Computers and Concrete, *Vol. 13, No. 5 (2014) 659-677*
DOI: http://dx.doi.org/10.12989/cac.2014.13.5.659

Bayesian updated correlation length of spatial concrete properties using limited data

Pieterjan Criel[1*], Robby Caspeele[a] and Luc Taerwe[b]

Ghent University, Department of Structural Engineering, Magnel Laboratory for Concrete Research, Ghent, Belgium

(*Received January 29, 2013, Revised March 11, 2014, Accepted March 22, 2014*)

Abstract. A Bayesian response surface updating procedure is applied in order to update the parameters of the covariance function of a random field for concrete properties based on a limited number of available measurements. Formulas as well as a numerical algorithm are presented in order to update the parameters of response surfaces using Markov Chain Monte Carlo simulations. The parameters of the covariance function are often based on some kind of expert judgment due the lack of sufficient measurement data. However, a Bayesian updating technique enables to estimate the parameters of the covariance function more rigorously and with less ambiguity. Prior information can be incorporated in the form of vague or informative priors. The proposed estimation procedure is evaluated through numerical simulations and compared to the commonly used least square method.

Keywords: Bayesian updating techniques; random fields; concrete properties; covariance function; variogram; Markov chain Monte Carlo simulations

1. Introduction

Material properties of concrete structures have a significant spatial variability due to e.g. effects of workmanship, material heterogeneity, environmental influences and other factors. Random field theory can be used to model this spatial variability if the covariance function of the random field is known. This allows modelling concrete properties in a more realistic way, which might have a significant effect on structural reliability quantification. In particular cases, it can be insufficient to base such reliability-based calculations on a random estimate of a spatially homogeneous material property. Based on an empirical semi-variogram, this spatial variability can be modelled, but sufficient data is seldom available to compose a reliable semi-variogram for concrete related properties, especially with respect to properties assessed through destructive testing. Hence, often assumptions with respect to random field characteristics are made.

Random fields are frequently used in geotechnical reliability calculations, such as slope stability (Griffiths *et al*. 2011), or bearing capacity of a rigid strip footing on cohesive soil

∗Corresponding author, Ph.D.Student, E-mail: Pieterjan.Criel@UGent.be
[a]Professor., E-mail: Robby.Caspeele@UGent.be
[b]Professor, E-mail: Luc.Taerwe@UGent.be

http://www.techno-press.org/?journal=cac&subpage=8 ISSN: 1598-8198 (Print), 1598-818X (Online)

(Griffiths *et al*. 2002).Applications are also found in mining engineering, e.g. mineral resources evaluation (Emery and Cornejo 2010). Usually a considerable amount of data is available to define the field properties in these applications. In the case of concrete structures it is often not possible to acquire an equivalent amount of data, especially if the data can only be acquired by destructive testing. However, random fields aremore frequently applied in structural engineering analyses related to concrete. Consider for example the effects of the spatial variability of chloride induced corrosion of concrete slabs (Keßler *et al*. 2010; Straub 2011; Daniel Straub and Fischer 2011), spatial uncertainty concerning corrosion-induced cracking or cracking of concrete slaps and beams (Firouzi and Rahai 2011; Most and Bucher 2006, 2007)or the spatial uncertainty concerning geometrical en material properties when performing structural analyses of reinforced concrete (Vasconcellos Real *et al*. 2003). In most of these references, a value for the correlation length is assumed. As can be concluded from Table 1, these assumptions for the correlation length are not consistent with eachother.

This paper deals with the problem of insufficient data to properly define the characteristics of a random field. Typically analysis of samples gives an estimate of the mean values and standard deviation of the field. However, a third parameter is necessary to describe the spatial variability of the field. A common parameter for this is the semi-variogram. The semi-variogram is a function of the distance in the field. Evaluated in an arbitrary distance it is a measure for the correlation over that distance in the random field.

In the scope of this paper it is deemed sufficient to consider only Gaussian random fields. Non-Gaussian fields can be created through transformations of Gaussian fields (Fenton, 1994). Gaussian fields are important for two main reasons. First, the Gaussian distribution is well-studied, which makes the calculations easier and second the central limit theorem states that the net result of many small-order effects is approximately Gaussian (Cressie 1993) even if those small-order effects are weakly correlated (Vanmarcke 2010). A Gaussian field can always be transformed to a standard normal distributed random field. As such, only standard normal distributed random fields are considered in this paper.

Methods such as the maximum likelihood estimation (MLE) or least-square fitting (LSQ) methods require large datasets in order to accurately determine the characteristics of the covariance structure of the random field. In cases where only a few measurements are available, often assumptions are made about the parameters of the covariance function. However, Bayesian updating techniques allow to estimate the parameters of the covariance function more rigorously and with less ambiguity as these can be used to update previously obtained information regarding parameters of similar random fields. Markov chain Monte Carlo (MCMC) simulations can be used to incorporate Bayesian updating based on limited samples in the parameter estimation. Prior information (vague or informative) can then be used to update the covariance function based on available monitoring data or measurement results.

In order to verify this method, Gaussian random fields with a known covariance structure need to be simulated. Different simulation methods are available and can be classified as exact algorithms, where the statistical properties of the field match the properties of the desired model, or approximate algorithms, where the statistical properties of the field approximate those of the desired model. Exact algorithms comprise a decomposition of the covariance matrix (typically using the Cholesky decomposition, which typically has a computational cost of O(n^3)(Trefethen and Bau 1997)). These exact algorithms tend to be time consuming for large random fields. The approximate algorithms are often more efficient for practical applications. In this paper the

Table 1 Suggested correlation lengths for various concrete properties by different authors

Concrete property	Correlation length [m]	Reference
Concrete cover (c)	1.0	(D. Straub 2011)
	2.0	(Li 2004)
	3.5	(Stewart & Mullard 2007)
Chloride surface concentration (C_0)	1.0	(Engelund 1997)
	1.96	(Vu 2003)
	2.0	(D. Straub 2011)
	3.5	(Duprat 2007)
Concrete strength (f_c)	3.5	(Duprat 2007)

Turning Bands Method (TBM) (Matheron, 1973) is used to generate random fields. This method allows the simulation of a random field with a zero-mean and a specified covariance function using a sequence of one-dimensional zero-mean processes along lines crossing the domain (Fenton, 1994).

The proposed parameter estimation method is based on Bayesian updating of nonlinear response surface parameters. A semi-variogram model is fitted on an empirical semi-variogram while considering prior information about the model parameters. Since it is typically difficult or even impossible to find an analytically closed expression for the posterior distribution, the update process relies on Markov chain Monte Carlo simulations. In the following, first some basic concepts are explained, an adapted Metropolis-Hasting algorithm is established in order to enable Bayesian updating of covariance functions and finally the proposed method is evaluated by numerical simulations and compared with the LSQ method.

2. Random field theory

2.1 Basic concept

A random field $\{X(\underline{t}), \underline{t} \in \Omega\}$ is a function whose values are random variables for any position $\underline{t}$in the domain $\Omega \subset \mathbb{R}^d$. In this paper equations are presented for two-dimensional random fields, i.e.$d = 2$. In general, the characteristics of those random variables can differ for each position $\underline{t}$ in the random field. A realisation of a random field is denoted by a deterministic function$x(\underline{t})$.Random fields can be decomposed into a mean value of a trend surface $m(\underline{t})$ and a residual variation with mean 0. This residual variation usually exhibits some spatial structuredescribed by a covariance function $B(\underline{t}_i, \underline{t}_j)$.

In general, the mean and the covariance function are not sufficient to specify a random field. However, as mentioned, Gaussian random fields are second order stationary.Thus, the mean and covariance function are enough to completely specify the field (Cressie 1993).

The covariance function $B(\underline{t}_i, \underline{t}_j)$of two random variables on location $\underline{t}_i$and $\underline{t}_j$ is defined as

$$B(\underline{t}_i,\underline{t}_j) = E[X(\underline{t}_i)\text{-}E[X(\underline{t}_i)])(X(\underline{t}_j)\text{-} E[X(\underline{t}_j)])]$$

$$= E[X(\underline{t}_i)\text{-}X(\underline{t}_j)])\text{-} E[X(\underline{t}_i)]E[X(\underline{t}_j)] \quad (1)$$

where $E[.]$ is the expectancy operator.
When only homogeneous, isotropic and ergodic fields are considered, the covariance function is only dependent on the distance between those two location vectors $\underline{t}_i$ and $\underline{t}_j$.

$$B((\underline{t}_i,\underline{t}_j) = B(||\underline{t}_i - \underline{t}_j||) = B(\tau) \tag{2}$$

Note that if $\tau = 0$ the distance between the two random variables in the random field is zero; For second-order stationary fields the covariance will be equal to the variance σ^2 of the random field.

$$B(0) = \sigma^2 \tag{3}$$

Different models are available for the covariance function. Commonly used models are the exponential or Gauss-Markov covariance function model, the squared exponential or Gaussian covariance function model and the Matérn covariance function model. The exponential and squared exponential covariance function models are described by only one parameterθ. The Matérn covariance function model additionally incorporates a smoothness parameter ν.

The correlation length ρ_l is a measure of the distance over which the field is significantly correlated. A large correlation length indicates that the field is varying slowly, while a small value for ρl indicates a field that is varying rapidly (D. V. Griffiths *et al*. 2011). The correlation length isdefined as the centroid of the surface under $B(\tau)$(Baecher and Christian 2003)

$$\rho_l = \frac{\int_0^\infty \tau B(\tau) \mathrm{d}\tau}{\int_0^\infty B(\tau) \mathrm{d}\tau} \tag{4}$$

Instead of assigning model parameters to define the covariance function it is more interesting to assign the correlation length ρ_l as a model parameter. For the exponential covariance function this is fairly easy since the correlation length is equal to the model parameter, i.e.,$\rho_l = \theta$. The model parameters for the other covariance functions can be replaced by a function $\theta(\rho_l)$.An overview is given in Fig. 1 and Table 2.

In spatial data analysis the variogram is used rather than the covariance function. The variogram is – like the covariance function – a function describing the degree of spatial dependence in a random field. The variogram$2\gamma(\underline{t}_j - \underline{t}_i)$ is defined as the variance of the difference between two values at $\underline{t}_i$ and $\underline{t}_j$ in the field.

$$\begin{aligned} 2\gamma(\underline{t}_j - \underline{t}_i) &= VAR\left[X(\underline{t}_i) - X(\underline{t}_j)\right] \\ &= VAR\left[X(\underline{t}_i)\right] + VAR\left[X(\underline{t}_j)\right] - 2COV\left[X(\underline{t}_i), X(\underline{t}_j)\right] \end{aligned} \tag{5}$$

where $COV\left[X(\underline{t}_i), X(\underline{t}_j)\right] = B(\underline{t}_i, \underline{t}_j)$is the covariance of $X(\underline{t}_i)$ with $X(\underline{t}_j)$

Under the assumption of homogeneous and isotropic fields, every position in the field has the same mean value. Hence, the variogram $2\gamma(\cdot)$ is a function of the increment $\tau = \|\underline{t}_j - \underline{t}_i\|$. For second-order stationary fields all random variables in the random field have the same variance, resulting the in following relation between the variogram and the covariance function (see Eqs. (1)-(2) and Eq. (3))

Table 2 Overview of common covariance functions for homogeneous, isotropic and ergodic random fields

Covariance function	$\boldsymbol{B(\tau)}$[a]	**Parameters**	$\boldsymbol{\theta(\rho_l)}$
Exponential	$\sigma^2 \exp\left(-\frac{\lvert\tau\rvert}{\theta}\right)$	$\theta > 0$	ρ_l
Squared exponential	$\sigma^2 \exp\left(-\frac{\lvert\tau\rvert^2}{\theta^2}\right)$	$\theta > 0$	$\rho_l\sqrt{\pi}$
Matérn[b]	$\sigma^2 \frac{1}{\Gamma(\upsilon)2^{\upsilon-1}}\left(2\sqrt{\upsilon}\frac{\lvert\tau\rvert}{\theta}\right)^{\upsilon} K_\upsilon\left(2\sqrt{\upsilon}\frac{\lvert\tau\rvert}{\theta}\right)$	$\theta > 0, \upsilon > 0$	$\rho_l \frac{\sqrt{\pi}\Gamma(\upsilon+1/2)}{\sqrt{\upsilon}\Gamma(\upsilon)}$

[a] σ^2 is the variance of the field
[b] $\Gamma(\cdot)$ is the Gamma function and $K_\upsilon(\cdot)$ the $BesselK(\upsilon,\cdot)$ function

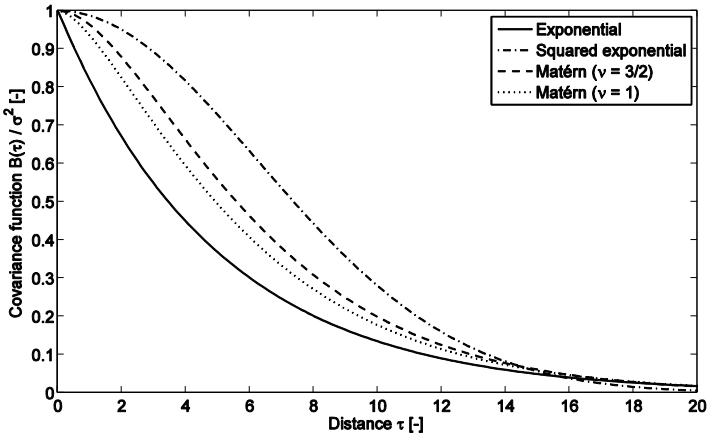


Fig. 1 Different covariance functions for $\rho_l = 5$

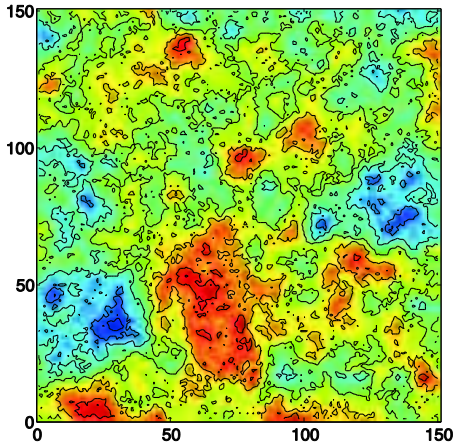


Fig. 2 Visual representation of a random realization of a Gaussian random field with a correlation length $\beta = 20$ on a 150×150 grid using the **TBSIM.M** program

$$2\gamma(\tau) = 2\big(B(0) - B(\tau)\big) \tag{6}$$

For numerical applications random fields are most often defined on a discrete domain, e.g., a random field on a regular lattice grid.A continuous random field can be obtained by interpolation methods, e.g. Kriging (Cressie 1993).

2.2 Simulation of random fields

The Turning Band Method is used in order to simulate random fields with known properties (Matheron 1973). The simulation of the random field $X(\underline{t})$using this method is based on a sequence of one-dimensional processes $Y_i(\xi)$, $\xi \in \mathbb{R}$ with a covariance function $B_1(\xi)$that cross the domain of $X(\underline{t})$. The direction of $Y_i(\xi)$ is given by a unit vector $\underline{u}_i$.ξ is measured along $\underline{u}_i$.

The value of the field in position $\underline{t}_k$ is given by

$$X(\underline{t}_k) = \frac{1}{\sqrt{L}} \sum_{i=1}^{L} Y_i\big(\langle \underline{t}_k, \underline{u}_i \rangle\big) = \frac{1}{\sqrt{L}} \sum_{i=1}^{L} Y_i(\zeta_k) \tag{7}$$

where L is the number of one-dimensional processes and the operator $\langle\ ,\ \rangle$determines the coordinate ξ_k of the orthogonal projection of $\underline{t}_k$ on $\underline{u}_i$.If $\underline{t}_k$ and $\underline{u}_i$ have the same origin then $\langle\ ,\ \rangle$ is defined by the dot product.

The covariance function $B_1(\xi)$ for the one-dimensional processes must be chosen in a manner that the covariance function of the field is $B(\tau)$. For two-dimensional isotropic processes the following relation is suggested (Mantoglou and Wilson 1982)

$$B(\tau) = \frac{2}{\pi} \int_0^{\tau} \frac{B_1(\zeta)}{\left(\tau^2 - \zeta^2\right)} \mathrm{d}\zeta \tag{8}$$

Eq. (8) is a Volterra integral of the first kind to be solved for $B_1(\xi)$which is usually complicated planar random fields ($d = 2$).However$B_1(\xi)$ has the following simple solution in three dimensions if spherical coordinates are used (Emery & Lantuéjoul, 2006)

$$B_1(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\big[\tau B(\tau)\big] \tag{9}$$

Emery and Lantuéjoul (2006) implemented this method in a program **TBSIM.M**, written in MATLAB,which will be used to simulateGaussian random fields in this paper. An example of such a simulated field is shown in Fig.2.

3. Bayesian estimation of response surface parameters

Ba yesian estimation of linear regression models is well-described in literature, e.g. in (Box and Tiao 1992; Gamerman and Lopes 2006; Gelman *et al*. 2003; Ghosh *et al*. 2010; Gregory 2005; Lee

2012). However, in most cases observational data is modelled as a nonlinear combination of multiple model parameters and variables. Available literat ure on Bayesian nonlinear regression is rather limited (see e.g. (Gelman *et al*. 2003; Gregory, 2005)). See also (Caspeele and Taerwe 2013, Van Der Vurst *et al*. 2014).

Assume that the true value of the response variable $\tilde{y}$ can be predicted by a mathematical model M which is a nonlinear function of R parameters β_r $(r = 1, \ldots, R)$ and depends on a vector $\underline{x}$ which represents a m-dimensional set of input variables. If this model would be "perfect" and the true values $\underline{\tilde{x}}$ are exactly known, the model would be able to predict the true response value $\underline{\tilde{y}}$ exactly. However, due to the existence of uncertainties, the true value is given by

$$\tilde{y} = y + \varepsilon = M\left(\underline{x}\right) + \varepsilon \tag{10}$$

where the error term can be considered as a realization of a Gaussian random variable with mean 0 and variance σ_ε^2, representing the measurement and model uncertainties. Hence

$$\varepsilon \infty N(0, \sigma_\varepsilon^2) \tag{11}$$

The variance of the error term is assumed to be constant in the domain of the input variables.

If N independent test results y_i are available for the response variable of N sets of corresponding input variables $\underline{x_i}$, the likelihood of the experimental data can in general be written as

$$L\left(y_1, \ldots, y_N \mid \sigma_\varepsilon, \beta_1, \ldots, \beta_R\right) = \prod_{i=1}^{N} \frac{1}{\sigma_\varepsilon} \phi\left(\frac{y_i - M\left(\underline{x}_i\right)}{\sigma_\varepsilon}\right) \tag{12}$$

where $\phi(\cdot)$ is the probability density function (PDF) of the standard normal distribution.

Based on the Bayesian principle, the prior information (either vague or informative) is given as the joint prior distribution $f_B'(\sigma_\varepsilon, \beta_1, \ldots, \beta_R)$ of the standard deviation σ_ε of the error term and the model parameters $(\beta_1, \ldots, \beta_R)$. This prior distribution can be updated towards a posterior distribution $f_B''(\sigma_\varepsilon, \beta_1, \ldots, \beta_R)$ by using the likelihood function as follows

$$\begin{aligned} f_B''\left(\sigma_\varepsilon, \beta_1, \ldots, \beta_R\right) &= \frac{f_B'\left(\sigma_\varepsilon, \beta_1, \ldots, \beta_R\right) L\left(y_1, \ldots, y_N \mid \sigma_\varepsilon, \beta_1, \ldots, \beta_R\right)}{\int_B f_B'\left(\sigma_\varepsilon, \beta_1, \ldots, \beta_R\right) L\left(y_1, \ldots, y_N \mid \sigma_\varepsilon, \beta_1, \ldots, \beta_R\right) \mathrm{d}\sigma_\varepsilon \, \mathrm{d}\beta_1 \ldots \mathrm{d}\beta_R} \\ &= c f_B'\left(\sigma_\varepsilon, \beta_1, \ldots, \beta_R\right) L\left(y_1, \ldots, y_N \mid \sigma_\varepsilon, \beta_1, \ldots, \beta_R\right) \end{aligned} \tag{13}$$

with c a normalizing constant and B the domain of the parameters $(\sigma_\varepsilon, \beta_1, \ldots, \beta_R)$ that have to be updated. It turns out to be difficult or even impossible to solve Eq. (13) analytically. Therefore MCMC simulations are applied (i.e. using the Metropolis-Hastings algorithm) to estimate values for the model parameters and the standard deviation of the error term.

4. Markov chain monte carlo bayesian updating of response surface parameters using an adapted metropolis-Hastings algorithm

Markov chain Monte Carlo methods (MCMC) form a class of numerical algorithms that allow obtaining samples from probability distributions based on the construction of a Markov chain. A Markov chain is defined in probability theory as a sequence of random variables x_i for which the distribution of x_i, conditioned on past realizations $x_{i-1}, x_{i-2}, ...,$ depends only on the previous sample x_{i-1}, i.e. not on $x_{i-2}, x_{i-3},$ etc. (Gelman *et al*. 2003).

These methods allow to draw a discrete-time homogeneous chain of samples from the posterior distribution(Perrin, Pendola, & De Rocquigny, 2007). The idea is to generate iteratively samples of a Markov chain, which asymptotically behaves as the probability density function (PDF) that has to be sampled. More specifically, the Metropolis-Hastings algorithm (Hastings, 1970; Nicholas, Arianna, Marshall, Augusta, & Edward, 1953) is commonly used for generating such Markov chains. The practical adaptation of this algorithm for the Bayesian estimation of response surface parameters is explained hereafter. More profound information on MCMC simulations can be found in e.g. (Gamerman & Lopes, 2006; Gelman *et al*. 2003; Ghosh *et al*. 2010; Liu 2008; Robert and Casella 2010; Santoso, Phoon and Quek 2011).

Considering a given PDF $f_X(\underline{x})$ that is a function of an input vector $\underline{x}$, MCMC realizations $\underline{x}_s$ are generated sequentially and independently, starting from an arbitrary chosen starting vector $\underline{x}_0$. In each step, the transition between the states $\underline{x}_s$ and $\underline{x}_{s+1}$ is given according to (see e.g. (Gelman *et al*. 2003))

$$x_{s+1} = \begin{cases} \tilde{\underline{x}} \propto q\left(\tilde{\underline{x}} \mid \underline{x}_s\right) & \text{with probability } \alpha\left(\underline{x}_s, \tilde{\underline{x}}\right) \\ \underline{x}_s & \text{else} \end{cases} \tag{14}$$

where $\tilde{x}$ is a candidate vector, $q\left(\tilde{\underline{x}} \mid \underline{x}_s\right)$ is called the transition or proposal distribution and the acceptance probability $\alpha\left(\underline{x}_s, \tilde{\underline{x}}\right)$ is given by (see e.g. (Gelman *et al*. 2003))

$$\alpha\left(\underline{x}_s, \tilde{\underline{x}}\right) = \min\left\{1, \frac{f_X\left(\tilde{\underline{x}}\right)}{f_X\left(\underline{x}_s\right)} \frac{q\left(\underline{x}_s \mid \tilde{\underline{x}}\right)}{q\left(\tilde{\underline{x}} \mid \underline{x}_s\right)}\right\} \tag{15}$$

Practically, in order to select a candidate $\tilde{\underline{x}}$ – calculated according to Eq. (14) – a random number r_s is generated (i.e. from a uniform distribution $U[0;1]$) and $\tilde{\underline{x}}$ is accepted as the next

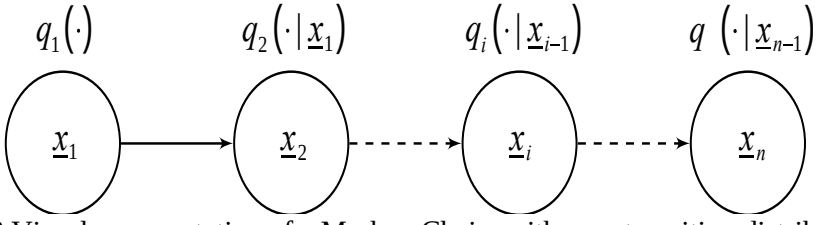


Fig. 3 Visual representation of a Markov Chain, with q a transition distribution

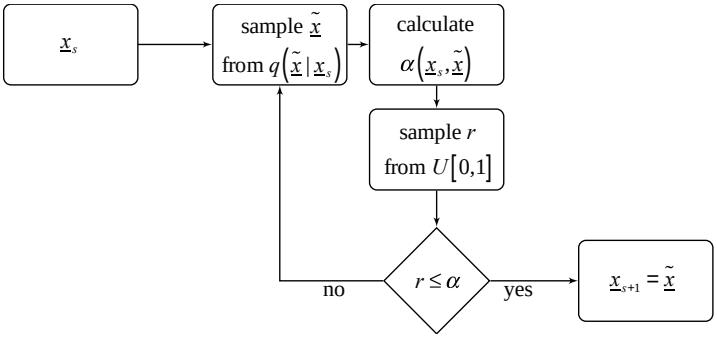


Fig. 4 Metropolis-Hasting algorithm flow chart. Transition step from $\underline{x}_s$ to $\underline{x}_{s+1}$

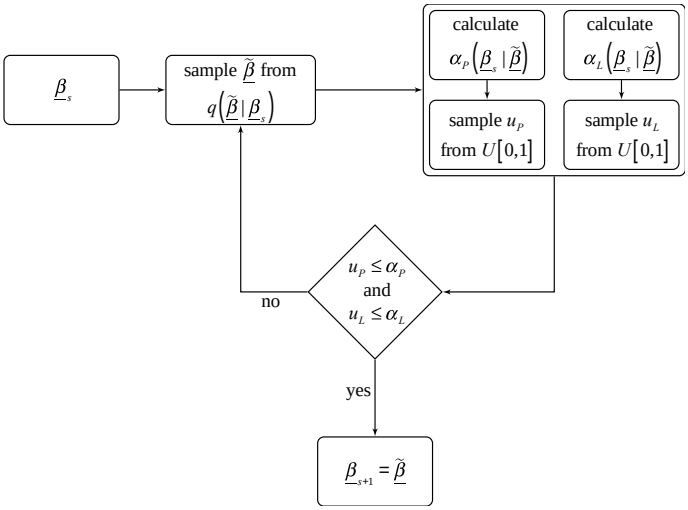


Fig. 5 Metropolis-Hasting algorithm flow diagram for Bayesian updating with $\underline{\beta}_s = \left(\sigma_{\epsilon,s}, \beta_{1,s}, \ldots, \beta_{R,s}\right)$. Transition step from $\underline{\beta}_s$ to $\underline{\beta}_{s+1}$

realization of $f_X\left(\underline{x}\right)$ with a probability $\alpha\left(\underline{x}_s, \tilde{\underline{x}}\right)$ in case $r_s \le \alpha\left(\underline{x}_s, \tilde{\underline{x}}\right)$ or rejected in the other case. In this way, a sequence of random draws $\underline{x}_s$ from $f_X\left(\underline{x}\right)$ is generated, even when no analytical solution is available for $f_X\left(\underline{x}\right)$. A flow chart of this algorithm is given in Fig.4.

In case of the Bayesian estimation of response surface parameters, the transition between two estimates $\left(\sigma_{\varepsilon,s}, \beta_{1,s}, \ldots, \beta_{R,s}\right)$ and $\left(\sigma_{\varepsilon,s+1}, \beta_{1,s+1}, \ldots, \beta_{R,s+1}\right)$ for the posterior set of response surface parameters can be rewritten as given in Eq. 16, considering an acceptance probability ψ which takes into account the posterior belief in the next sampling vector of estimates.

$$\left(\sigma_{\varepsilon,s+1}, \beta_{1,s+1}, \ldots, \beta_{R,s+1}\right) = \begin{cases} \left(\tilde{\sigma}_\varepsilon, \tilde{\beta}_1, \ldots, \tilde{\beta}_R\right) \propto q\left(\tilde{\sigma}_\varepsilon, \tilde{\beta}_1, \ldots, \tilde{\beta}_R \mid \sigma_{\varepsilon,s}, \beta_{1,s}, \ldots, \beta_{R,s}\right) & \text{with probability } \psi \\ \left(\sigma_{\varepsilon,s}, \beta_{1,s}, \ldots, \beta_{R,s}\right) & \text{else} \end{cases} \tag{16}$$

where$q\left(\tilde{\sigma}_\varepsilon,\tilde{\beta}_1,\ldots,\tilde{\beta}_R|\sigma_{\varepsilon,s},\beta_{1,s},\ldots,\beta_{R,s}\right)$ is the transition distribution. A common choice for this transition distribution is the random walk algorithm, more specifically by adding a random increment$\underline{\varsigma}=(\varsigma_0,\varsigma_1,\ldots,\varsigma_R)$ to the previous estimate according to

$$\left(\widetilde{\sigma}_\varepsilon,\widetilde{\beta}_1,\ldots,\widetilde{\beta}_R\right)^T=\left(\sigma_{\varepsilon,s},\beta_{1,s},\ldots,\beta_{R,s}\right)^T+\left(\varsigma_0,\varsigma_1,\ldots,\varsigma_R\right)^T \tag{17}$$

with$(\varsigma_0,\varsigma_1,\ldots,\varsigma_R)$ a random vector that does not depend on the previous chain. In practice, it is common to choose the values ς_i according to a normal or uniform distribution with mean 0 and variance σ_ς^2. The random walk algorithm results in a symmetrical transition distribution$q\left(\sigma_{\varepsilon,s},\underline{\beta}_s|\tilde{\sigma}_\varepsilon,\underline{\tilde{\beta}}\right)=q\left(\tilde{\sigma}_\varepsilon,\underline{\tilde{\beta}}|\sigma_{\varepsilon,s},\underline{\beta}_s\right)$. Hence,the transition distribution does not appear in the expression for the joint acceptance probability:

$$\psi\left(\sigma_{\varepsilon,s},\underline{\beta}_s|\widetilde{\sigma}_\varepsilon,\underline{\tilde{\beta}}\right)=\min\left\{1,\frac{f_B''\left(\widetilde{\sigma}_\varepsilon,\underline{\tilde{\beta}}\right)}{f_B''\left(\sigma_{\varepsilon,s},\underline{\beta}_s\right)}\right\} \tag{18}$$

Further, the probabilityψ is the joint acceptance probability based on the prior probability and the likelihood function or in other words the probability that a random sample$u_P\propto \mathrm{U}[0;1]$ from a uniform distribution (defined for values between 0 and 1) is accepted according to the prior distribution and that a random sample $u_L\propto \mathrm{U}[0;1]$ is accepted according to the likelihood function. This probability is generalized for Bayesian estimation of response surface parameters according the following equations

$$\psi=\mathrm{Prob}\left[\left(u_p\le\alpha_p\right)\cap\left(u_L\le\alpha_L\right)\right] \tag{19}$$

$$\begin{aligned}\alpha_p&\equiv\alpha_p\left[\left(\sigma_{\varepsilon,s},\beta_{1,s},\ldots,\beta_{R,S}\right),\left(\widetilde{\sigma}_\varepsilon,\widetilde{\beta}_1,\ldots,\widetilde{\beta}_R\right)\right]\\&=\min\left\{1,\frac{f_B'\left(\widetilde{\sigma}_\varepsilon,\widetilde{\beta}_1,\ldots,\widetilde{\beta}_R\right)}{f_B'\left(\sigma_{\varepsilon,s},\beta_{1,s},\ldots,\beta_{R,S}\right)}\right\}\end{aligned} \tag{20}$$

$$\begin{aligned}\alpha_L&\equiv\alpha_L\left[\left(\sigma_{\varepsilon,s},\beta_{1,s},\ldots,\beta_{R,S}\right),\left(\widetilde{\sigma}_\varepsilon,\widetilde{\beta}_1,\ldots,\widetilde{\beta}_R\right)\right]\\&=\min\left\{1,\frac{L\left(y_1,\ldots,y_N|\widetilde{\sigma}_\varepsilon,\widetilde{\beta}_1,\ldots,\widetilde{\beta}_R\right)}{L\left(y_1,\ldots,y_N|\sigma_{\varepsilon,s},\beta_{1,s},\ldots,\beta_{R,S}\right)}\right\}\end{aligned} \tag{21}$$

with the likelihood $L(y_1,\ldots,y_N\,|\ldots)$ according to Eq. (12) in case of independent response measurements. A flow diagram of this adaptation for Bayesian updating is given in Fig. 5.

The convergence speed of the Markov chain is strongly influenced by the random increment $\underline{\varsigma}$(each $\varsigma_i$either uniform or standard normally distributed) and more specifically by its standard deviationσ_ς. This standard deviation has the property of a scale parameter and has to make sure

that the domain over which $f_B^{"}$ is sampled is properly explored. If the scale parameter is small, then the Markov chain will converge slowly, because more iterations are needed to explore the entire domain of $f_B^{"}$. However, if the scale parameter is too large, then the chain will also converge slowly, because the acceptance probability is too low. Thus, it is important to make sure that theMarkov chain has enough iterations to represent the function $f_B^{"}$ accurately and to detect the length of the burn-in period (the period in the Markov chain where stationary is not yet obtained) so that these initial values can be deleted from the samples for $f_B^{"}$.

5. Bayesian estimation of the correlation length based on limited measurements

The method described in Section 3 and Section 4 can be used to estimate the covariance function of a random field based on limited measurements by fitting a semi-variogram model to the empirical semi-variogram. In case of an exponential of squared exponential covariance function the only unknown parameter is the correlation length ρ_l.

Rewriting Eq. (10)in terms of the semi-variogram yields

$$\bar{\gamma}(\tau) = M(\tau) + \varepsilon = \gamma(\tau \mid \rho_l) + \varepsilon \tag{22}$$

where $\bar{\gamma}(\tau)$ is the empirical semi-variogram, $\gamma(\tau \mid \rho_l)$ the semi-variogram model and ε the error term as defined in Eq. (11).

There are different methods available to compose an empirical semi-variogram based on measurement data (Cressie, 1993). In this paper the method-of-moments estimator defined by Matheron (1962) is adopted

$$\bar{\gamma}(\tau) = \frac{1}{2|T(\tau)|} \sum_{T(\tau)} \left(x(\underline{t}_i) - x(\underline{t}_j) \right)^2 \tag{23}$$

$$T(\tau) = \left\{ \left(x(\underline{t}_i), x(\underline{t}_j) \right) \; ; \; \left\| x(\underline{t}_i) - x(\underline{t}_j) \right\| = \tau \; ; \; i, j = 1 \ldots M \right\} \tag{24}$$

where $|T(\tau)|$ is the number of elements in the set $T(\tau)$ and M the amount of samples taken.

$\bar{\gamma}(\tau)$ is not continuous as there are only a finite number of distances between the measurement points. In order to obtain sufficient samples per set $T(\tau)$ to compose a semi-variogram in practice, similar distances are grouped in distance classes by adding a tolerance on the distance τ. Hence, the set of elements corresponding to a distance class τ are described as follows

$$T(\tau) = \left\{ \left(x(\underline{t}_i), x(\underline{t}_j) \right) \; ; \; \tau - e \le \left\| x(\underline{t}_i) - x(\underline{t}_j) \right\| \le \tau + e \; ; \; i, j = 1 \ldots M \right\} \tag{25}$$

The tolerance e should be carefully chosen so that the empirical semi-variogram is not biased and there are enough elements for each distance classτ_i. It is pointed out that due the nature of this estimator the semi-variogram can show considerable variation for different realisations ofa field as is shown in Fig. 6. Of course less measurement points will lead to more variation of the semi-variogram.

The likelihood function defined in Eq. (12) in terms of the semi-variogram can be rewritten as

$$L\left(\bar{\gamma}_1,\ldots,\bar{\gamma}_N \mid \sigma_\varepsilon,\rho_l\right)=\prod_{i=1}^{N}\frac{1}{\sqrt{2\pi}\sigma_\varepsilon}\exp\left(-\frac{1}{2}\frac{\left(\gamma\left(\tau_i\right)-\bar{\gamma}\left(\tau_i \mid \rho_l\right)\right)^2}{\sigma_\varepsilon^2}\right) \tag{26}$$

The acceptance probabilities α_P and α_L as given in Eqs. (21)-(21) respectively can then be rewritten as

$$\alpha_p=\min\left\{1,\frac{f'\left(\tilde{\sigma}_\varepsilon,\tilde{\rho}_l\right)}{f'\left(\sigma_{\varepsilon,s},\rho_{l,s}\right)}\right\} \tag{27}$$

$$\alpha_L=\min\left\{1,\frac{\prod_{i=1}^{N}\frac{1}{\sqrt{2\pi}\tilde{\sigma}_\varepsilon}\exp\left(-\frac{1}{2}\frac{\left(\gamma\left(\tau_i\right)-\bar{\gamma}\left(\tau_i \mid \tilde{\rho}_l\right)\right)^2}{\tilde{\sigma}_\varepsilon^2}\right)}{\prod_{i=1}^{N}\frac{1}{\sqrt{2\pi}\sigma_{\varepsilon,s}}\exp\left(-\frac{1}{2}\frac{\left(\gamma\left(\tau_i\right)-\bar{\gamma}\left(\tau_i \mid \rho_{l,s}\right)\right)^2}{\sigma_{\varepsilon,s}^2}\right)}\right\} \tag{28}$$

where $\boldsymbol{f'(\sigma_\varepsilon,\rho_l)}$ is the joint prior distribution of the correlation length and the error term.

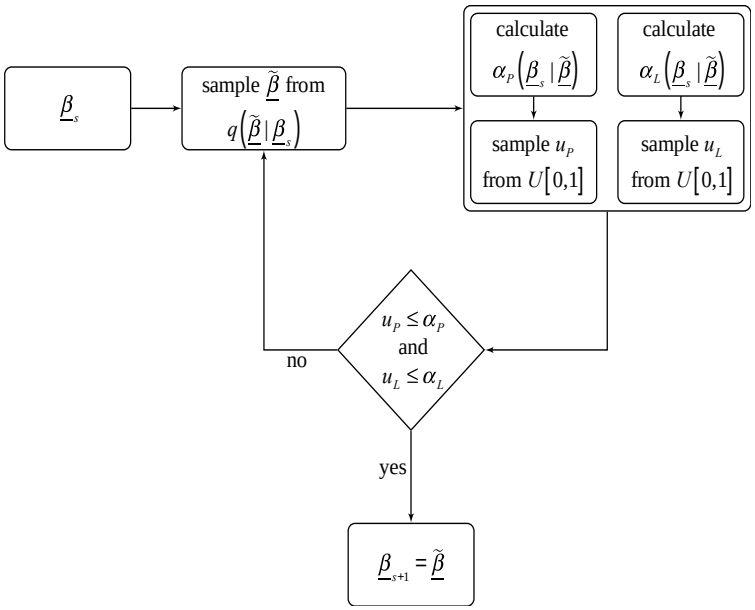


Fig. 5 Metropolis-Hasting algorithm flow diagram for Bayesian updating with $\underline{\beta}_s=\left(\sigma_{\epsilon,s},\beta_{1,s},\ldots,\beta_{R,s}\right)$. Transition step from $\underline{\beta}_s$ to $\underline{\beta}_{s+1}$

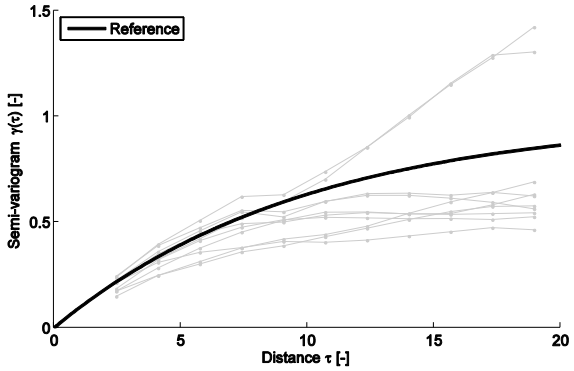
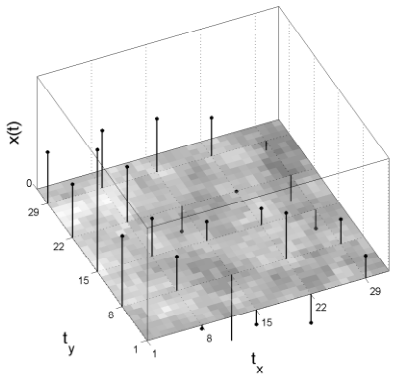
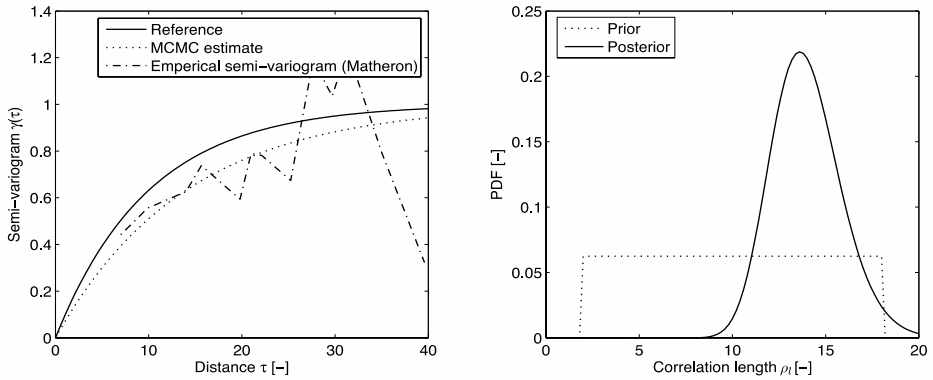


Fig. 6 Estimated semi-variograms for 10 realisations of a random field on a lattice grid (32 by 32 positions) with an exponential covariance function and a correlation length $\rho_l = 10$. In each field 225 measurement points where sampled over the area of the field)



t_x	t_y	$x(t)$	t_x	t_y	$x(t)$
1	1	-0.152	15	22	-0.419
1	8	1.188	15	29	0.901
1	15	2.057	22	1	-0.491
1	22	0.907	22	8	0.784
1	29	0.857	22	15	0.282
8	1	-0.062	22	22	-0.004
8	8	0.573	22	29	0.653
8	15	0.64	29	1	0.376
8	22	0.93	29	8	0.413
8	29	0.966	29	15	-0.32
15	1	-0.254	29	22	-1.748
15	8	-1.263	29	29	-1.181
15	15	0.322			

Fig. 7 Sample locations and values of a standard normal random field on a lattice grid (32 by 32 positions) with an exponential covariance function and a correlation length $\rho_l = 10$



(a) (b)

Fig. 8 (a) Estimated semi-variogram using the MCMC method with a vague prior for ρ_l and σ_ε. (b) Prior and Posterior distribution for the correlation length

6. Example of the bayesian estimation of the correlation length

To demonstrate the algorithm, consider a standard normal random field with an exponential covariance function and a known correlation length $\rho_l = 10$ defined on a lattice grid (32 by 32 positions). Only 25 points of a realisation of the random field are evaluated. These points span the area of the fields and form a regular pattern to increase the amount of points with a same distance apart, see Fig. 7.

These 25 samples are used to calculate the correlation length of this random field. First the empirical semi-variogram$\overline{\gamma}(\tau)$ is calculated using Eq. (23).A tolerance of $e = 1$ was allowed on the distance classes τ_i.With this empirical semi-variogram, the likelihood function can be calculated using Eq. (26).First a vague prior is chosen for the correlation length and the standard deviation of the error term within reasonable intervals, respectively $\rho_l \propto \mathrm{U}[2;18]$ and $\sigma_\varepsilon \propto \mathrm{U}[0;3]$. The joint priordensity function is hence given by

$$f'(\sigma_\varepsilon,\rho_l) = \begin{cases} \frac{1}{3-0}\frac{1}{18-2} & \text{if } 0 \le \sigma_\varepsilon \le 3 \text{ and } 2 \le \rho_l \le 18 \\ 0 & \text{else} \end{cases} \tag{29}$$

The above-mentioned MCMC method is used to sample the posterior distribution for the correlation length ρ_l and the standard deviation of the error term σ_ε. For the transition distribution a normal distributed increment with a standard deviation σ_ς = 0.2is chosen. 10000 iterations were performed and the first 2000 were withdrawn from the samples. Results are shown in Fig. 8.

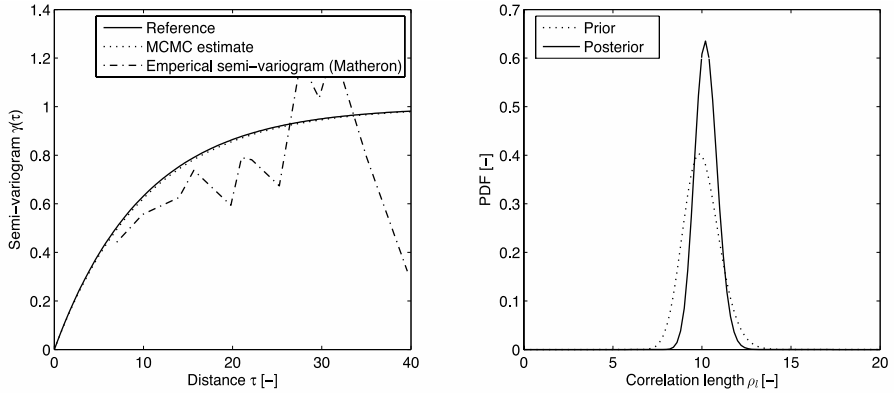
The MCMC method results in ρ_l=14.0 (compared to the reference value ρ_l=10). As seen on Fig. 8(a) this is a good fit to the empirical semi-variogram. In this particular case the estimated variogram lies beneath the real semi-variogram. If more measurement points were considered, the estimation can be shown to converge to the reference value.

The above example is repeated, now using an informative prior for the correlation length. As prior for the correlation length a lognormal distribution is chosen with a mean value of 10 and a standard deviation of 1. For the standard deviation of the error term the same vague prior was used. The joint priordensity function is given by:

$$f'(\sigma_\varepsilon,\rho_l) = \begin{cases} \frac{1}{3-0}\frac{1}{\delta\rho_l\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{\ln(\rho_l)-\xi}{\delta}\right)^2\right) & \text{if } 0 \le \sigma_\varepsilon \le 3 \\ 0 & \text{else} \end{cases} \tag{30}$$

with $\xi = \ln(\mu_{\rho_l}) - 1/2\,\delta^2$, $\delta = \sqrt{\ln\left(1+\left(\sigma_{\rho_l}/\mu_{\rho_l}\right)^2\right)}$ the parameters of the lognormal distribution corresponding with $\mu_{\rho_l} = 10 \text{and} \sigma_{\rho_l} = 1$, i.e. $\xi = 2.2976$ and $\delta = 0.0998$. The same amount of iterations and the same standard deviation σ_ς for the candidate values of the random walk algorithm were used. Some results are shown in Fig. 9.

Compared to the first results the estimated semi-variogram with the MCMC method is already very accurate with only 25 measurement points. Because an informative prior was used for the correlation length, relatively more samples were accepted where the corresponding probability density of the prior distribution was high



(a) (b)

Fig. 9 (a) Estimated semi-variogram using the MCMC method with an informative prior for ρ_l and a vague prior for σ_ε. (b) Prior and Posterior distribution for the correlation length

7. Comparison between the LSQ and the MCMC method

The traditional LSQ method and the currently developed MCMC method are compared. Both methods are applied to find the correlation length ρ_l of a random field on a lattice grid (32 by 32 positions). In case of the MCMC method 25 measurements are considered, while for the LSQ measurements either 25 or 1024 measurements are considered. 100 standard normal distributed random fields with a correlation length $\rho_l = 10$ are generated and both methods are applied on these simulations in order to find the confidence interval on the estimate of ρ_l. Notice that in the case of 1024 measurements all available points were used to compose the empirical semi-variogram and calculate the correlation length with the LSQ method.

The LSQ method solves Eq.(31) for estimating the correlation length ρ_l.

$$\rho_l = \text{argmn}\sum_{i=1}^{N}\left(\gamma(\tau_i)\right) - \bar{\gamma}(\tau_i|\rho_l)^2 \tag{31}$$

where $\gamma(\cdot)$ is the empirical semi-variogram and $\bar{\gamma}(\cdot\,|\rho_l)$ is the semi-variogram model. No prior information about the correlation length is considered.

In case of the MCMC method a lognormal prior distribution for ρ_l is considered with $\mu'_{\rho_l} = 9$ and $\sigma'_{\rho_l} = 2$ (coefficient of variation $\delta'_{\rho_l} = \sigma'_{\rho_l}/\mu'_{\rho_l} = 0.22$). Results are shown in Fig. 10. Hence, the prior distribution used for the correlation length was biased.

Compared to the LSQ method, the MCMC method provides a less uncertain estimation (due to the incorporation of prior information) even when the assumed prior distribution for β was biased. The LSQ estimation improves when more measurement points are available, however a large number of measurement points are necessary for achieving a similar accuracy as provided by the MCMC method. Such large datasets are often impossible to achieve for certain material properties in common structural engineering applications (especially in case of concrete properties assessed by destructive testing), hence indicating the importance of the proposed Bayesian response surface updating method.

The confidence interval of the posterior distribution for the correlation length resulting from the

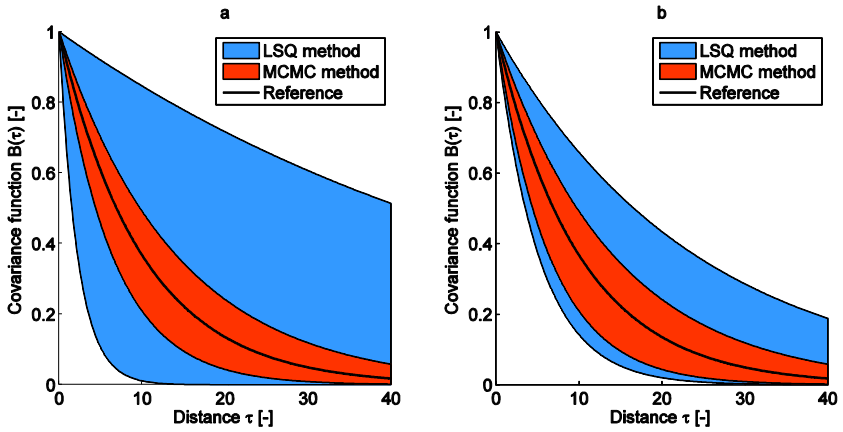


Fig. 10 (a) 90% confidence interval for the LSQ and MCMC method using 25 points. (b) 90% confidence interval for the LSQ method using all 1024 points en the MCMC method using 25 points.

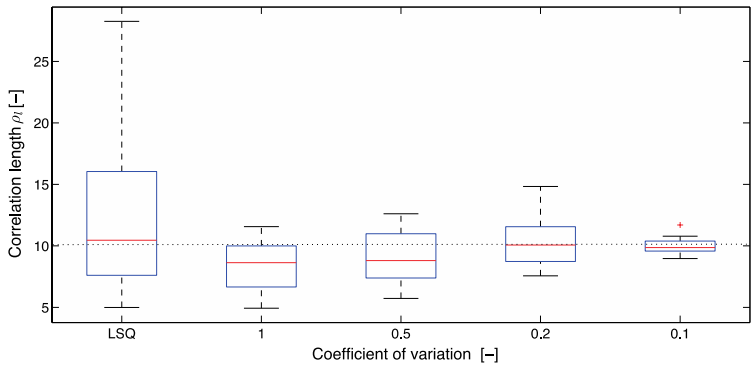


Fig. 11 Box plots of the LSQ and MCMC results. Different coefficients of variation δ'_{ρ_l} are considered in case of the MCMC method. An unbiased prior distribution for the correlation length was used

MCMC method is of course dependent on the uncertainty of the prior distribution. In Fig. 11 box plots are given for different coefficients of variation for the prior distribution. When the coefficient of variation of the prior decreases, the posterior distribution becomes more precise. Furthermore also the results in case of the LSQ method are provided.

In case of rather precise prior information the confidence interval of the posterior distribution for the correlation length will be narrow. The influence of the additional measurement data, used to compose a likelihood functions, is in this case negligible. Moreover, in the case of informative prior the likelihood function will not be able to correct the prior estimation.

Results of the estimated correlation length in case of the MCMC method using biased prior distributions for the correlation length are shown in Fig. 12. The calculations were executed for different coefficients of variation for a lognormal prior distribution with mean μ'_{ρ_l}. As the coefficient of variation increases, the prior distribution becomes less informative and the influence of thebias decreases.

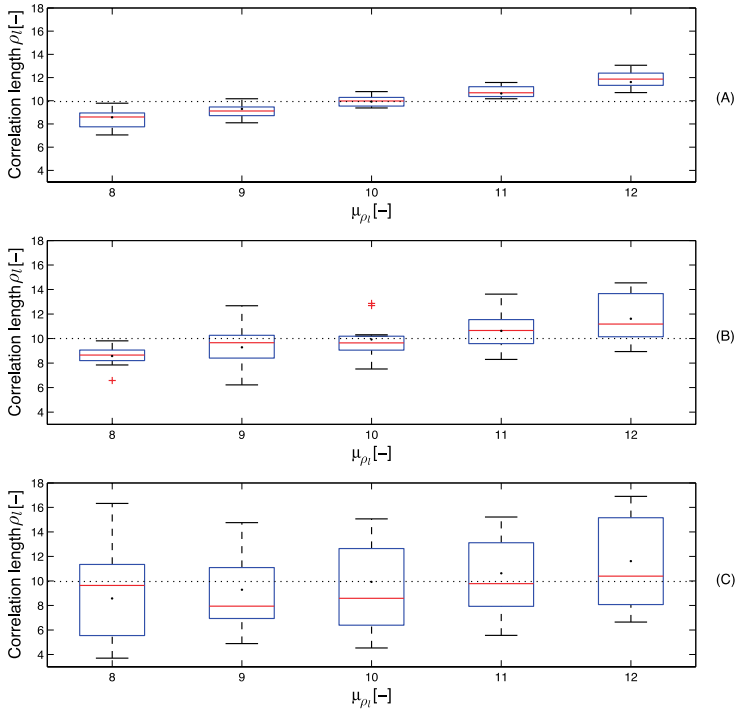


Fig. 12 Box plot results for the correlation length using the MCMC method considering biased prior distributions for different coefficients of variation; (A) $\delta'_{\rho_l} = 0.1$ (B) $\delta'_{\rho_l} = 0.2$ (A) $\delta'_{\rho_l} = 0.5$

8. Conclusions

- A common problem when estimating the correlation length in common concrete engineering problems is a lack of available data. Due to this problem estimated semi-empirical variograms show considerable variation. The LSQ method is sensitive to this variation, which results in a rather uncertain estimate of the covariance function. The LSQ cannot be a reliable method to calculate the correlation length in such cases.Hence,in practice correlation lengths are oftenassumed based on literature.
- Random fields model concrete properties more realistic but the characteristics of these fields are based on empirical assumptions. Moreover, theseassumed values for the correlation length of concrete properties are not consistent with each other. Available measurement data, even if limited, should be considered to update these values. The suggested values for the correlation length could be used as prior information in the Bayesian updating framework.
- A methodology based on Markov Chain Monte Carlo (MCMC) simulations is developed in order to estimate the correlation length, (i.e. covariance function), of random fields from empirical semi-variograms using Bayesian updating of prior (vague of informative) information. This prior information should be based on previous test results or expert judgment, as the uncertainty of the prior information of course influences the reliability of the outcome. As the

coefficient of variation of the prior distribution of the correlation length increases, the prior distribution becomes less decisive and reduces the influence of a possible bias in the prior information.

- The MCMC method enables to obtain a more accurate estimation of the correlation length of random fields using considerably less data compared to the commonly used LSQ, which is of particular importance when random field characteristics have to be obtained from limited data in practical concrete engineering applications.

Reference

Baecher, G.B. and Christian, J.T. (2003), Reliability and statistics in geotechnical engineering: John Wiley & Sons.

Box, G.E.P. and Tiao, G.C. (1992), Bayesian inference in statistical analysis: Wiley.

Caspeele, R. and Taerwe, L. (2013), "Numerical Bayesian updating of prior distributions for concrete strength properties considering conformity control", *Adv. Concrete Construct.*, **1**(1), 85-102.

Cressie, N.A.C. (1993), Statistics for spatial data: J. Wiley.

Duprat, F. (2007), "Reliability of RC beams under chloride-ingress", *Construct. Build. Mater.*, **21**(8), 1605-1616.

Emery, X. and Cornejo, J. (2010), "Truncated Gaussian simulation of discrete-valued, ordinal coregionalized variables", *Comput. Geosci.*, **36**(10), 1325-1338.

Emery, X. and Lantuéjoul, C. (2006), "TBSIM: A computer program for conditional simulation of three-dimensional Gaussian random fields via the turning bands method", *Comput. Geosci.*, **32**(10), 1615-1628.

Engelund, S. (1997), Probabilistic Models and Computational Methods for Chloride Ingress in Concrete. . Aalborg University, Aalborg

Fenton, G. (1994), "Error evaluation of three random field generators", *J. Eng. Mech.*, **120**(12), 2478-2497.

Firouzi, A. and Rahai, A.R. (2011), Prediction of extent and liklihood of corrosion-induced cracking in reinforced concrete bridge decks, *Int. J. Civil Eng.*, **9**(3), 183-192.

Gamerman, D. and Lopes, H.F. (2006), Markov Chain Monte Carlo: Stochastic Simulation for Bayesian Inference, Second Edition: Taylor & Francis.

Gelman, A., Carlin, J.B., Stern, H.S. and Rubin, D.B. (2003), Bayesian Data Analysis: Chapman & Hall/CRC.

Ghosh, J.K., Delampady, M. and Samanta, T. (2010), An Introduction to Bayesian Analysis: Theory and Methods: Springer.

Gregory, P. (2005), Bayesian Logical Data Analysis for the Physical Sciences: A Comparative Approach with Mathematica® Support: Cambridge University Press.

Griffiths, D., Fenton, G. and Manoharan, N. (2002), Bearing Capacity of Rough Rigid Strip Footing on Cohesive Soil: Probabilistic Stud, *J. Geotech. Geoenviron. Eng.*, **128**(9), 743-755.

Griffiths, D.V., Huang, J. and Fenton, G.A. (2011), "Probabilistic infinite slope analysis", *Comput. Geotech.*, **38**(4), 577-584.

Hastings, W.K. (1970), "Monte Carlo sampling methods using Markov chains and their applications", *Biometrika*, **57**(1), 97-109.

Keßler, S., Huber, M., Straub, D., Gehlen, C. and Moormann, C. (2010), Stochastic Evaluation of Active Corroding Areas in Concrete Structures, Paper presented at the 8th International Probabilistic Workshop, Szczecin.

Lee, P.M. (2012), Bayesian Statistics: An Introduction: Wiley.

Li, Y. (2004), Effect of Spatial Variability on Maintenance and Repair Decisions for Concrete Structures. Delft University Press.

Liu, J.S. (2008), Monte Carlo Strategies in Scientific Computing: Springer.

Mantoglou, A. and Wilson, J.L. (1982), **"**The Turning Bands Method for simulation of random fields using

line generation by a spectral method, *Water Resour. Res.*, **18**(5), 1379-1394.
Matheron, G. (1962), Traité de géostatistique appliquée (Vol. 14), Éditions Technip.
Matheron, G. (1973), "The itrinsic rndom fnctions and their aplications", *Adv. Appl. Prob.*, **5**(3), 439-468.
Most, T. and Bucher, C. (2006), Stochastic simulation of cracking in concrete structures using multiparameter random fields, *Int. J. Reliab. Safety*, **1**(1/2), 168-187.
Most, T. and Bucher, C. (2007), Probabilistic analysis of concrete cracking using neural networks and random fields, *Prob. Eng. Mech.*, **22**(2), 219-229.
Nicholas, M., Arianna, W.R., Marshall, N.R., Augusta, H.T. and Edward, T. (1953), "Equation of State Calculations by Fast Computing Machines", *J. Chem. Phys.*, **21**(6), 1087-1092.
Perrin, F.S., Pendola, M. and De Rocquigny, E. (2007), Comparison of Markov chain Monte Carlo simulation and FORM-based approach for Bayesian updating of mechanical models. Paper presented at the 10th int. Conf. on Applications of Statistics and Probaility in Civil Engineering, Tokyo.
Robert, C. and Casella, G. (2010), Monte Carlo Statistical Methods: Springer.
Santoso, A.M., Phoon, K.K. and Quek, S.T. (2011), Modified Metropolis-Hastings algorithm with reduced chain correlation for efficient subset simulation, *Prob.Eng. Mech.*, 26(2), 331-341.
Stewart, M.G. and Mullard, J.A. (2007), "Spatial time-dependent reliability analysis of corrosion damage and the timing of first repair for RC structures", *Eng. Struct.*, **29**(7), 1457-1464.
Straub, D. (2011), Reliability updating with inspection and monitoring data in deteriorating reinforced concrete slabs. Paper presented at the Application of Statistics and Probability in Civil Engineering, London.
Straub, D. and Fischer, J. (2011), Reliability Assessment of Corroding Reinforced Concrete Slabs with Inspection Data. Paper presented at the 9th International Probabilistic Workshop, TU Braunschweig, germany.
Trefethen, L.N. and Bau, D. (1997), Numerical Linear Algebra: SIAM.
Van Der Vurst, F., Caspeele, R., Desnerck, P., Schutter, G.D. and Peirs, J. (2014), Modification of existing shape factor models for self- compacting concrete strength by means of Bayesian updating techniques, Materials and Structures.
Vanmarcke, E. (2010), Random Fields: Analysis and Synthesis: World Scientific.
Vasconcellos Real, M.D., Filho, A.C. and Maestrini, S.R. (2003), "Response variability in reinforced concrete structures with uncertain geometrical and material properties", *Nuclear Eng. Des.*, **226**(3), 205-220.
Vu, K.A.T. (2003), Corrosion-induced cracking and spatial timedependent reliability analysis of reinforced concrete structures, The University of Newcastle, New South Wales, Australia.

CC