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Parametric surface and properties defined on parallelogrammic domain

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Abstract

Similar to the essential components of many mechanical systems, the geometrical properties of the teeth of spiral bevel gears greatly influence the kinematic and dynamic behaviors of mechanical systems. Logarithmic spiral bevel gears show a unique advantage in transmission due to their constant spiral angle property. However, a mathematical model suitable for accurate digital modeling, differential geometrical characteristics, and related contact analysis methods for tooth surfaces have not been deeply investigated, since such gears are not convenient in traditional cutting manufacturing in the gear industry. Accurate mathematical modeling of the tooth surface geometry for logarithmic spiral bevel gears is developed in this study, based on the basic gearing kinematics and spherical involute geometry along with the tangent planes geometry; actually, the tooth surface is a parametric surface defined on a parallelogrammic domain. Equivalence proof of the tooth surface geometry is then given in order to greatly simplify the mathematical model. As major factors affecting the lubrication, surface fatigue, contact stress, wear, and manufacturability of gear teeth, the differential geometrical characteristics of the tooth surface are summarized using classical fundamental forms. By using the geometrical properties mentioned, manufacturability (and its limitation in logarithmic spiral bevel gears) is analyzed using precision forging and multi-axis freeform milling, rather than classical cradle-type machine tool based milling or hobbing. Geometry and manufacturability analysis results show that logarithmic spiral gears have many application advantages, but many urgent issues such as contact tooth analysis for precision plastic forming and multi-axis freeform milling also need to be solved in a further study.

Keywords: Spiral bevel gear; Mathematical modeling; Parametric surface; Geometrical characteristics; Manufacturability

1. Introduction

Parametric surfaces in computer aided geometric design (CAGD) are commonly defined on a triangular, rectangular or N-sided domain. The most important surface, the non-uniform rational B-spline (NURBS) surface, which is defined on a rectangular domain, is mainly used to describe the shape of industrial products. However, due to its intrinsic properties, the NURBS surface cannot accurately depict a class of kinematic or dynamic shape, such as the tooth surfaces of spiral bevel gears.

Spiral bevel gears, the teeth of which are curved and angled away from the shaft centerline, are widely used in the power transmission of intersection axes. Unlike spur and helical gears in which teeth are generated from a cylinder blank, in spiral bevel gears, teeth are generated on a conical surface, which allows the teeth to come into contact with each other gradually. Since these gears provide excellent smoothness and load capacity, they are one of the most essential components in modern mechanical engineering. Theoretically, the tooth surfaces of spiral bevel gears are spherical involute surfaces [1]; actually, the tooth flank geometry almost completely depends on the related cutting processes. More precisely, spiral bevel gears are manufactured using cradle-type milling or hobbing machine tools; their geometrical and functional properties are thus determined by the kinematic and dynamic characteristics of different machine tools. This is why standardized spiral bevel gears are not manufactured. Park and Lee [2] utilized the spherical involute tooth profile to standardize bevel gear systems and explained the geometric characteristics and kinematic behavior of the standardized bevel gears.

Based on the milling or hobbing process, several practical approaches have been taken [3-6] to design the tooth surface of a spiral bevel gear using NURBS. Since the tooth surface is constructed from actual tooth surface sampling points [3, 4] or machining simulation points [5, 6], in the final digital model, the parametric feature information such as spiral angle, nominal pressure angle, module, etc. is completely lost. The NURBS based approach cannot be conveniently used for the parametric modeling of a spiral bevel gear.

Computer numerical control (CNC) cradle-type machine

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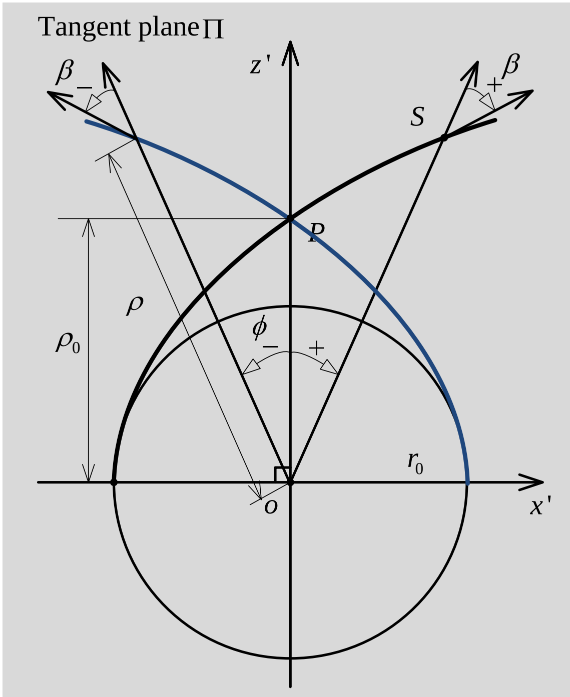

Figure 6. Planar spiral in different direction.

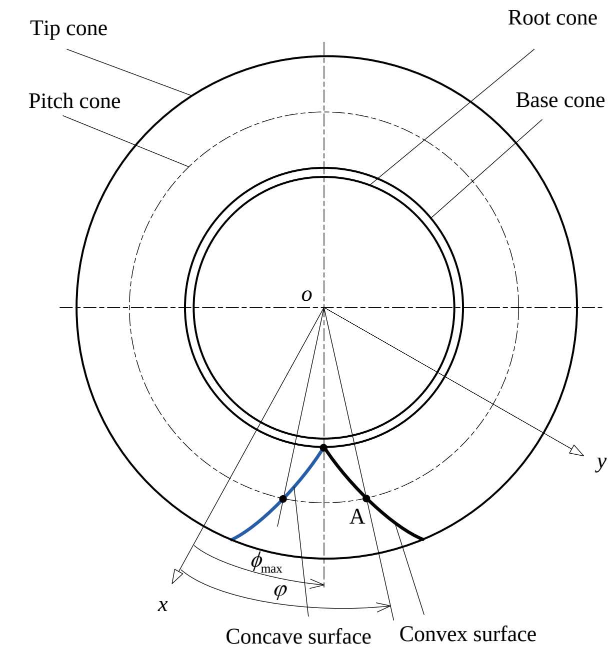

Figure 7. Planar spiral in different directions.

parametric surfaces that can be used for bevel gears design.

3.1 Unified description for planar spiral

The direction of the spiral angle β determines whether the gear tooth is left-hand or right-hand. In the left-hand/right-hand gear tooth, the outer half of the tooth is inclined in the counterclockwise/clockwise direction from the axial plane through the midpoint of the tooth, as viewed by an observer looking at the face of the gear.

Assume that the counterclockwise direction of spiral angle β is positive (-), and vice versa, as illustrated in Figure 6. The planar logarithmic spiral can then be formulated as

$$\rho = r_0 e^{(-\pi/2 + \phi')\cot\beta} = \rho_0 e^{\phi' \cot\beta} \tag{18}$$

where $\rho_0 = r_0 e^{-\pi\cot\beta/2}$ and $\beta \le 0$, $\phi' \le 0$. Eq. (18) has the same style as Eq. (7); regardless of the direction of the spiral angle, the planar logarithmic spiral has a unified definition.

3.2 Convex/concave tooth surface

Obviously, Eq. (15) represents the convex tooth surface of different direction spiral bevel gears. Compared with the convex surface generating principle, the concave surface can be considered as the trajectory in which the logarithmic spiral curve in the tangent plane Π rolls over the base cone without slipping in the negative (-) direction (clockwise direction).

Consequently, the equation of the concave surface is the same as the convex surface equation. The difference is the definition domain of parameter φ, as shown in Figure 7.

3.3 Surface on parallelogrammic domain

According to the above analysis, we can depict the tooth surfaces using a unified parametric surface Eq. (15). All surfaces are defined on different parallelogrammic domains that depend on the direction of φ and ϕ. Figure 8 shows the category of the tooth surfaces.

4. Differential geometrical characteristics

The elastohydrodynamic lubrication, surface fatigue, contact stress, wear, life and manufacturability of the spiral bevel gears heavily rely on the differential geometrical characteristics of the tooth surface, such as normal vectors, principal curvatures and directions. The many advantageous properties of the logarithmic spiral bevel gears should be revealed by using classical differential geometry tools.

4.1 First fundamental form

The tooth surface Σ is described by a pair of parameters φ and ϕ through the vector equation $\boldsymbol{r}(\varphi,\phi)$, where $\boldsymbol{r}$ is the position vector of a typical point P on Σ. The base vectors of Σ at any point P are then given by

$$\frac{\partial \boldsymbol{r}}{\partial \varphi} = \rho \sin\psi \cos\delta \begin{bmatrix} \cos\varphi\cos\delta \\ \sin\varphi\cos\delta \\ -\sin\delta \end{bmatrix}, \tag{19}$$

and

$$\frac{\partial \boldsymbol{r}}{\partial \phi} = \frac{\rho \sin\delta}{\sin\beta}\begin{bmatrix} \cos(\psi-\beta)\cos\varphi\sin\delta + \sin(\psi-\beta)\sin\varphi \\ \cos(\psi-\beta)\sin\varphi\sin\delta - \sin(\psi-\beta)\cos\varphi \\ \cos\delta\cos(\psi-\beta) \end{bmatrix}. \tag{20}$$

Hence, the unit normal vector of P can be given by

$$n = \left(\frac{\partial \boldsymbol{r}}{\partial \varphi} \times \frac{\partial \boldsymbol{r}}{\partial \emptyset}\right) \Big/ \left\| \frac{\partial \boldsymbol{r}}{\partial \varphi} \times \frac{\partial \boldsymbol{r}}{\partial \phi} \right\|$$

$$= \begin{bmatrix} -\sin(\psi-\beta)\cos\varphi\cos\delta + \cos(\psi-\beta)\sin\varphi \\ -\sin(\psi-\beta)\sin\varphi\sin\delta - \cos(\psi-\beta)\cos\varphi \\ -\sin(\psi-\beta)\cos\delta \end{bmatrix}. \tag{21}$$

The first fundamental form is formulated as

$$I = Ed\varphi^2 + 2Fd\varphi d\phi + Gd\phi^2 \tag{22}$$

where

$$E = \frac{\partial \boldsymbol{r}}{\partial \varphi} \cdot \frac{\partial \boldsymbol{r}}{\partial \varphi} = \rho^2 \sin^2\psi \cos^2\delta \,, \tag{23}$$

$$F = \frac{\partial \boldsymbol{r}}{\partial \varphi} \cdot \frac{\partial \boldsymbol{r}}{\partial \emptyset} = 0 \,, \tag{24}$$

and

$$G = \frac{\partial \boldsymbol{r}}{\partial \emptyset} \cdot \frac{\partial \boldsymbol{r}}{\partial \emptyset} = \frac{\rho^2 \sin^2\delta}{\sin^2\beta}. \tag{25}$$

$F = 0$ means that the iso-parametric curves, more precisely the φ-curve and ϕ-curve shown in Figure 5, are mutually orthogonal anywhere on the tooth surface Σ. Thus, the iso-parametric curves are the principal curve lines, and the base vectors coincide with the principal directions.

Yet another important conclusion is hidden behind the first fundamental form of the tooth surface; i.e., the angle β' between the tangent vector and its radius vector of ϕ-curve at any point P is always equal to β because

$$\cos\beta' = r \cdot \left(\frac{\partial \boldsymbol{r}}{\partial \phi}\right) \Big/ \left\| \boldsymbol{r} \cdot \left(\frac{\partial \boldsymbol{r}}{\partial}\emptyset\right) \right\| = \cos\beta\,. \tag{26}$$

This conclusion implies that every ϕ-curve is a spatial logarithmic spiral curve. In comparison with the c curve, the basic differences are the cone angle γ and its initial point P_0. Based on the spherical triangle sine theorem, it is easy to obtain the relation between δ and γ. In other words,

$$\sin\delta = \sin\gamma\cos\alpha \tag{27}$$

where α is an instantaneous pressure angle. If γ is equal to the pitch angle γ_p, then α is the nominal pressure α_n, of which the typical value is 20° in gear transmission.

4.2 Second fundamental form

Differentiate the normal vector Eq. (21) with parameters φ and ϕ, and obtain the following formulas

$$\frac{\partial \boldsymbol{n}}{\partial \varphi} = \cos\delta \begin{bmatrix} \cos(\psi-\beta)\cos\varphi\cos\delta \\ \cos(\psi-\beta)\sin\varphi\cos\delta \\ -\cos(\psi-\beta)\sin\delta \end{bmatrix} \tag{28}$$

$$\frac{\partial \boldsymbol{n}}{\partial \varphi} = \sin\delta \begin{bmatrix} \cos(\psi-\beta)\cos\varphi\sin\delta + \sin(\psi-\beta)\sin\varphi \\ \cos(\psi-\beta)\sin\varphi\sin\delta - \sin(\psi-\beta)\cos\varphi \\ \cos(\psi-\beta)\cos\delta \end{bmatrix}. \tag{29}$$

The second fundamental form is given by

$$II = Ld\varphi^2 + 2Md\varphi d\phi + Nd\phi^2 \tag{30}$$

where

$$L = -\frac{\partial \boldsymbol{r}}{\partial \varphi} \cdot \frac{\partial \boldsymbol{n}}{\partial \varphi} = -\rho \sin\psi \cos(\psi-\beta)\cos^2\delta, \tag{31}$$

$$M = -\frac{\partial \boldsymbol{r}}{\partial \emptyset} \cdot \frac{\partial \boldsymbol{n}}{\partial \varphi} = -\frac{\partial \boldsymbol{r}}{\partial \varphi} \cdot \frac{\partial \boldsymbol{n}}{\partial \emptyset} = 0, \tag{32}$$

and

$$N = -\frac{\partial \boldsymbol{r}}{\partial \emptyset} \cdot \frac{\partial \boldsymbol{n}}{\partial \emptyset} = -\frac{\rho \sin^2\delta}{\sin\beta}. \tag{33}$$

In the second fundamental form, we focus on the principal curvatures and directions distribution on the tooth surface. Since $F = 0$ and $M = 0$, the principal curvature expressions can be derived respectively as

$$k_1 = \frac{L}{E} = -\frac{\cos(\psi-\beta)}{\rho\sin\psi} \tag{34}$$

and

$$k_2 = \frac{N}{G} = -\frac{\sin\beta}{\rho}. \tag{35}$$

From Eqs. (19)-(20), the principal directions can be depicted as

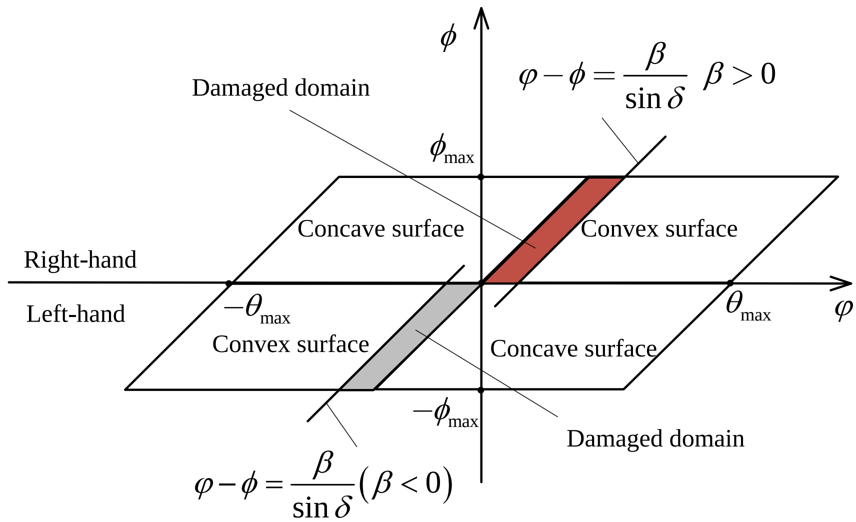

Figure 8. Pattern draft limitation for precision forging.

$$\boldsymbol{e}_1=\begin{bmatrix}cos\,\varphi\,cos\,\delta\\ sin\,\varphi\,cos\,\delta\\ -\,sin\,\delta\end{bmatrix} \tag{36}$$

and

$$\boldsymbol{e}_2=\begin{bmatrix}cos(\psi-\beta)\,cos\,\varphi\,sin\,\delta+sin(\psi-\beta)\,sin\,\varphi\\ cos(\psi-\beta)\,sin\,\varphi\,sin\,\delta-sin(\psi-\beta)\,cos\,\varphi\\ cos\,\delta\,cos(\psi-\beta)\end{bmatrix}. \tag{37}$$

The above explicit expressions allow us to compute the secondary characteristics of the tooth surface accurately from the gear parameters. On the other hand, we cannot accurately determine the principal curvatures and principal directions if the tooth surface is approximated by NURBS.

5. Manufacturability analysis

In the traditional cutting process, it is not possible to produce logarithmic spiral bevel gears, since the milling or hobbing will inevitably change the spiral angle during the movement of the machine tools. However, a suitable method for manufacturing logarithmic spiral bevel gears remains uncertain.

5.1 Precision forging

Among the various plastic forming methods, precision forging offers the possibility of obtaining high quality parts. It allows better material utilization in comparison to cutting, a reduction of the costs due to the shorter cycle times, and new possibilities concerning the tooth surface geometry of the forged gears. Precision forging also contributes to fulfill the demand of the production of highly loaded and small module gears widely used in the automobile industry, because of the fiber orientation which is favorable for carrying high oscillating loads [20].

However, precision forging technology is typically only applied for manufacturing spur gears and straight bevel gears [16]. For logarithmic spiral bevel gears, the basic limitation is the pattern draft of the forging die. The die geometry is obtained for logarithmic spiral bevel gears from their theoretical geometry; the manufacturability can thus be analyzed according to the above mentioned tooth surface geometry.

A suitable pattern draft along the *z*-axis, of which the unit vector $\boldsymbol{v}_d$ is (0,0,1), must satisfy

$$\boldsymbol{n}\cdot v_d\leq 0 \tag{38}$$

where $\boldsymbol{n}$ is the unit normal vector of the tooth surface. For the convex tooth surface, according to Eq. (20) and Eq. (38), the following formula must work.

$$\varphi-\phi\geq\frac{\beta}{sin\,\delta} \tag{39}$$

Eq. (39) shows it is impossible to remove the forged part from the forging die without any damage if the shape of the theoretical convex tooth surface has not been modified, as seen in the parametric domain shown in Figure 8. Eq. (39) also implies that precision forging technology is only suitable for manufacturing spiral bevel gears with a smaller spiral angle. However, according to the shape modification of the damaged domain and optimization of the contact zone by TCA or the function-oriented active design method, it is en-

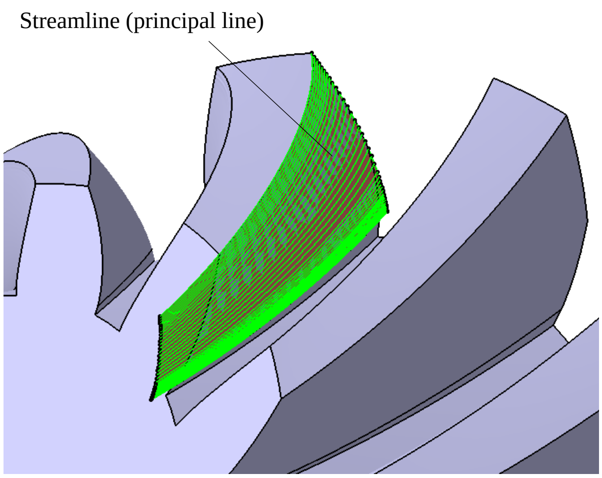

Figure 9. Streamline field orientation.

tirely possible to manufacture logarithmic spiral bevel gears using precision forging technology.

For the concave surface, the normal vector ***n*** should be reversed according to the parametric direction. Thus,

$$\varphi - \phi \leq \frac{\beta}{\sin \delta} \tag{40}$$

always exists due to $-\theta_{max} \leq \varphi - \phi \leq 0$ $(\beta > 0)$ or $0 \leq \varphi - \phi \leq \theta_{max}$ $(\beta < 0)$. Thus, it is not necessary to modify the shape of theoretical concave tooth surface, as shown in Figure 8.

5.2 Multi-axis freeform milling

In terms of the manufacturing process, in almost all works it is assumed that the gears are machined using special types of machine tools, such as CNC based hobbing and milling machines. However, the kinematic structure and dynamics behavior of the CNC based gear manufacturing machine tools still inherently differ from the industrial multi-axis milling machine tools. Although freeform milling by widely used industrial multi-axis machines has an obviously lower production rate than cutting using special types of machine tools for spiral bevel gear manufacturing, in single piece and small batch productions, it is advantageous to have a broad range of size change due to unnecessary equipment investment, especially in the manufacture of substantially large gears with diameters of over 1,000 mm.

Similar to the method used for manufacturing integral impellers, tool path planning is the key to obtaining successful results for logarithmic spiral bevel gears with multi-axis freeform milling. Besides tool interference, we particularly focus on the curvature field of the tooth surface coupled with the tool path. The tooth surface geometry (in particular its principal curvature field), deeply influences its contact mechanical properties. An unsuitable tool path will damage its streamline field orientation; the tool path should thus coincide with one of the principle curvature lines of the tooth surface. More precisely, the tooth path should be ϕ-curve, since ϕ-curve is also a principal curvature line, as shown in Figure 9.

6. Conclusions

- The tooth surface of logarithmic spiral bevel gears is a parametric surface defined on a parallelogrammic domain. It undoubtedly offers many advantageous geometrical characteristics by differential geometry analysis. Analyzing the tooth surface geometry helps us to fully understand its manufacturability and possible kinematic and dynamic behavior in application.
- Because logarithmic spiral bevel gears cannot be manufactured using traditional hobbing and milling machines, analysis is carried out on their manufacturability with precision forging and multi-axis freeform milling technology. The result shows that tooth surface shape modification is inevitable for precision forging. However, tooth shape modification can be easily controlled by two simple feature parameters. In addition, the curvature streamline should be maintained for multi-axis freeform milling to obtain a high quality tooth surface.
- In theory, the truly conjugate spiral gears have a line contact. More precisely, the line contact is a spatial logarithmic spiral. However, in order to decrease the sensitivity of the gear pair to errors in tooth surfaces and to the relative positions of the mating members, a set of carefully chosen modifications must be applied to the teeth of one or both mating gears. As a result of these modifications, the logarithmic spiral bevel pair becomes mismatched, and a point contact replaces the theoretical line contact. Regardless of the method used to manufacture the mismatched logarithmic spiral bevel gears, such as precision forging and general multi-axis freeform milling, the emergent practice challenge is how to generate the optimal tooth surfaces of the pinion and the gear in order to reduce transmission error.

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