}H&=\left(-\frac{\dot{\theta}^2}{\lambda}+\dot{\theta}^2-\tfrac{1}{2}\dot{\theta}^2\theta^2+\tfrac{1}{24}\dot{\theta}^2\theta^4+\frac{g\lambda}{2}\theta^2-\frac{g\lambda}{24}\theta^4\right)\\&=\tfrac{1}{2}\lambda A^2-\tfrac{1}{24}g\lambda A^4\end{aligned} \tag{33}$$

or

$$\begin{aligned}R(t)&=-\frac{\dot{\theta}^2}{\lambda}+\dot{\theta}^2-\tfrac{1}{2}\dot{\theta}^2\theta^2+\tfrac{1}{24}\dot{\theta}^2\theta^4+\frac{g\lambda}{2}\theta^2\\&\quad-\frac{g\lambda}{24}\theta^4-\tfrac{1}{2}\lambda A^2+\tfrac{1}{24}g\lambda A^4,\end{aligned} \tag{34}$$

We will use the trial function to determine the angular frequency ω, i.e.

$$\theta(t) = A\cos(\omega t) \tag{35}$$

If we substitute (35) into (34), it results the following residual equation

$$\begin{aligned} R(t) &= -\frac{\dot{\theta}^2}{\lambda} + \dot{\theta}^2 - \tfrac{1}{2}\dot{\theta}^2\theta^2 + \tfrac{1}{24}\dot{\theta}^2\theta^4 + \frac{g\lambda}{2}\theta^2 - \frac{g\lambda}{24}\theta^4 \\ &= -\frac{A^2\omega^2\sin^2(\omega t)}{\lambda} - A^2\omega^2\sin^2(\omega t) \\ &\quad + \tfrac{1}{2}A^4\omega^2\sin^2(\omega t)\cos^2(\omega t) \\ &\quad - \tfrac{1}{24}A^6\omega^2\sin^2(\omega t)\cos^4(\omega t) + \tfrac{1}{2}g\lambda A^2\cos^2(\omega t) \\ &\quad - \tfrac{1}{24}g\lambda A^4\cos^4(\omega t) - \tfrac{1}{2}\lambda A^2 + \tfrac{1}{24}g\lambda A^4, \end{aligned} \tag{36}$$

If we collocate at $\omega t = \pi/4$ we obtain

$$\begin{aligned} &\tfrac{1}{2}\frac{A^2\omega^2}{\lambda} - \tfrac{1}{2}A^2\omega^2 + \tfrac{1}{8}A^4\omega^2 \\ &- \tfrac{1}{192}A^6\omega^2 - \tfrac{1}{4}g\lambda A^2 + \tfrac{1}{32}g\lambda A^4 = 0 \end{aligned} \tag{37}$$

This leads to the following result

$$\omega_{EBM} = \frac{\lambda\sqrt{6g\left(-8 + A^2\right)}}{\sqrt{\left(-24A^2\lambda + A^4\lambda - 96 + 96\lambda\right)}} \tag{38}$$

According to Eqs. (35) and (38), we can obtain the following approximate solution

$$\theta(t) = A\cos\left(\frac{\lambda\sqrt{6g\left(A^2 - 8\right)}}{\sqrt{\left(A^4\lambda - 24A^2\lambda + 96\lambda - 96\right)}}t\right) \tag{39}$$

6. Results and discussions

The important step in this paper is comparison of the analytical solutions and numerical ones. The results of Hamiltonian approach and Energy balance method are compared with the numerical one (Runge-kutta algorithm). Table 1 is the comparison of important parameters on the frequency of the systems (A, α, g). The maximum error between the results are less than 1.7 % for different values of (A, α, g). Fig. 2 is the comparison of the behavior of the system for two different cases:

(a): $A = \pi/6,\ \alpha = \pi/2,\ g = 9.81$

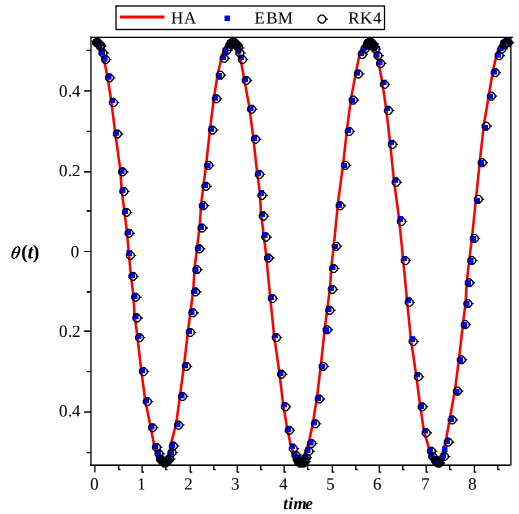
(b): $A = \pi/4,\ \alpha = \pi/12,\ g = 9.81$

From the figure, the behavior of the oscillation is cyclic.

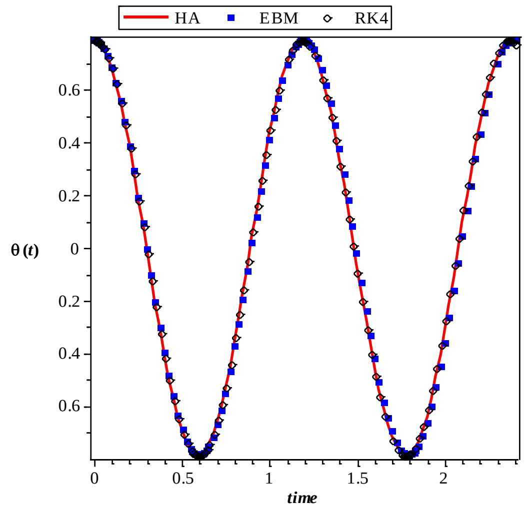
To show the effects of the important parameters such as angel and amplitude, we have developed the Fig 3. Different angels from $\pi/12$ to $\pi/2$ are studied in this figure. Fig. 4, is the sensitive analysis between the important parameters (A, α, ω).

Table 1 Comparison of frequency corresponding to various parameters of system

A	α	g	(HA)	(EBM)	RK4	Error % HA	Error % EBM
$\pi/12$	$\pi/6$	9.81	8.50100	8.49939	8.51288	0.13948	0.15849
$\pi/8$	$\pi/4$	9.81	5.39311	5.39087	5.40454	0.21136	0.25284
$\pi/6$	$\pi/2$	9.81	2.17381	2.17315	2.17422	0.01860	0.04925
$\pi/4$	$\pi/12$	9.81	5.29230	5.25979	5.32158	0.55009	1.16116
$\pi/3$	$\pi/8$	9.81	3.82760	3.79573	3.84757	0.51898	1.34753
$\pi/2$	$\pi/3$	9.81	1.95909	1.93793	1.97154	0.63169	1.70495



(a)



(b)

Fig. 2 Comparison of time history response of the analytical solution and numerical solution for

(a): $A = \pi/6,\ \alpha = \pi/2,\ g = 9.81$,

(b): $A = \pi/4,\ \alpha = \pi/12,\ g = 9.81$

It is obvious from the figures and tables the Hamiltonian approach and Energy balance method have an excellent agreement with the numerical solution and quickly convergent and valid for a wide range of vibration amplitudes and initial conditions.

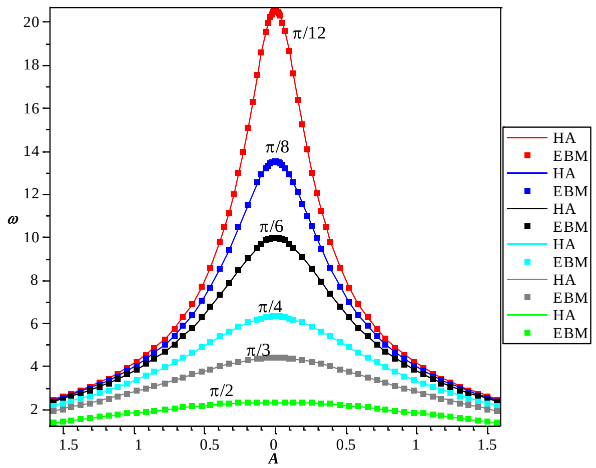


Fig. 3 The effects of angel and amplitude on the nonlinear frequency

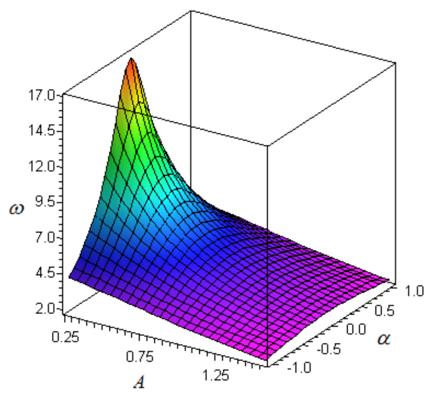


Fig. 4 Sensitivity analysis of nonlinear frequency

7. Conclusions

In this paper, two new analytical approaches called: Hamiltonian approaches and Energy balance method were studied in detailed. High nonlinear problem of the thin circular cylinder had been studied. It has been proved that the Hamiltonian approach and Energy Balance Method are clearly effective, convenient and does not require any linearization or small perturbation, and adequately accurate for nonlinear problems in physics and engineering. It has illustrated that the results of HA and EBM are in an excellent agreement with those obtained by the numerical one. These method scan be easily extended to any nonlinear oscillator without any difficulty.

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Appendix A: Basic idea of Runge-Kutta's Method

For the numerical approach to verify the analytic solution, the fourth RK (Runge-Kutta) method has been used. This iterative algorithm is written in the form of the following formulation

$$\dot{\theta} = f\,(t, \theta), \quad \theta(t_0) = \theta_0 \tag{A.1}$$

θ is an unknown function of time t which we would like to approximate. Then RK4 method is given for this problem as below

$$\begin{aligned} \theta_{n+1} &= \theta_n + \tfrac{1}{6} h \left(k_1 + 2k_2 + 2k_3 + k_4\right), \\ t_{n+1} &= t_n + h. \end{aligned} \tag{A.2}$$

forn=0,1,2,3,..., using

$$\begin{aligned} k_1 &= f\,(t_n, \theta_n), \\ k_2 &= f\left(t_n + \tfrac{1}{2}h,\ \theta_n + \tfrac{1}{2}hk_1\right), \\ k_3 &= f\left(t_n + \tfrac{1}{2}h,\ \theta_n + \tfrac{1}{2}hk_3\right), \\ k_4 &= f\,(t_n + h,\ \theta_n + hk_3). \end{aligned} \tag{A.3}$$

Where θ_{n+1} is the RK4 approximation of $\theta(t_{n+1})$. The fourth-order Runge-Kutta method requires four evaluations of the right hand side per step h.