Earthquakes and Structures, *Vol. 10, No. 3 (2016) 681-697*
DOI: http://dx.doi.org/10.12989/eas.2016.10.3.681

Non-simple magnetothermoelastic solid cylinder with variable thermal conductivity due to harmonically varying heat

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(Received July 18, 2014, Revised November 1, 2015, Accepted December 8, 2015)

Abstract. The model of two-temperature magneto-thermoelasticity for a non-simple variable-thermal-conductivity infinitely-long solid cylinder is established. The present cylinder is made of an isotropic homogeneous thermoelastic material and its bounding plane is traction-free and subjected to a time-dependent temperature. An exact solution is firstly obtained in Laplace transform space to obtain the displacement, incremental temperature, and thermal stresses. The inversion of Laplace transforms has been carried out numerically since the response is of more interest in the transient state. A detailed analysis of the effects of phase-lags, an angular frequency of thermal vibration and the variability of thermal conductivity parameter on the field quantities is presented.

Keywords: thermoelasticity; phase-lags; non-simple; solid cylinder; variable thermal conductivity

1. Introduction

The two-temperature model is one of the non-classical thermoelasticity theories of elastic solids. The thermal dependence is considered as the main difference of this model with respect to the classical. Chen and Gurtin (1968) proposed a theory of non-simple rigid materials for which the two temperatures introduced; the thermodynamic temperature and the conductive temperature are not identical. This theory was further extended to deformable bodies by Chen *et al*. (1969). They formulated a theory of heat conduction in deformable bodies, which depends on the two distinct temperatures. For time-independent problems, the difference between these two temperatures is proportional to the heat supply. In absence of heat supply, these two temperatures are identical. However, for time-dependent problems, the two temperatures are different, independently of the presence of a heat source (Boit 1956, Warren and Chen 1973, Quintanilla 2004, Das and Kanoria 2012, Lotfy 2014, Zenkour and Abouelregal 2014 a, b).

The generalized theories of thermoelasticity have been developed to overcome the infinite propagation speed of thermal signals predicted by classical coupled dynamical theory of

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http://www.techno-press.com/journals/eas&subpage=7 ISSN: 2092-7614 (Print), 2092-7622 (Online)

thermoelasticity. Biot (1956) developed the coupled thermoelasticity theory to deal with the first defect of the uncoupled one but the other defect of the uncoupled theory still as it is. Lord and Shulman (1967) introduced the theory of generalized thermoelasticity with one relaxation time to deal with both the first and second defects of the uncoupled theory. They postulated a new law of heat conduction to replace the classical Fourier's law. This new law contains the heat flux vector as well as its time derivative. Green and Lindsay (1972) developed a theory of temperature-rate-dependent thermoelasticity that does not violate the classical Fourier law of heat conduction. They included the temperature rate among the constitutive variables to predict a finite speed for heat propagation. Tzou (1995) proposed a dual-phase-lag heat conduction model (DPL) to incorporate the effects of microscopic interactions in the fast-transient process of heat transport mechanism in a macroscopic formulation. Two different phase-lags have been introduced in the constitutive relations between heat flux vector and the temperature gradient (see also, Prasad *et al.* 2010 and Zenkour *et al.* 2013).

A general theory of simple force and stress multi-poles which were defined with the help of velocity components and their spatial derivatives is developed by Green and Rivlin (1964). Constitutive theory for non-simple materials has not been studied in detail. Therefore, it was so difficult to obtain solutions of problems in this theory. Wozniak (1967) stated that the notion of oriented material is generalized to thermal problems by introducing additional fields describing the temperature distribution. Carroll (1969) obtained a form of the constitutive equation in non-simple solids to describe the propagation of plane waves of finite amplitude. Leşana (1983) used an entropy production inequality to derive a linear theory of thermoelasticity for non-simple materials. Dhar (1985) discussed the effect due to mechanical shock in a non-simple elastic material. Ciarletta (1996) derived a theory of thermoelasticity for non-simple materials within the framework of extended thermodynamics. Quintanilla (2003) proposed a model of non-simple thermoelastic theory without energy dissipation.

The present paper is devoted to estimate the influence of DPL model of generalized thermoelasticity on an infinitely non-simple long annular cylinder under variable thermal conductivity and magnetic fields (see Abbas and Zenkour 2013, Zenkour and Abbas 2014, 2015 and Zenkour 2014). It is known that the heat conduction equation for non-simple materials contains an additional term involving the time derivative of the Laplacian of the conductive temperature. Also, the equation of motion contains an additional term involving space derivative of the Laplacian of the conductive temperature. The present cylinder is subjected to a time-dependent thermal shock and its surface is considered to be traction-free. The solution has been obtained by the application of the Laplace transform in a direct approach. Numerical results are obtained in the physical domain by employing a numerical technique. Finally, the case of results obtained in a simple medium is compared with that of a non-simple one.

2. Two-temperature thermoelasticity model with phase lags

In the present section, we formulate a generalized two-temperature thermoelasticity theory with dual-phase-lags (DPL) model for non-simple materials. The field equations for a linear, homogeneous and thermoelastic material, take the following forms:

The constitutive equations

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e_{kk}\delta_{ij} - \gamma\theta\delta_{ij} \tag{1}$$

where σ_{ij} are the stresses, e_{ij} are the strains, e_{kk} is the strain dilatation, u_i are the displacements, $\gamma = (3\lambda + 2\mu)\alpha_t$ is the coupling parameter, in which α_t being the coefficient of linear thermal expansion and λ, μ are Lame's constants, and $\theta = T - T_0$ is the dynamical temperature increment of the resonator, in which T_0 is the environmental temperature.

The Cauchy relations

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{2}$$

The equations of motion

$$\sigma_{ji,j} + F_i = \rho \ddot{u}_i \tag{3}$$

where ρ denotes the material density of the medium and F_i denote the Lorentz force.

The heat conduction equation for simple materials

$$\left(1 + \tau_\theta \frac{\partial}{\partial t}\right)(K\theta_{,i})_{,i} = \left(1 + \tau_q \frac{\partial}{\partial t}\right)\left(\rho C_e \frac{\partial \theta}{\partial t} + \gamma T_0 \frac{\partial e}{\partial t}\right) \tag{4}$$

where K denotes the variable thermal conductivity, C_e denotes the specific heat per unit mass at constant strain, τ_θ and τ_q denote finite times (the phase-lags of temperature gradient and heat flux, respectively).

Eq. (3) using Eq. (1) will be in the form

$$(\lambda + \mu)u_{i,ij} + \mu u_{i,jj} - \gamma\theta_{,i} + F_i = \rho \ddot{u}_i \tag{5}$$

The classification of real material into simple and non-simple materials is proposed by Chen and Gurtin (1968). The thermodynamics and conductive temperatures are not identical for non-simple materials while they are identical for simple materials. The relation relates the two temperatures is given by

$$\varphi - \theta = b\varphi_{,ii} \tag{6}$$

where φ denotes the conductive temperature, θ denotes thermodynamic temperature and $b > 0$ is the temperature-discrepancy factor. Therefore, in the case of non-simple medium, Eq. (4) takes the form (Dhar 1985)

$$\left(1 + \tau_\theta \frac{\partial}{\partial t}\right)(K\varphi_{,i})_{,i} = \left(1 + \tau_q \frac{\partial}{\partial t}\right)\left(\rho C_e \frac{\partial \theta}{\partial t} + \gamma T_0 \frac{\partial e}{\partial t}\right) \tag{7}$$

In all of the above equations, the comma followed by a suffix denotes derivative with respect to this suffix and the superposed dot denotes derivative with respect to time t. The key element that sets the two-temperature thermoelasticity theory apart from the classical theory is the material parameter b. Specifically, in the limit as $b \to 0$, $\varphi \to \theta$ and the classical theory (one-temperature generalized thermoelasticity theory 1TT) is recovered.

3. Maxwell's relations

The application of the initial magnetic field $\vec{H}$ always produces an induced magnetic field $\vec{h}$ and an induced electric field $\vec{E}$. The linear equations of electrodynamics of a slow-moving medium for a homogeneous and thermally and electrically perfect conducting elastic solid may be simplified in the following form (Maxwell's electromagnetic field equations without the inclusion of the charge density)

$$\vec{J} = \nabla \times \vec{h}, \quad \nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{h}}{\partial t}, \quad \vec{E} = -\mu_0 \left(\frac{\partial \vec{u}}{\partial t} \times \vec{H} \right), \quad \nabla \cdot \vec{h} = 0. \tag{8}$$

where ∇ is the Hamilton arithmetic operator (nabla), $\vec{J}$ is the current density and μ_0 is the magnetic permeability. The Maxwell's stress equations are given by

$$\tau_{ij} = \mu_e [H_i h_j + H_j h_i - H_k h_k \delta_{ij}] \tag{9}$$

The Lorentz force F_i (for a perfect conductor) induced by the magnetic field $\vec{H}$ is

$$F_i = \mu_0 (\vec{J} \times \vec{B}) \tag{10}$$

4. Formulation of the problem

Let us consider a long solid cylinder of radius R with traction-free surface and subjected to a time-dependent thermal shock. The cylinder is initially exposed to an axial magnetic field $\vec{H} = (0,0,H_0)$. The cylindrical coordinates system (r,ζ,z) with the z-axis lying along the axis of the cylinder is used. Now the problem is reduced to one-dimensional due to the symmetry occurred. Therefore, all functions considered hear are depending on the radial distance r and the time t, only.

The displacements are reduced to one component. That is

$$u_r = u(r,t), \quad u_\zeta(r,t) = u_z(r,t) = 0. \tag{11}$$

The constitutive relations, given in Eq. (1), take the form

$$\sigma_{rr} = 2\mu \frac{\partial u}{\partial r} + \lambda e - \gamma\theta, \tag{12}$$

$$\sigma_{\zeta\zeta} = 2\mu \frac{u}{r} + \lambda e - \gamma\theta \tag{13}$$

$$\sigma_{zz} = \lambda e - \gamma\theta. \tag{14}$$

$$\sigma_{rz} = \sigma_{r\zeta} = \sigma_{\zeta z} = 0 \tag{15}$$

The electromagnetic equation of motion of the solid cylinder is expressed as

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\zeta\zeta}}{r} + F_r = \rho \frac{\partial^2 u}{\partial t^2} \tag{16}$$

The initial magnetic field vector $\vec{H} = (0,0,H_0)$ is applied to Eq. (8). That is

$$\vec{E} = \mu_0 H_0 \left(0, \frac{\partial u}{\partial t}, 0\right), \quad \vec{J} = \left(0, \frac{\partial e}{\partial r}, 0\right), \quad \vec{h} = -H_0(0,0,e) \tag{17}$$

where

$$e = \frac{1}{r}\frac{\partial (ru)}{\partial r} \tag{18}$$

Thus, from Eqs. (9), (10), and (17), one obtains

$$F_r = \mu_0 H_0{}^2 \frac{\partial e}{\partial r}, \quad \tau_{rr} = \mu_0 H_0^2 e. \tag{19}$$

where τ_{rr} is the Maxwell's stress.

Generally, the thermal conductivity of most materials should be temperature-dependent. An acceptable approximation in limited temperature intervals is obtained by considering the thermal conductivity K to depend linearly on the change of temperature. It is, assumed that (Nowinski 1978)

$$K = K(\theta) = K_0(1 + K_1\theta) \tag{20}$$

where K_0 denotes the thermal conductivity at reference temperature T_0 and K_1 denotes the slope of the thermal conductivity-temperature θ curve divided by the intercept K_0. It is to be noted that K_1 is usually negative experimental coefficient (Nowinski 1978).

Using Eqs. (6) and (20), we approximate the thermal conductivity of materials as conductive temperature dependent in the form

$$K = K(\varphi) = K_0(1 + K_1\varphi) \tag{21}$$

Now, we will consider the mappings

$$\Phi = \frac{1}{K_0}\int_0^{\varphi} K(\xi)\,\mathrm{d}\xi \tag{22}$$

$$\Theta = \frac{1}{K_0}\int_0^{\theta} K(\xi)\,\mathrm{d}\xi. \tag{23}$$

where Φ and Θ are new functions expressing the heat conduction. Using Eq. (21) into Eq. (22), one gets

$$\Phi = \varphi(1 + \tfrac{1}{2}K_1\varphi) \tag{24}$$

Differentiating Eqs. (22) and (23) with respect to r, one obtains

$$K_0\Phi_{,r} = K(\varphi)\varphi_{,r} \tag{25}$$

$$K_0\Theta_{,r} = K(\theta)\theta_{,r} \tag{26}$$

Differentiating Eq. (25) once again with respect to r gives

$$K_0\Phi_{,rr} = [K(\varphi)\varphi_{,r}]_{,r}. \tag{27}$$

Also, differentiating Eq. (23) with respect to t gives

$$K_0\dot{\Theta} = K(\theta)\dot{\theta} \tag{28}$$

Substituting from Eqs. (25) and (27) into Eq. (7), one gets the heat equation for non-simple materials in the form

$$\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\Phi_{,rr} = \left(1+\tau_q\frac{\partial}{\partial t}\right)\left(\frac{1}{k}\frac{\partial\Theta}{\partial t}+\frac{\gamma T_0}{K_0}\frac{\partial e}{\partial t}\right) \tag{29}$$

where $\rho C_e(\varphi) = K(\varphi)/k$ and k is known as the diffusivity of the material.

With the same manner, using the Eqs. (20)-(28), we have the approximation of Eq. (6) in the simplest form as

$$\Phi - \Theta = b\Phi_{,rr} \tag{30}$$

From Eqs. (12), (13), (16) and (19), the equation of motion will be in the form

$$(\lambda+2\mu+\mu_0H_0^2)\frac{\partial e}{\partial r} - \gamma\frac{\partial\theta}{\partial r} = \rho\frac{\partial^2 u}{\partial t^2} \tag{31}$$

By using Eq. (26), one obtains

$$(\lambda+2\mu+\mu_0H_0^2)\frac{\partial e}{\partial r} - \frac{\gamma}{(1+K_1\theta)}\frac{\partial\Theta}{\partial r} = \rho\frac{\partial^2 u}{\partial t^2} \tag{32}$$

For linearity, where $\theta = T - T_0$ such that $|\theta/T_0| << 1$, the equation of motion in the radial direction will be in the form

$$(\lambda+2\mu+\mu_0H_0^2)\frac{\partial e}{\partial r} - \gamma\frac{\partial\Theta}{\partial r} = \rho\frac{\partial^2 u}{\partial t^2} \tag{33}$$

We will use the following dimensionless variables

$$\begin{gathered}\{r',u',R_i'\} = \frac{c_0}{k}\{r,u,R_i\},\quad \{t',\tau_q',\tau_\theta'\} = \frac{c_0^2}{k}\{t,\tau_q,\tau_\theta\},\quad \{\theta',\Phi',\Theta'\} = \frac{1}{T_0}\{\theta,\Phi,\Theta\}\\ \sigma_{ij}' = \frac{\sigma_{ij}}{\mu},\quad K_1' = T_0K_1,\quad b' = \left(\frac{c_0}{k}\right)^2 b,\quad c_0^2 = \frac{\lambda+2\mu}{\rho}.\end{gathered} \tag{34}$$

The governing equations are summarized here by using the above dimensionless variables and dropping the primes for convenience. They are given by

$$\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\nabla^2\Phi = \left(1+\tau_q\frac{\partial}{\partial t}\right)\left(\frac{\partial\Theta}{\partial t}+g\frac{\partial e}{\partial t}\right) \tag{35}$$

$$\nabla^2 e - a_1 \nabla^2 \Theta = a_2 \frac{\partial^2 e}{\partial t^2}, \tag{36}$$

$$\Theta = \Phi - b\nabla^2 \Phi, \tag{37}$$

$$\sigma_{rr} = 2\frac{\partial u}{\partial r} + (\beta^2 - 2)e - n\theta, \tag{38}$$

$$\sigma_{\zeta\zeta} = 2\frac{u}{r} + (\beta^2 - 2)e - n\theta, \tag{39}$$

$$\sigma_{zz} = (\beta^2 - 2)e - n\theta, \tag{40}$$

where

$$\begin{gathered} n = \frac{\gamma T_0}{\mu}, \quad g = \frac{\gamma T_0 k}{K_0}, \quad \beta^2 = \frac{\lambda + 2\mu}{\mu}, \\ a_1 = \frac{\gamma T_0}{\lambda + 2\mu + \mu_0 H_0^2}, \quad a_2 = \frac{\rho c_0^2}{\lambda + 2\mu + \mu_0 H_0^2} \end{gathered} \tag{41}$$

4. Initial/boundary conditions

The initial and boundary conditions should be considered to solve the present problem. The initial conditions of the problem are taken in the form

$$\begin{gathered} u(r,t)\big|_{t=0} = \frac{\partial u(r,t)}{\partial t}\bigg|_{t=0} = 0, \\ \theta(r,t)\big|_{t=0} = \frac{\partial \theta(r,t)}{\partial t}\bigg|_{t=0} = 0, \\ \varphi(r,t)\big|_{t=0} = \frac{\partial \varphi(r,t)}{\partial t}\bigg|_{t=0} = 0. \end{gathered} \tag{42}$$

The boundary of the cylinder is constrained and subjected to a harmonically varying heat. So the following boundary conditions hold:

• The surface of the cylinder is subjected to a harmonically varying heat

$$\varphi(R,t) = \varphi_0 \cos(\omega t), \quad \omega > 0, \tag{43}$$

where ω is the angular frequency of thermal vibration ($\omega = 0$ for a thermal shock) and φ_0 is constant. Using Eq. (23), then one gets

$$\Phi(R,t) = \varphi_0 \cos(\omega t) + \tfrac{1}{2} K_1 [\varphi_0 \cos(\omega t)]^2 \tag{44}$$

• The mechanical boundary condition on the surface is traction-free at $r = R$. That is

$$\sigma_{rr}(R,t)+\tau_{rr}(R,t)=0. \tag{45}$$

5. Solution of the problem in the Laplace transform domain

Taking the Laplace transform of Eqs. (35)-(40), under the above homogeneous initial conditions, we obtain the following equations

$$(\nabla^2-\alpha)\overline{\Phi}=\alpha g\overline{e} \tag{46}$$

$$(\nabla^2-a_2s^2)\overline{e}=a_1\nabla^2\overline{\Theta}, \tag{47}$$

$$\Theta=\Phi-b\nabla^2\Phi \tag{48}$$

$$\overline{\sigma}_{rr}=2\frac{\partial\overline{u}}{\partial r}+(\beta^2-2)\overline{e}-n\overline{\theta} \tag{49}$$

$$\overline{\sigma}_{\zeta\zeta}=2\frac{\overline{u}}{r}+(\beta^2-2)\overline{e}-n\overline{\theta} \tag{50}$$

$$\overline{\sigma}_{zz}=(\beta^2-2)\overline{e}-n\overline{\theta} \tag{51}$$

where

$$\alpha=\frac{s(1+\tau_q s)}{1+\tau_\theta s+bs(1+\tau_q s)} \tag{52}$$

Here, an over bar denotes the Laplace transform of the corresponding function and s is the transform parameter. Eliminating $\overline{e}$ or $\overline{\Phi}$ from Eqs. (46) and (47), one gets

$$[(1+bg\alpha a_1)\nabla^4-[a_2s^2+\alpha(1+a_1g)]\nabla^2+a_2s^2\alpha]\{\overline{\Phi},\overline{e}\}=0 \tag{53}$$

Introducing m_i $(i=1,2)$ into Eq. (53), one gets

$$(\nabla^2-m_1^2)(\nabla^2-m_2^2)\{\overline{\Phi},\overline{e}\}=0 \tag{54}$$

where m_1^2 and m_2^2 are the roots of the characteristic equation

$$[1+bg\alpha a_1]m^4-[a_2s^2+\alpha(1+a_1g)]m^2+a_2s^2\alpha=0 \tag{55}$$

The roots of the above characteristic equation are given by

$$m_1^2=\tfrac{1}{2}[2A+\sqrt{A^2-4B}],\quad m_2^2=\tfrac{1}{2}[2A+\sqrt{A^2-4B}] \tag{56}$$

where

$$A=\frac{a_2s^2+\alpha(1+a_1g)}{1+bg\alpha a_1},\quad B=\frac{a_2s^2\alpha}{1+bg\alpha a_1} \tag{57}$$

The solutions of Eq. (54) under the regularity conditions that $u,\theta,\varphi \to 0$ as $r \to 0$ can be written in the form

$$\overline{\Phi} = \alpha g \sum_{i=1}^{2} A_i I_0(m_i r). \tag{58}$$

where $I_0(\cdot)$ are the modified Bessel functions of the second kinds of order zero, and A_i are parameters depending on the parameter s of the Laplace transform. Using Eqs. (46) and (58), we obtain

$$\overline{e} = \sum_{i=1}^{2} (m_i^2 - \alpha) A_i I_0(m_i r) \tag{59}$$

Substituting the above equation into the Laplace transform of Eq. (18) gives

$$\overline{u} = \sum_{i=1}^{2} \left(\frac{m_i^2 - \alpha}{m_i} \right) A_i I_1(m_i r). \tag{60}$$

In deriving the above equation, we have used the following well-known relation of the Bessel function

$$\int z\, I_0(z) dz = z\, I_1(z). \tag{61}$$

After using Laplace transform, the boundary conditions (43) and (45) take the forms

$$\overline{\Phi}(R,s) = \theta_0 \left(\frac{s}{s^2 + \omega^2} + \frac{K_1(2\omega^2 + s^2)}{2s(s^2 + 4\omega^2)} \right) = \overline{G}(s) \tag{62}$$

$$\overline{\sigma}_{rr}(R,s) + \overline{\tau}_{rr}(R,s) = 0 \tag{63}$$

Substituting Eqs. (19), (49) and (58) into the above boundary conditions, one obtains four equations in the unknown parameters A_i, B_i, $i = 1,2$. After solving the above equations, we have the values of the two constants A_i, $i = 1,2$ whose solution complete the solution of the problem in the Laplace transform domain. Hence, we obtain the expressions for the displacement, the stress components and other physical quantities of the medium.

After obtaining $\overline{\Phi}$, the conductive temperature $\overline{\varphi}$ can be obtained by solving Eq. (24) in the Laplace domain as follows

$$\overline{\varphi} = \frac{-1 + \sqrt{1 + 2K_1 \overline{\Phi}}}{K_1} \tag{64}$$

From Eqs. (6) and (64) we can get the thermodynamic temperature in the Laplace domain in the following form

$$\overline{\theta}(r,s) = \frac{1}{K_1}(1 - b\nabla^2)\left(-1 + \sqrt{1 + 2K_1 \overline{\Phi}}\right). \tag{65}$$

Finally, substituting from Eqs. (55)-(57) into Eqs. (46)-(48) and using the relations

$$\begin{aligned} rI'_\nu(r) &= xI_\nu(r) - \nu I_{\nu-1}(r), \\ rI'_\nu(r) &= rI_{\nu+1}(r) + \nu I_{\nu-1}(r), \end{aligned} \tag{66}$$

to obtain the stresses in the form

$$\bar{\sigma}_{rr} = \sum_{i=1}^{2}(m_i^2-\alpha)\left(\beta^2 I_0(m_i r) - \frac{2}{rm_i} I_1(m_i r)\right)A_i - n\frac{1}{K_1}(1-b\nabla^2)\left(-1+\sqrt{1+2K_1\bar{\Phi}}\right), \tag{67}$$

$$\bar{\sigma}_{\zeta\zeta} = \sum_{i=1}^{2}(m_i^2-\alpha)\left((\beta^2-2) I_0(m_i r) - \frac{2}{rm_i} I_1(m_i r)\right)A_i - \frac{n}{K_1}(1-b\nabla^2)\left(-1+\sqrt{1+2K_1\bar{\Phi}}\right), \tag{68}$$

$$\sigma_{zz} = \sum_{i=1}^{2}(m_i^2-\alpha)(\beta^2-2) I_0(m_i r)A_i - n\frac{1}{K_1}(1-b\nabla^2)\left(-1+\sqrt{1+2K_1\bar{\Phi}}\right). \tag{69}$$

At this end, the solution of the problem in the Laplace transform domain is completed.

6. Numerical inversion of the Laplace-transformed equations

Here we will determine the field quantities. The conductive and thermal temperatures, the displacement and the stress in the time domain represent these quantities. A numerical inversion method is adopted based on a Fourier series expansion (Honig and Hirdes 1984). This method gives the approximate value of the inverse $f(t)$ of the Laplace transform $\bar{f}(s)$ by the relation

$$f(t) = \frac{e^{\xi t}}{t_1}\left(\tfrac{1}{2}\bar{f}(\xi) + \mathrm{Re}\left\{\sum_{k=0}^{N} \bar{f}\left(\xi + \frac{ik\pi}{t_1}\right)e^{ik\pi t/t_1}\right\}\right), \quad 0 \le t \le t_1 \tag{70}$$

where Re denotes the real part, i denotes an imaginary number unit and N is a sufficiently large integer representing the number of terms in the truncated infinite Fourier series. The number N must be chosen such that

$$e^{\xi t} e^{iN\pi t/t_1}\,\mathrm{Re}\left\{\bar{f}\left(\xi + \frac{iN\pi}{t_1}\right)\right\} \le \varepsilon_1 \tag{71}$$

where ε_1 denotes a persecuted small positive number that corresponds to the degree of accuracy to be achieved. The parameter ξ is a positive free parameter that must be greater than the real parts of all singularities of $\bar{f}(s)$. The optimal choice of ξ is obtained according to the criteria described in (Honig and Hirdes 1984).

7. Numerical results

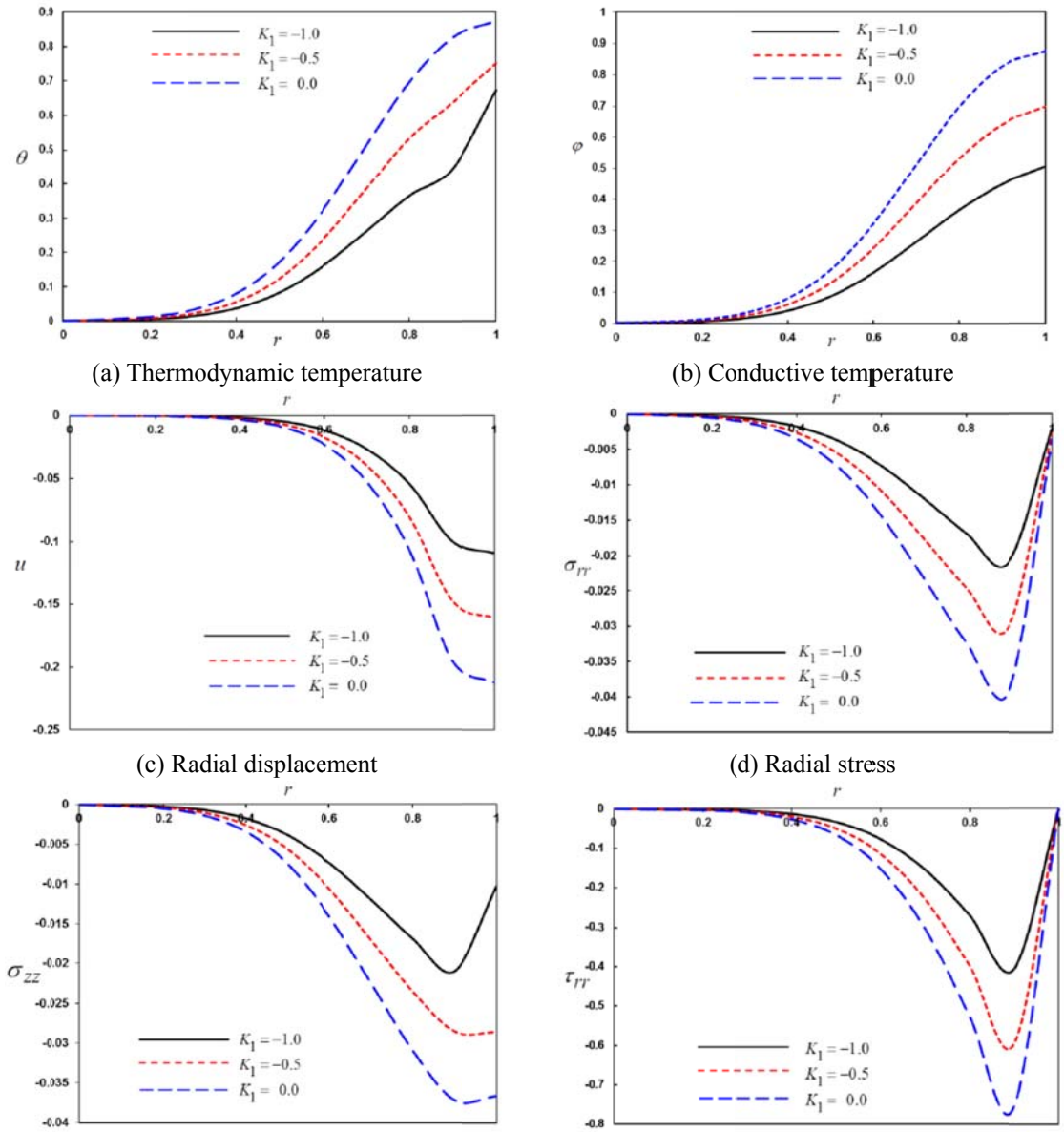
A numerical example for computational results is considered here to illustrate the analytical procedure presented earlier. The results for field quantities are presented graphically. For this

purpose, the following values of physical constants are taken

$$K_0 = 368\mathrm{Wm^{-1}K^{-1}},\quad \alpha_t = 1.78\times10^{-5}\mathrm{K^{-1}},\quad C_e = 383.1\,\mathrm{Jkg^{-1}K^{-1}},$$

$$\rho = 8954\ \mathrm{kgm^{-3}},\quad \lambda = 7.76\times10^{10}\,\mathrm{Nm^{-2}},\quad \mu = 3.86\times10^{10}\,\mathrm{Nm^{-2}},$$

$$T_0 = 293\mathrm{K},\quad g = 1.61,\quad a = 0.0105,\quad b = 0.042,\quad \beta = 2.$$

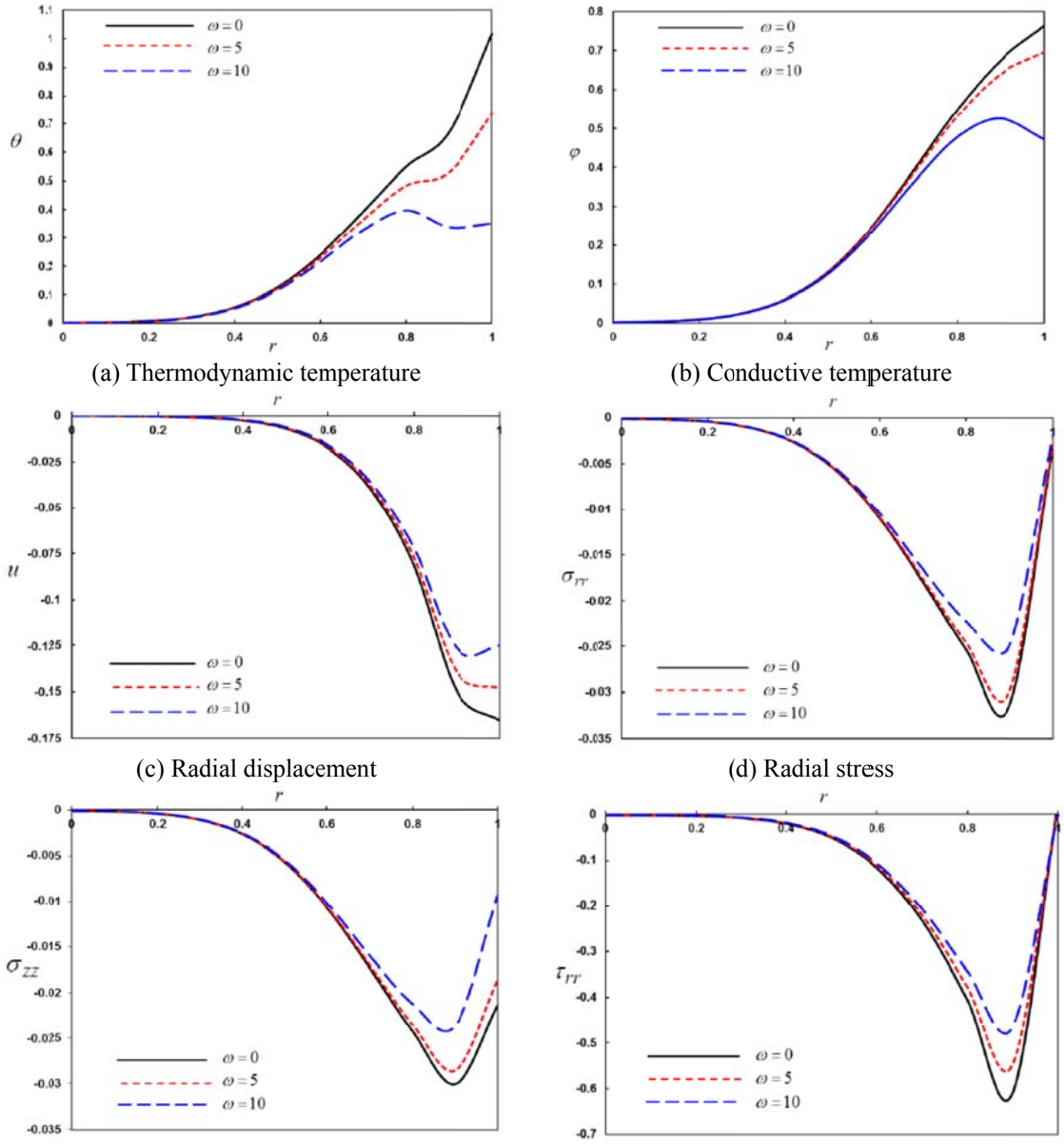


(a) Thermodynamic temperature (b) Conductive temperature

(c) Radial displacement (d) Radial stress

(e) Axial stress (f) Maxwell's stress

Fig. 1 Distribution of the field quantities for various thermal conductivity parameters



(a) Thermodynamic temperature (b) Conductive temperature

(c) Radial displacement (d) Radial stress

(e) Axial stress (f) Maxwell's stress

Fig. 2 Distribution of the field quantities for various angular frequencies of thermal vibration

The results are illustrated graphically in Figs. 1-4, for the thermodynamic temperature θ, the conductive temperature φ, the displacement u, the thermal stresses σ_{rr} and σ_{zz} and Maxwell's stress τ_{rr}. The field quantities depend not only on the time t and space coordinate r, but also depend on the two-temperature parameter b, the variability thermal conductivity parameter, angular frequency of thermal vibration and phase lags parameters. Numerical calculations are carried out for the next four cases.

Case 1 investigates how the dimensionless thermodynamic temperature, conductive temperature, displacement, stresses and Maxwell's stress vary with different values of variability thermal conductivity parameter K_1 when the angular frequency of thermal vibration ω and phase-lag of the heat flux τ_q and the phase-lag of temperature gradient τ_θ remain constants.

In this case, three different values of the variability thermal conductivity parameter K_1 are considered to discuss the effect of temperature on thermal conductivity. We take $K_1 = -1, -0.5$ for variable thermal conductivity and $K_1 = 0$ when thermal conductivity is independent of temperature. Here, the angular frequency of thermal vibration $\omega = 5$, the phase-lag of heat flux $\tau_q = 0.1$ and the phase-lag of temperature gradient $\tau_\theta = 0.05$. From Figs. 1(a-f), the parameter K_1 has significant effects on all the field quantities. Figs. 1a and 1b show that the two temperatures increase as K_1 increases. They also increase along the radial direction. From Fig. 1(c), it can be found that the displacement decreases as K_1 increases. It is also decreasing along the radial direction. In Figs. 1(d)-(f) it can be observed that the distributions of thermal redial stress σ_{rr}, the axial stress σ_{zz} and the Maxwell's stress τ_{rr} decrease as K_1 increases. These stresses are np longer decreasing through the radial direction and have their minimums near the outer surface of the cylinder. The medium close to the cylinder surface suffers from tensile stress which becomes larger with the time passing.

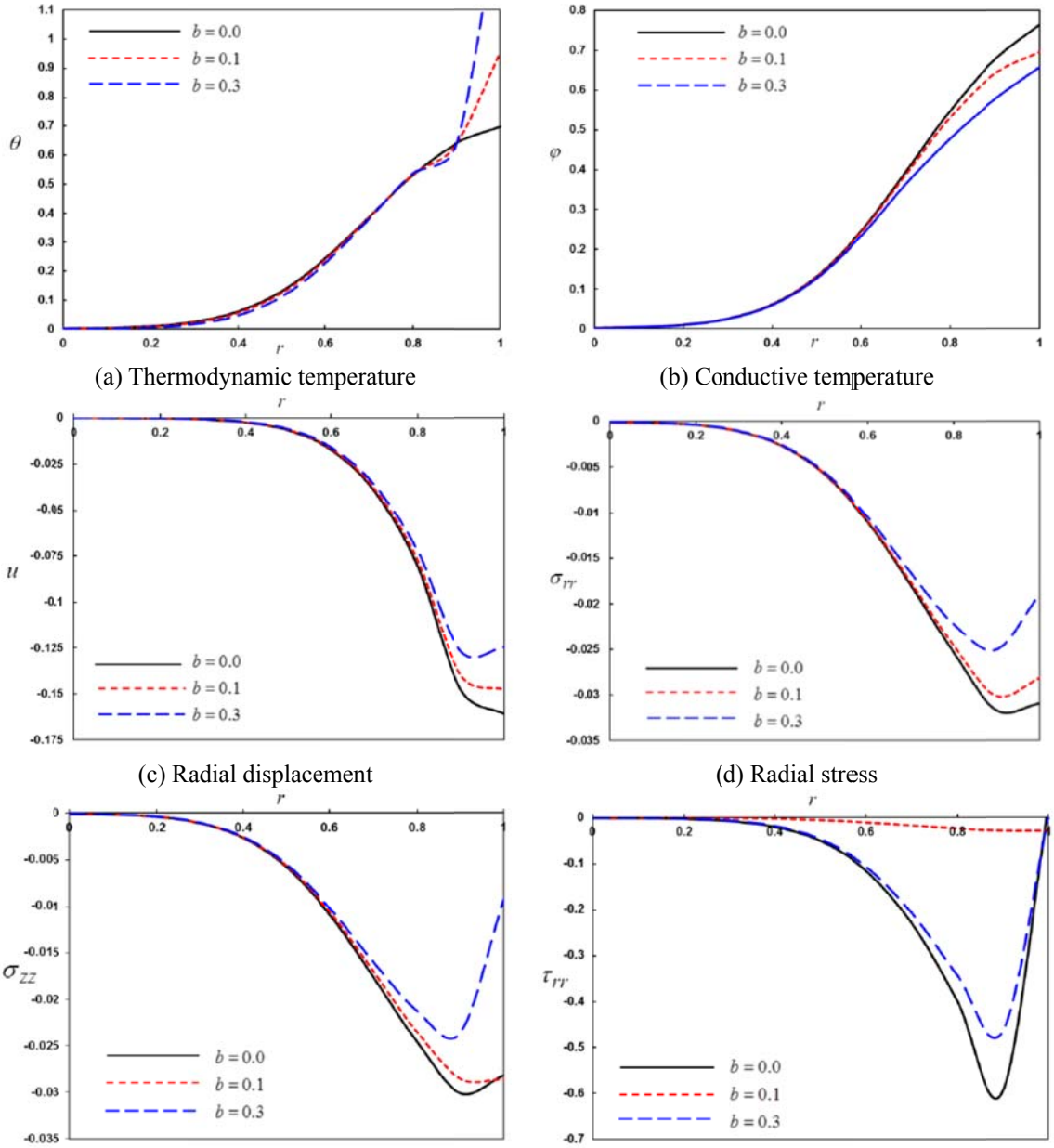
Case 2 illustrates how the field quantities vary with different values of the angular frequency of thermal vibration ω when the variability thermal conductivity parameter K_1, the phase-lag of heat flux τ_q and the phase-lag of temperature gradient τ_θ remain constants. Three different values of the angular frequency of thermal vibration are considered. For thermal shock problem, we put $\omega = 0$ and for harmonically heat it is set to be $\omega = 5,10$. In this case, the variability thermal conductivity parameter K_1 is fixed to -0.5. Figs. 2(a)-(f) illustrate that, the angular frequency parameter ω has significant effects on all studied fields. It is clear that the maximum values for temperatures occur at the surface of the cylinder and their magnitude increase with the increase of *r*. The thermodynamical temperature θ and the conductive temperature φ decrease as ω increases. Also, it can be seen that as ω increases all of the radial displacement u and the thermal stresses σ_{rr}, σ_{zz} and τ_{rr} increase. However, these quantities decrease along the radial directions. The radial displacement reaches its minimum at the surface of the cylinder while the stresses get their minimums near the outer surface.

In Case 3, the dimensionless thermodynamic temperature, conductive temperature, displacement, stresses and Maxwell's stress distributions with different values of the two-temperature parameter *b* are considered to stand on the effect of this parameter on all the studied fields. The value of $b = 0$ indicates the old situation (one temperature theory 1TT) while $b = 0.1$ or 0.3 indicates the two-temperature theory (2TT). In this case one takes τ_q=0.1, τ_θ=0.05, K_1=−0.5, and $\omega = 5$.

Figs. 3(a)-(f) plots the field quantities for different values of the two-temperature parameter b to stand on the effect of this parameter. The wave-amplitude of the displacement, stresses and Maxwell's stress decreases as *b* increases. For $r > 0.9$, the thermodynamical temperature θ increases as b increase. Also, as *b* increases the conductive temperature φ decreases in the interval $0 < r < 1$. This shows the difference between the one temperature thermoelasticity of DPL model $(b = 0)$ and the two-temperature generalized thermoelasticity $(b = 0.1 \text{ or } 0.3)$. The figures show that this parameter has significant effect on all the fields. The waves reach the steady state depending on the value of the temperature discrepancy *b*. Also these figures indicate that, the two-temperature generalized theory of thermoelasticity describes the behavior of the particles of

an elastic body more realistically than the one-temperature theory of generalized thermoelasticity.

Finally, the last case discusses how the dimensionless temperatures, displacement, stresses and Maxwell's stress vary with the phase-lag of the heat flux τ_q and the phase-lag of temperature gradient τ_θ when the angular frequency ω and the variability thermal conductivity parameter K_1 remain constants.

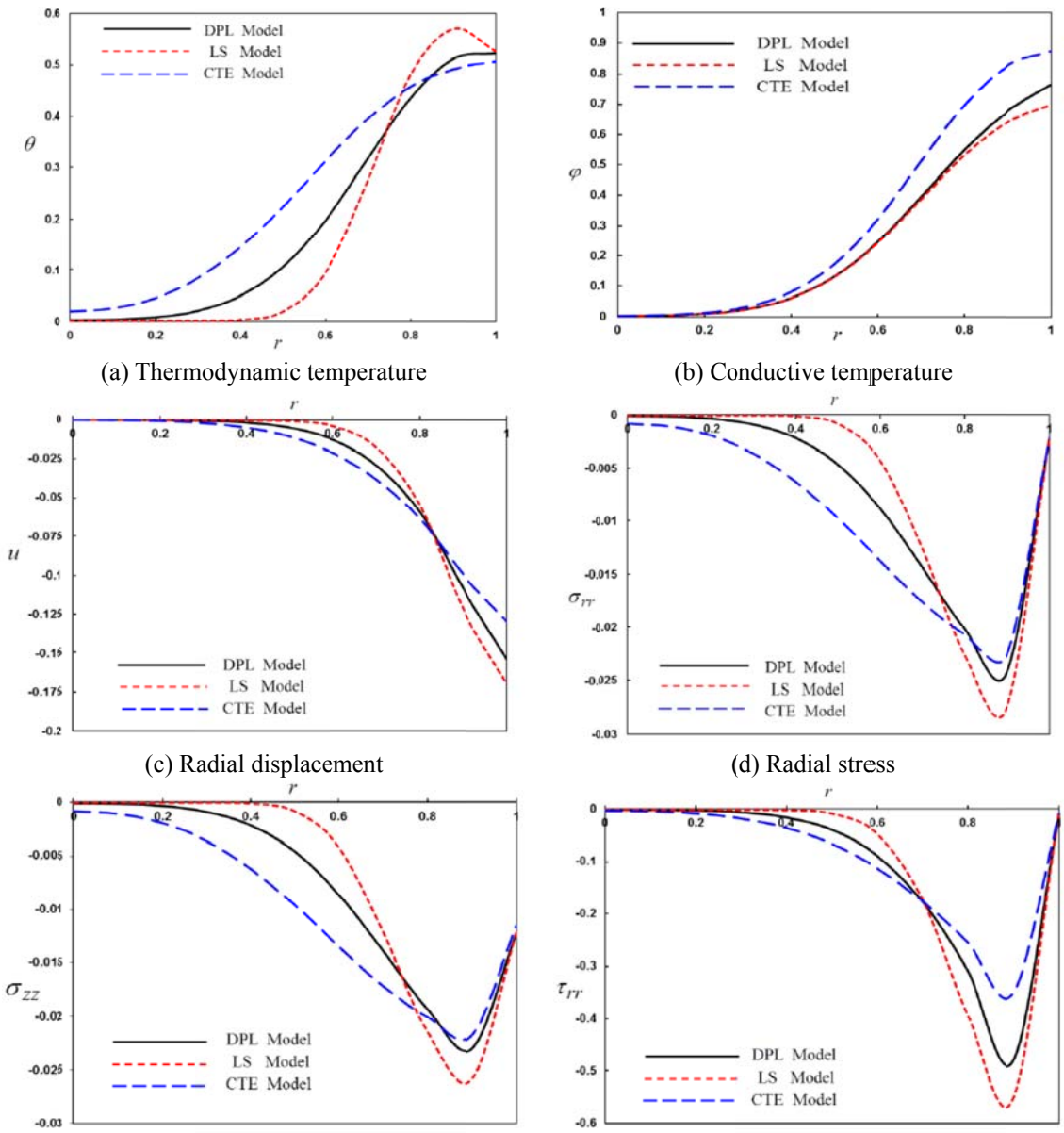


(a) Thermodynamic temperature (b) Conductive temperature

(c) Radial displacement (d) Radial stress

(e) Axial stress (f) Maxwell's stress

Fig. 3 Distribution of the field quantities for various two-temperature parameters



(a) Thermodynamic temperature (b) Conductive temperature

(c) Radial displacement (d) Radial stress

(e) Axial stress (f) Maxwell's stress

Fig. 4 Distribution of the field quantities for different theories of thermoelasticity

It is considered here different values of dual-phase-lag of the heat flux and the temperature gradient τ_q and τ_θ respectively. The graphs in Figs. 4(a)-(f) represent the curves predicted by three different theories of thermoelasticity obtained as special cases of the present dual-phase-lag model. The computations were performed for various values of the parameters τ_q and τ_θ to obtain the coupled theory (CTE) ($\tau_q = \tau_\theta = 0$), the Lord-Shulman (LS) theory ($\tau_\theta = 0$, $\tau_q = 0.1$), and the generalized theory of thermoelasticity proposed by Tzou (DPL) ($\tau_q = 0.1$, $\tau_\theta = 0.05$).

It can be observed that the phase-lag parameters have a great effects on the distribution of field quantities. The mechanical distributions indicate that the wave propagates as a wave with finite velocity in medium. The values in classical theory of thermoelasticity (CTE model) are different compared to those of other theories. The fact that in generalized thermoelasticity theories (DPL and LS), the waves propagate with finite speeds is evident in all these figures. The behavior of the three theories is generally quite similar.

8. Conclusions

The influence of different items such as phase lags, magnetic field, thermal shock and variable thermal conductivity is considered. The DPL model of generalized thermoelasticity is presented to an infinitely long cylinder with traction-free surface and subjected to time-dependent thermal shock and magnetic field. The solution is obtained in the Laplace transform domain by a direct approach. A numerical technique is employed to obtain the solution in the physical domain. From the numerical results, concluded that:

• The variability thermal conductivity parameter has significant effects on the speed of the wave propagation of all the studied fields.

• The dependence of the thermal conductivity on the temperature has a significant effect on thermal and mechanical interactions.

• The thermoelastic stresses, displacement and temperature have a strong dependency on the angular frequency parameter.

• It is seen that the values of all the field variables are significantly dependent on the two-temperature parameter.

• According to the theory of thermoelasticity with two-temperatures, we have to construct a new classification for materials according to their fractional parameter a where this parameter becomes a new indicator of its ability to conduct heat under the effect of thermoelastic properties.

• In generalized magneto-thermoelasticity theory with phase-lags heat propagates as a wave with finite velocity instead of infinite velocity in medium.

• The phase-lag of the heat flux and a phase-lag of temperature gradient have a great effect on the field quantities.

• The theories of coupled thermoelasticity, generalized thermoelasticity with one relaxation time can extracted as special cases.

• In addition, the near behavior between LS and DPL models and differ to CTE theory that indicates to originate of the thermoelastic theory.

• Results show that the dual-phase-lag model of thermoelasticity predicts a finite speed of wave propagation that made the generalized theorem of thermoelasticity more consistent with the physical properties of the material.

• The results presented in this paper should prove useful for researchers in scientific and engineering, as well as for those working on the development of mechanics of solids.

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