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Single and multi-material topology optimization of CFRP composites to retrofit beam-column connection

Hoang V. Dang, Dongkyu Lee and Kihak Lee*

Department of Architectural Engineering, Sejong University, 98 Gunja-dong, Gwangjin-gu, Seoul 143-747, Republic of Korea

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Abstract. Carbon Fiber Reinforced Plastic (CFRP) has commonly been used to strengthen existing RC structures. Wrapping the whole component with CFRP is an effective method and simple to execute. Besides, specific configuration of CFRP sheets (L, X and T shape) has also been considered in some experiments to examine CFRP effects in advance. This study aimed to provide an optimal CFRP configuration to effectively retrofit the beam-column connection using continuous material topology optimization procedure. In addition, Moved and Regularized Heaviside Functions and penalization factors were also considered. Furthermore, a multi-material procedure was also used to compare with the results from the single material procedure.

Keywords: topology optimization; 99-line matlab code; multi-material; stress concentration; beam-column joint; retrofitted

1. Introduction

Carbon Fiber Reinforced Plastic (CFRP), a composite material that is superior to others in terms of high strength and stiffness-to-weight ratio, simple implementation, excellent fatigue behavior and corrosion resistance, has been widely utilized as an effective material to retrofit structures. Many experiments have previously been carried out to investigate the impact of CFRP on structural components. Kim *et al.* (2015) investigated the shear behavior of reinforced concrete T-beams retrofitted with CFRP strips and anchors, while Song *et al.* (2015) examined the fatigue performance of corroded concrete beams reinforced with CFRP sheets.

Recently, many existing reinforced concrete (RC) structures, which were designed and constructed according to conventional codes, have been found to be deficient in the seismic detail of the beam-column connection. That deficiency could lead to structure failure when sustaining seismic loadings during severe earthquakes. A common solution is to use extra CFRP to strengthen old existing RC structures to improve their seismic performance. Among studies on the effect of CFRP on structure components, many experimental investigations have been performed on the CFRP-retrofitted beam-column connection, including by Pantelides *et al.* (1999), Yao *et al.* (2005), Antonopoulos and Triantafillou (2003), Ghobarah and Said (2002), Yurdakul and Avsar (2015), and Rahimipour *et al.* (2016). In particular, Le-Trung *et al.* (2010, 2011) used different configurations of CFRP (L, T and X shapes) to experimentally determine which would significantly strengthen the beam-column connection. A macro-scale model was also developed and proposed to simulate the behavior of the CFRP-strengthened beam-column connection mentioned above. Both experimental and analytical results were compared to determine the base CFRP effect on each configuration. However, investigating CFRP configurations is largely a process of trial and error, and mostly depends on the experience and intuition of the structural engineers. Inspired by such works, this study aims to provide an optimal 2D CFRP configuration for the beam column connection, obtained from both single and multi-material topology optimization procedures.

Topology optimization has become very popular in several fields seeking to obtain an optimal material distribution within a prescribed set of design variables. Researchers have focused on topology optimization using various approaches. Both 2D and 3D structures have been deeply investigated using single and multi-material topology optimization procedures. Wang *et al.* (2004), Wang *et al.* (2015), Luo *et al.* (2008), Xia *et al.* (2014, 2015, 2016) presented level set-based methods for topology and shape optimization. Park and Sutradhar (2015) proposed a multi-resolution implementation in 3D for the multi-material topology optimization problem. These methods demonstrated both ease and effectiveness through several numerical examples. Several approaches with many considerations in the field of topology optimization have been addressed by many researchers for decades Sigmund and Petersson (1998), Bendsøe and Sigmund (1999), Sigmund (2001), Buehler *et al.* (2004), Zhou *et al.* (2007), Stainko (2006), Lee *et al.* (2012), Luo *et al.* (2012), Luo *et al.* (2013), Xia *et al.* (2013), Bruggi and Taliercio (2013), Tavakoli (2014).

This research aimed to find the most effective arrangement of CFRP for retrofitting the beam-column connection, using a procedure of continuous material topology optimization. Single and multi-material

*Corresponding author, Professor
E-mail: kihaklee@sejong.ac.kr

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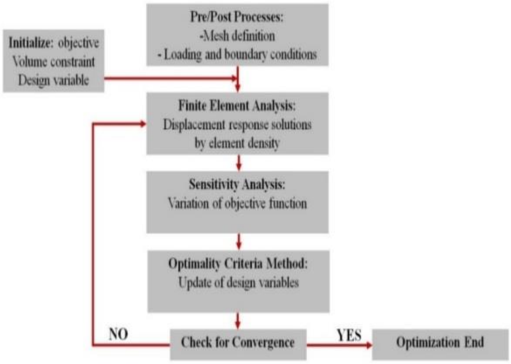


Fig. 1 Flowchart of general topology optimization procedure

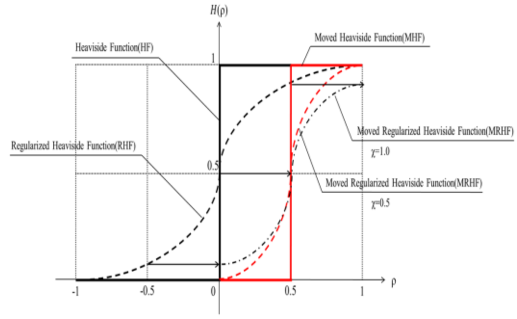


Fig. 2 Moved and Regularized Heaviside Functions (MRHF)

optimization procedures were extended from the source of the 99 line Matlab implementation developed by Sigmund (2001) and the 115 line Matlab code by Tavakoli and Mohseni (2014), respectively. The 99-line code has been extended by Andreassen *et al.* (2011) to an 88-line code with more improvements in speed and code length to solve large scaled problems. The resulting configuration arrangement obtained for the CFRP is expected to be beneficial in terms of structure shear capacity, overall connection damage tolerance, and economic aspects. Additionally, Heaviside functions and a penalization factor, which affect the topology results in terms of the existence of the physical material, were also considered. The results obtained from the single material and multi-material topology optimization procedures were then compared and discussed.

2. Topology optimization problem

2.1 Single and multi-material topology optimization

To date, many approaches have been developed and proposed for solving topology optimization problems. Bendsøe and Kikuchi (1988) used so-called microstructure or homogenization based approaches, which investigate the existence of solutions. While the approaches are important for providing bounds on the theoretical performance of structures, the determination and evaluation of optimal microstructures and their orientation still have some disadvantages.

An alternative approach is the so-called "power-law approach" or SIMP approach (Solid Isotropic Material with Penalization), which was proposed by Bendsøe (1989), Zhou and Rozvany (1991) and Mlejnek (1992). This mathematical programming based approach is simple to implement, and computationally is equally efficient compared to other methods. Furthermore, many extended non-compliance objectives of this approach can be easily done, such as multi-physics and multi-constraints problems. A topology optimization problem based on the SIMP approach can be mathematically expressed as follows.

Minimize: $c(x) = U^T K U = \sum_{e=1}^{N} x_e{}^p \, u_e^T k_0 u_e$

Subject to: $\frac{V(x)}{V_0} = f$

: $KU=F$

: $0 < x_{\min} \le x \le 1$

U: global displacement

F: force vectors

K: global stiffness matrix

u_e: element displacement vector

k_e: stiffness matrix

x: vector of design variables

x_{min}: vector of minimum relative densities (non-zero to avoid singularity).

N: number of elements in the design domain

P: penalization power

$V(x)$: material volume

V_0: design domain volume

f: volume fraction

Fig. 1 presents the flowchart of the general topology optimization procedure. Further interpretation of a solution for this topology optimization problem can be seen in Sigmund (2001). A 99 line-Matlab code implementation proposed by Sigmund (2001), which is based on the SIMP approach, was adopted and modified to topology optimize the beam-column connection with a single material. Furthermore, Moved and Regularized Heaviside Functions (MRHF) were also considered in the topology optimization and compared with the originals. The MRHF which were used in this procedure are discontinuous functions, whose value is toward zero for below 0.5 argument and toward one for above 0.5 argument. The purpose of using MRHF is to filter the topologies or density of elements in the design domain which have values close to zero and 1. Fig. 2 illustrates comparison between the original Heaviside Functions and the MRHF, graphically. There are 4 MRHF that are usually applied, which be expressed in Eqs. (1), (2), (3), (4). In addition, the penalization factor p was also considered in this topology optimization. The SIMP approach has been criticized and provoked arguments about whether a physical material exists with properties described by the power-law interpolation. However, it has proved that the power-law approach is physically permissible, as long as simple conditions on the power are satisfied ($p \ge 3$ for a material with a Poisson's ratio equal to $\frac{1}{3}$). In order to see the effect of the penalization factor p, cases of penalization factor $p=3$ and $p=1$ are both considered and compared.

$$F_1(x) = \frac{3}{4}\left[\frac{x-0.5}{\rho} - \frac{1}{3}\left(\frac{x-0.5}{\rho}\right)^3\right] + \frac{1}{2} \qquad (1)$$

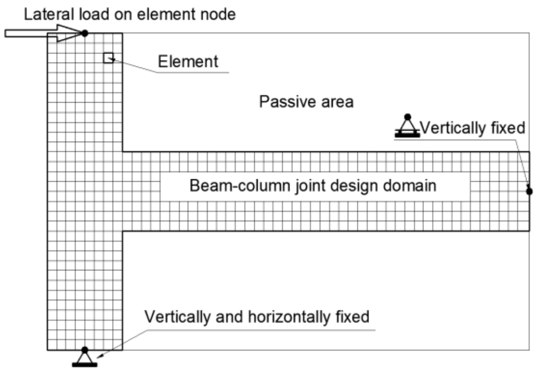


Fig. 3 Beam-column joint design domain and boundary conditions

$$F_2(x) = \frac{1}{2} + \frac{2}{\pi} arctan\left(\frac{x-0.5}{\rho}\right) \tag{2}$$

$$F_3(x) = \frac{1}{2}\left(1 + \frac{x-0.5}{\rho} + \frac{1}{\pi} sin\left(\pi \frac{x-0.5}{\rho}\right)\right) \tag{3}$$

$$F_4(x) = \frac{1}{2}\left(1 + sin\left(\pi \frac{x-0.5}{2\rho}\right)\right) \tag{4}$$

Where ρ=0.5

For the multi-material topology optimization of the beam-column connection, an alternating active-phase algorithm was adopted using a modified version of the 115-line Matlab implementation by Tavakoli and Mohseni (2014). A multi-material topology optimization problem can be expressed as follows.

Minimize: $J^h(\alpha^h, (U^h(\alpha)^h))$ where $\alpha^h \in A^h$

Subject to: $R^h\big(M(\alpha^h), (U^h(\alpha)^h)\big) = 0$ in Ω^h

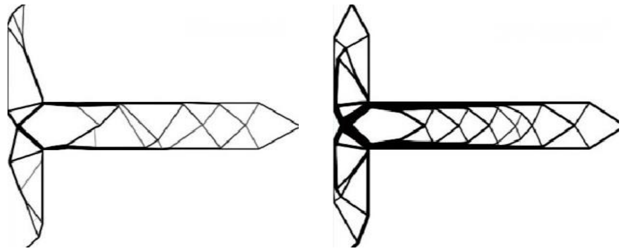
2.2 Design domain, load and boundary conditions of the beam-column connection

Voids were put in the initial rectangular design domain to obtain the geometry of the beam-column connection. The initial design domain was discretized by many square elements with 4 nodes at the corners. Each node has two horizontal and vertical degrees of freedom, which can be imposed with loads and/or restrained to create boundary conditions. In the beam-column connection case, a lateral load is imposed on the center node at the top of the column. The boundary condition of the beam-column connection is illustrated in Fig. 3.

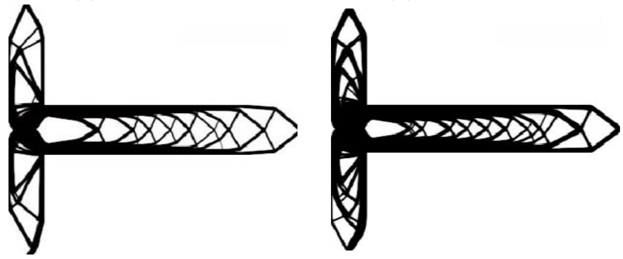
3. Results for beam-column connection

3.1 Single material topology optimization results

In this single material topology optimization for the beam-column connection, the Young's Modulus and Poison's ratio of the material are assumed to be 1 and 0.3, respectively. The minimum length scale (filter size) is 1.5 and the penalization factor is 3. As mentioned before, the penalization factor was chosen to satisfy the conditions for

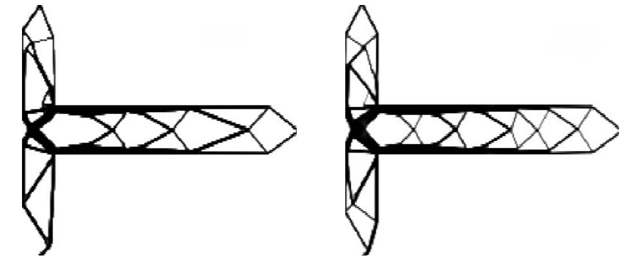


(a) 5% material (b) 10% material

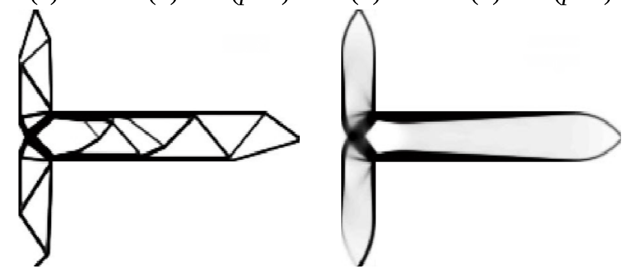


(c) 15% material (d) 30% material

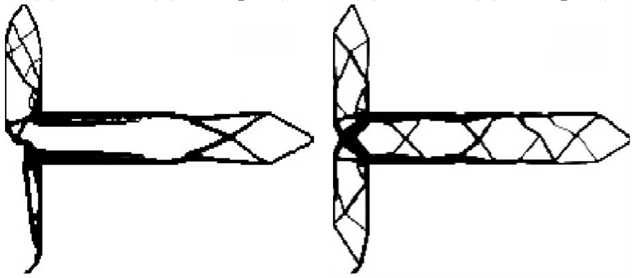
Fig. 4 Detail density arrangement of material



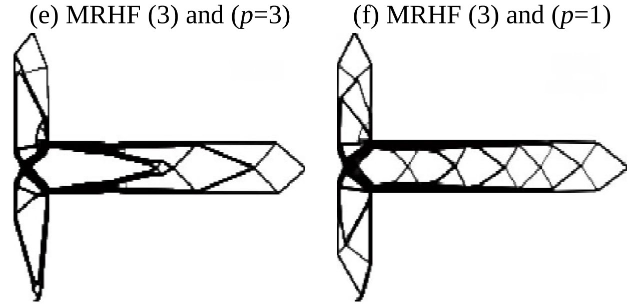
(a) MRHF (1) and (p=3) (b) MRHF (1) and (p=1)



(c) MRHF (2) and (p=3) (d) MRHF (2) and (p=1)



(e) MRHF (3) and (p=3) (f) MRHF (3) and (p=1)



(g) MRHF (4) and (p=3) (h) MRHF (4) and (p=1)

Fig. 5 Detailed results after using MRHF

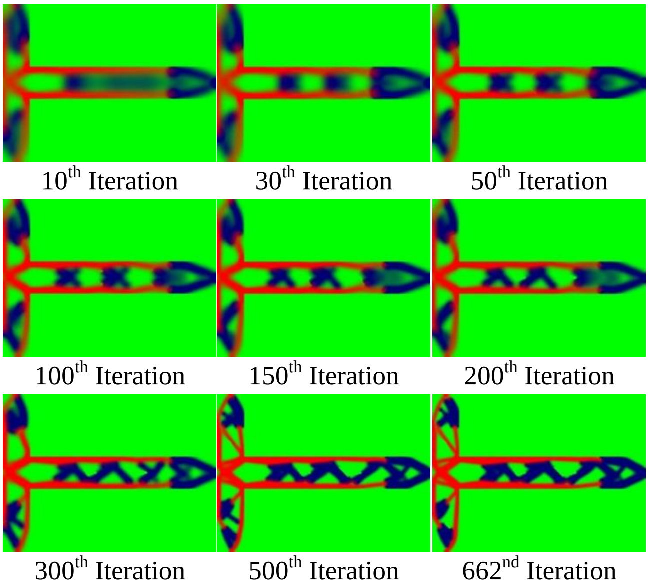
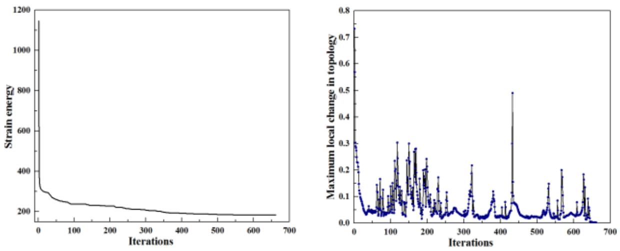


Fig. 6 Results of material distribution in selected iteration

E=8 and *V*=0.1 (CFRP)
E=1 and *V*=0.1 (Concrete)
E=1e-9 and *V*=0.8 (Void)



(a) Strain energy (b) Maximum local change in topology

Fig. 7 Iteration history of strain energy and topological change

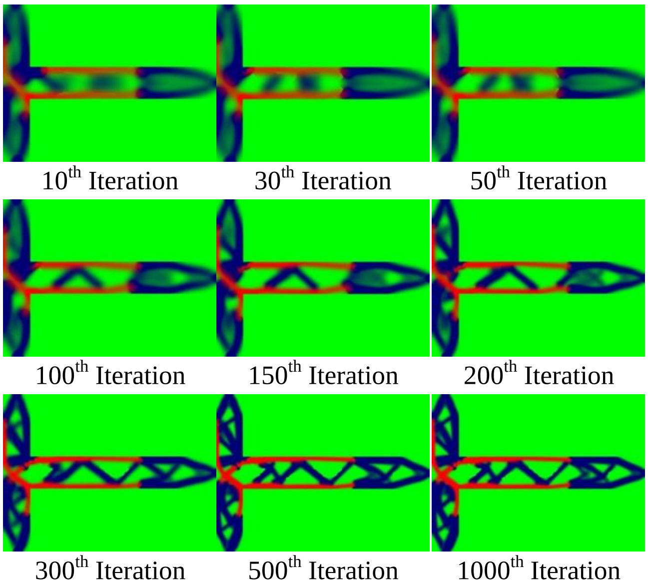
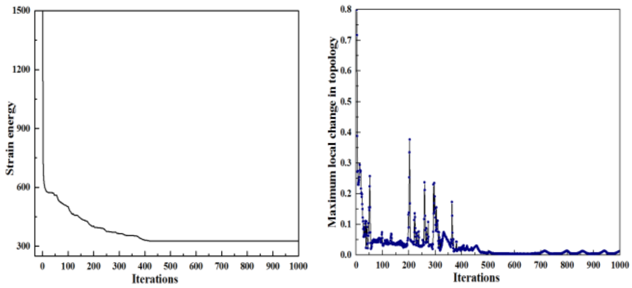


Fig. 8 Results of material distribution in selected iteration

E=8 and *V*=0.05 (CFRP)
E=1 and *V*=0.15 (Concrete)
E=1e-9 and *V*=0.8 (Void)



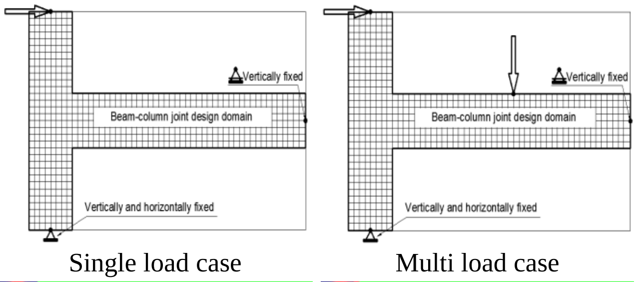
(a) Strain energy (b) Maximum local change in topology

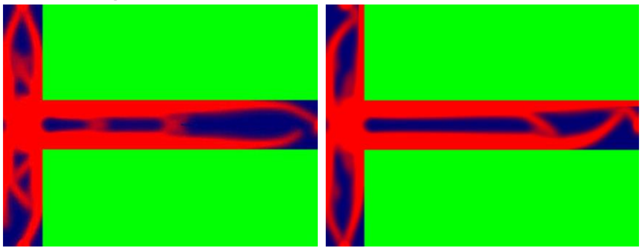
Fig. 9 Iteration history of strain energy and topological change

the SIMP approach, to obtain a physical material that exists. The amount of material used to retrofit the beam-column connection was sequentially chosen to be 5%, 10%, 15% and 30% of the whole design domain volume. The difference in topologies of those cases might show the most vulnerable location of the beam-column connection, where the attachment of the retrofitting material should be considered. Detailed material distributions of these cases are depicted in Fig. 4. It can be observed that with the increasing material volume, the rearrangement mostly occurs at the joint and along the edges of the beam-column connection. Since the joint of the beam-column connection is the place where stress concentrates the most, thick material density around the joint is a very common result, as expected.

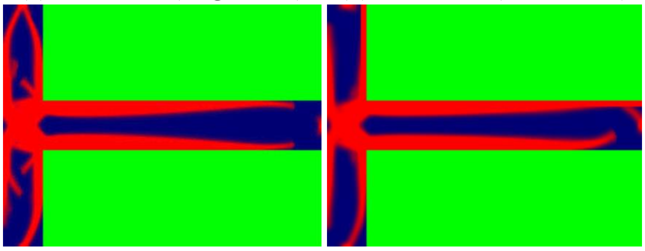
However, the material distributions on the beam and column part are similar to a complicated truss structure. Bruggi (2010) addressed similar optimal truss-like layouts for strut-and-tie models of concrete structures. This could be a motivation for many CFRP retrofitting applications, since to date, the truss has always been considered to be a main structure but not a truss-shaped CFRP retrofitting pattern. As material volume increases, the material strips the along beam and column edges becomes thicker, and the group of diagonal material strips are more complicated and overlap.

The case of 10% volume fraction was applied with the MRHF and a penalization factor of 1. The four MRHF mentioned above were all employed. Afterward, the penalization factor was changed (p=1) in these four cases. All detailed results are presented in Fig. 5. The material distributions, which were filtered with the MRHF, exhibit a major change compare to the original ones. Especially in the cases of MRHF (3) and MRHF (4), the materials in the bottom of the column and the beam were severely relocated. Among them all, MRHF (1) was the least changed from the material topology of the initial one. In cases where the penalization factor equal to 1, only MRHF (2) performed a non-physical material distribution. It can be also observed that with penalization equal to 1, the other cases of MRHF (1), (3), (4) give a similar topology. Hence, the MRHF significantly affect the topology results as long as the penalization factor is well considered (p=3).

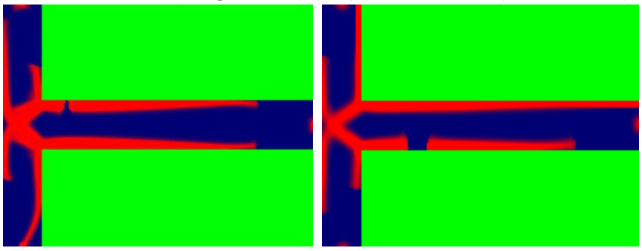




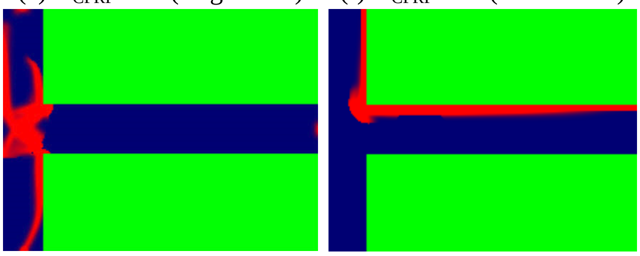
(a) V_{CFRP}=0.2 (single load) (b) V_{CFRP}=0.2 (multi loads)



(c) V_{CFRP}=0.15 (single load) (d) V_{CFRP}=0.15 (multi loads)



(e) V_{CFRP}=0.1 (single load) (f) V_{CFRP}=0.1 (multi loads)



(g) V_{CFRP}=0.05 (single load) (h) V_{CFRP}=0.05 (multi loads)

Fig. 10 Material distribution in cases with reduced volume of CFRP

3.2 Multi material topology optimization results

In this multi-material topology optimization, concrete and CFRP were chosen to be the main and the retrofit material, respectively. Hence, the elasticity modulus was assumed to be E=8 for CFRP and E=1 for concrete, which is in approximate ratio to their real elasticity modulus value. The void has E=1e-9. The Poisson's ratio of 0.3 was employed for both materials. Several cases with different volumes of concrete and CFRP were considered. By reducing the volume of CFRP, we can find the location to effectively retrofit the beam-column connection with CFRP. In order to see the significant topological changes of the beam-column connection during the solution process, several topologies at levels of iteration are shown.

Fig. 6 shows the topologies at selected iterations for the case where CFRP and concrete share the same volume fraction of 10%. Convergence of the compliance (or strain energy) of this case is also shown in Fig. 7(a). Fig. 7(b) shows the maximum local change in topology, which strongly oscillates during iterations and resulted in the change in details of material distribution. The strain energy in quickly converged, however, hundreds of iterations is required to obtain a sharp topology. The results of topologies and strain energy convergence for the other case, where the volume fraction of CFRP was reduced, are shown in Figs. 8 and 9, respectively. It can be seen that in both cases, the combination of topologies of the two materials shows a similarity to the topology of the single material case. As the stiffer material, CFRP mostly resides in the joint area and along the beam and column part of the beam-column connection. Meanwhile, the concrete forms truss-shaped patterns, which are located at the inner parts of the beam and column. These results show good agreement with the single material topology optimization, and the joint area and strips along the beam and column should be well retrofitted, especially the joint area and the strips along the beam (see Fig. 8) when less material is used.

However, these previous cases considered a small amount of concrete material, which led to an impractical concrete topology of truss-shaped patterns. The purpose of this research was to find an optimal material (CFRP) configuration to effectively retrofit the concrete beam-column connection. Therefore, bigger volume fractions of concrete were subsequently considered in the optimization procedure. Four more cases with descending volumes of CFRP are shown in Fig. 10(a), (c), (e), (g). Results of these cases also show good agreement with previous ones.

Multi load cases are also considered in this research to observe the change in topology. Fig. 10(b), (d), (f), (h) presents the material distribution when an additional vertical load was put in the middle of the beam. In cases of volume fraction of CFRP from 0.2 to 0.15, the change in topology is not significant, slightly change in details can be observed. However, in case of volume fraction of CFRP is 0.05, most of the material distributes along the upper edge of the beam, lacking of material in the beam-column joint.

4. Conclusions

This paper presented several results for single and multi-material topology optimizations for the beam-column connection. The results might be helpful for retrofitting the beam-column connection by providing some calculated foundations, instead of only experience and intuition. Both the single and multi-material topology optimization procedures shared similar patterns of retrofitting material. Besides the joint area of the beam-column connection, results showed that edges along the beam and column should also be well retrofitted. The complicated diagonal material strips located at the inner part of the beam and column of the structure resembled a truss-shaped pattern. Although it would be complicated to create approximately the same material pattern as that obtained with the topology optimization procedure, it could be a new motivation for retrofitting applications in the near future, due to the thriving potential offered by the 3D printing method nowadays. The results can also be adopted to build an analytical assessment to ensure the accuracy of the topology optimizations.

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