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Analytical solution for nonlocal buckling characteristics of higher-order inhomogeneous nanosize beams embedded in elastic medium

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Abstract. In this paper, buckling characteristics of nonhomogeneous functionally graded (FG) nanobeams embedded on elastic foundations are investigated based on third order shear deformation (Reddy) without using shear correction factors. Third-order shear deformation beam theory accounts for shear deformation effects by a parabolic variation of all displacements through the thickness, and verifies the stress-free boundary conditions on the top and bottom surfaces of the FG nanobeam. A two parameters elastic foundation including the linear Winkler springs along with the Pasternak shear layer is in contact with beam in deformation, which acts in tension as well as in compression. The material properties of FG nanobeam are supposed to vary gradually along the thickness and are estimated through the power-law and Mori–Tanaka models. The small scale effect is taken into consideration based on nonlocal elasticity theory of Eringen. Nonlocal equations of motion are derived through Hamilton's principle and they are solved applying analytical solution. Comparison between results of the present work and those available in literature shows the accuracy of this method. The obtained results are presented for the buckling analysis of the FG nanobeams such as the effects of foundation parameters, gradient index, nonlocal parameter and slenderness ratio in detail.

Keywords: buckling; third-order shear deformation beam theory; embedded functionally graded nanobeam; nonlocal elasticity theory

1. Introduction

Developments in materials engineering led to microscopically inhomogeneous spatial composite materials named Functionally graded materials (FGMs) which provide huge potential applications for various systems and devices, such as aerospace, aircraft, automobile and defense structures and most recently the electronic devices. According to the fact that FG materials have been placed in the category of composite materials, the volume fractions of two or more material constituents such as a pair of ceramic–metal are supposed to change smoothly and continuously throughout the gradient directions. The FGM materials are made to take advantage of desirable features of its constituent phases, for example, in a thermal protection system, the ceramic

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constituents are capable to withstand extreme temperature environments due to their better thermal resistance characteristics, while the metal constituents provide stronger mechanical performance and diminishes the possibility of catastrophic fracture (Kettaf *et al*. 2013, Kocaturk and Akbas 2013). Hence, presenting novel mechanical properties, FGMs have gained its applicability in several engineering fields, such as biomedical engineering, nuclear engineering and mechanical engineering (Bouremana *et al*. 2013).

In addition, fast growing progress in the application of structural elements such as beams and plates with micro or nanolength scale in micro/nano electro-mechanical systems (MEMS/ NEMS), due to their outstanding chemical, mechanical, and electrical properties, led to a provocation in modelling of micro/nano scale structures. In such applications, it is observed that the size effect has a major role on dynamic behavior of material. After the invention of carbon nanotubes (CNTs) by Iijima (1991), nanoscale engineering materials have exposed to considerable attention in modern science and technology. These structures possess extraordinary mechanical, thermal, electrical and chemical performances that are superior to the conventional structural materials. Therefore nanostructures attract great interest by researchers based on molecular dynamics and continuum mechanics. The problem in using the classical theory is that the classical continuum mechanics theory does not take into account the size effects in micro/nano scale structures. The classical continuum mechanics over predicts the responses of micro/nano structures. Another way to capture the size effects is to rely on molecular dynamic simulations (MD) which is considered as a powerful and accurate implement to study of structural components at nanoscale. But even the molecular dynamic simulation at nano scale is computationally exorbitant for modeling the nanostructures with large numbers of atoms. So a conventional form of continuum mechanics that can capture the small scale effect is required. Eringen's nonlocal elasticity theory is the most commonly used continuum mechanics theory that includes small scale effects with good accuracy to model micro/nano scale devices and systems. The nonlocal elasticity theory assumes that the stress state at a reference point is a function of the strain at all neighbor points of the body. Hence, this theory could take into consideration the effects of small scales. For proper design of nanostructures, it is very important to take all essential characteristics of their mechanical behaviors at this submicron size. To achieve this goal, based on the nonlocal constitutive relation of Eringen, a number of studies have been conducted attempting to develop nonlocal beam models for predicting the mechanical responses of nanobeams. The potential of application of nonlocal Euler–Bernoulli beam theory to materials in micro and nano scale proposed by Peddieson *et al* (2003) as the first researchers to propose nonlocal elasticity theory to nano structures. Then, the nonlocal elasticity theory gained considerable attention among the nanotechnology society and utilization of this theory extended in various mechanical analyses. Reddy (2007) formulated various available beam theories, including the Euler–Bernoulli, Timoshenko, Reddy, and Levinson beam theories through nonlocal differential relations of Eringen. In other scientific work, Wang and Liew (2007) carried out the static analysis of micro and nano scale structures based on nonlocal continuum mechanics using Euler–Bernoulli and Timoshenko beam theory. Aydogdu (2009) presented a general nonlocal beam model for analysis bending, buckling, and vibration of nanobeams using different beam theories. Pradhan and Murmu (2010) investigated the flapwise bending–vibration of rotating nanocantilevers by using Differential quadrature method (DQM). They noticed that size effects have a main role in the vibration behavior of rotating nanostructures. Civalek *et al*. (2010) proposed formulation of the governing equations of nonlocal Euler–Bernoulli beams to investigate bending of cantilever microtubules via the differential quadrature method. Thai (2012) suggested a nonlocal higher order beam theory to study mechanical responses of nanobeams. Simsek (2014) proposed a non-classical beam model based on the Eringen's nonlocal elasticity theory for nonlinear vibration of nanobeams with various boundary conditions. Size-dependent nonlinear forced vibration analysis of magneto-electro-thermo-elastic Timoshenko nanobeams based upon the nonlocal elasticity theory is studied by Ansari *et al*. (2015).

Since FG nanostructures are extensively used in MEMS and NEMS due to the rapid developments in nanotechnology. To applying accurately these kinds of novel materials in micro/nano electromechanical systems (MEMS/NEMS), their dynamic behaviors should be examined. Recently, Eltaher *et al*. (2012, 2013a) presented a finite element analysis for free vibration of FG nanobeams using nonlocal EBT. They also exploited the static and stability responses of FG nanobeams based on nonlocal continuum theory (Eltaher *et al*. 2013b). Thickness stretching effect on static and stability behavior of functionally graded material (FGM) nanoscale beams is studied by Mahmoud *et al*. (2015). Also, Ebrahimi and Salari (2015a, b, c) presented a semi-analytical method for vibrational and buckling analysis of FG nanobeams considering the physical neutral axis position. Niknam and Aghdam (2015) presented a closed form solution for both natural frequency and buckling load of nonlocal FG beams resting on nonlinear elastic foundation. Ebrahimi and Barati (2016 a-g) presented static and dynamic modeling of a thermo–piezo-electrically actuated nanosize beam subjected to a magnetic field. Also, Ebrahimi and Barati (2016h) investigated small scale effects on hygro-thermo-mechanical vibration of temperature dependent nonhomogeneous nanoscale beams. Ebrahimi *et al*. (2016) presented a nonlocal strain gradient theory for wave propagation analysis in temperature-dependent FG nanoplates.

Therefore, a survey in literature reveals that buckling analysis of FG nanobeams, especially for those on elastic foundations are very limited. Various kinds of elastic foundation models for the sake of describing the interactions of the beam and foundation have proposed via scientists (Tebboune *et al*. 2015). Winkler or one-parameter elastic foundation is known as the simplest model which regards the foundation as a series of separated linear elastic springs without coupling effects between each other. The defect of Winkler's formulation is the behavioral inconsistency associated to the discontinuous deflections on the interacted surface area of the beam (Khoshnevisrad *et al*. 2014). Pasternak (1954) later introduced an incompressible vertical element as a shear layer which is physically realistic representation of the elastic medium and can take into account the transverse shear stresses due to interaction of shear deformation of the surrounding elastic medium. Thus, a more realistic and generalized representation of the elastic foundation is expected through a two-parameter foundation model.

Also, it is understood that most of the previous studies on mechanical analysis of FG nanobeams have been carried out based on Euler-Bernoulli or classical beam theory and Timoshenko beam theoy. It should be noted that the classical theories fail to consider the influences of shear deformation and thickness stretching (Neves *et al*. 2012). Classical theory is only applicable for slender beams and should not be applied for thick beams, and also it suppose that the transverse perpendicular to the neutral surface stays normal during and after bending, which indicates that the transversal shear strain is equal to zero Hence, the buckling loads and natural frequencies of thick beams are overestimated in which shear deformation effects are prominent. Timoshenko Beam Theory can enumerate the influences of shear deformations for thick beams with presumption of a constant shear strain state in the direction of beam thickness. So, as a disadvantage of this theory, a shear correction factor is required to properly demonstration of the deformation strain energy. To prevent using the shear correction factors, many higher-order shear deformation theories have been developed such as the third-order shear deformation theory proposed by Reddy (2007), the generalized beam theory proposed by Aydogdu (2009) and sinusoidal shear deformation theory of Touratier (1991). Reddy's third order beam theory (RBT) can be used with supposing the higher order longitudinal displacement variations of beam along the thickness. By verifying zero transverse shear stresses at the upper and lower surfaces of the beam, this theory captures both the microstructural and shear deformation effects. Therefore, The Reddy beam theory is more exact and provides better representation of the physics of the problem, which does not need any shear correction factors. This theory relaxes the limitation on the warping of the cross sections and allows cubic variations in the longitudinal direction of the beam, so it can produce adequate accuracy when applying for beam analysis.

Therefore, a few studies have been performed to investigate the mechanical responses of FG micro/nano beams by using higher shear deformation beam theories. Rahmani and Jandaghian (2015) presented Buckling analysis of functionally graded nanobeams based on a nonlocal third-order shear deformation theory. Sahmani *et al*. (2014) investigated the free vibration response of third-order shear deformable nanobeams made of functionally graded materials (FGMs) around the postbuckling domain incorporating the effects of surface free energy Based on the modified couple stress theory (MCST), a unified higher order beam theory which contains various beam theories as special cases for buckling of a functionally graded (FG) microbeam embedded in elastic Pasternak medium is proposed by Simsek and Reddy (2013). Zhang *et al.* (2014) developed a size-dependent FG beam model resting on Winkler-Pasternak elastic foundation based on an improved third-order shear deformation theory and provided the analytical solutions for the bending, buckling and free vibration problems. Ebrahimi and Barati (2016i) proposed a nonlocal higher-order shear deformation beam theory for vibration analysis of size-dependent functionally graded nanobeams. By searching the literature, it is found that a work analyzing the buckling of embedded Mori-Tanaka based FG nanobeams using the third order shear deformation beam theory hasn’t been yet published.

In the present study the non-classical beam model within the framework of third order shear deformation beam theory is developed for analysis of buckling of FG nanobeam embedded on elastic foundations. Material properties of FG nanobeam are assumed to change continuously along the thickness according to two kinds of micromechanics models, namely, power-law model and Mori–Tanaka model. By using the Hamilton’s principle the governing equations of motion are derived and Navier type solution method is used to solve the equations. The obtained results based on third order shear deformation beam theory are compared with those predicted by the previously published works to verify the accuracy of the present solution. Numerical results are presented to show the effects of the gradient index, nonlocality and foundation parameters on the buckling behavior of FG nanobeams.

2. Theory and formulation

2.1 Power-law (PL) and Mori-Tanaka (MT) FGM beam models

One of the most favorable models for FGMs is the power-law model, in which material properties of FGMs are supposed to change according to a power law about spatial coordinates. The coordinate system for FG nano beam is shown in Fig. 1. The FG nanobeam is assumed to be combination of ceramic and metal and effective material properties (P_f) of the FG beam such as Young’s modulus E_f is supposed to change continuously in the direction of z-axis (thickness direction) according to an power function of the volume fractions of the material constituents. So,

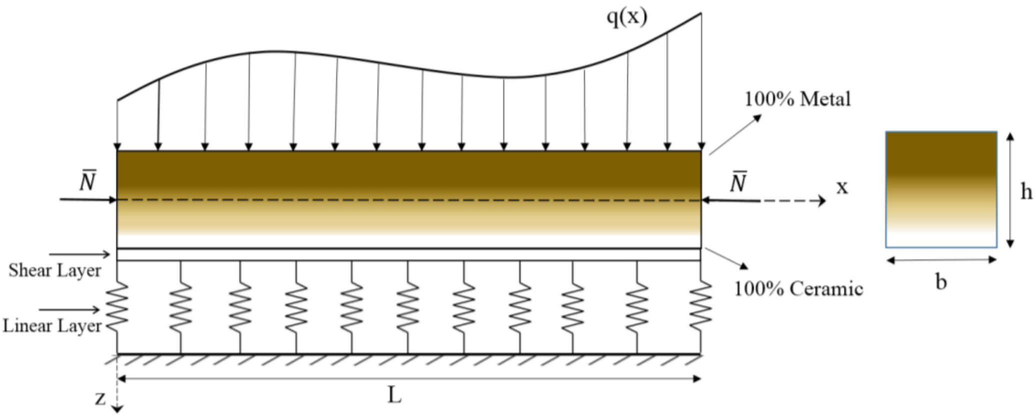


Fig. 1 Geometry and coordinates of functionally graded nanobeam embedded on elastic foundation

the effective material properties, P_f can be stated as

$$P_f = P_c V_c + P_m V_m \tag{1}$$

where subscripts m and c denote metal and ceramic, respectively and the volume fraction of the ceramic is associated to that of the metal in the following relation

$$V_c + V_m = 1 \tag{2a}$$

The volume fraction of the ceramic constituent of the beam is assumed to be given by

$$V_c = \left(\frac{z}{h} + \frac{1}{2}\right)^P \tag{2b}$$

Here p is the power-law exponent which determines the material distribution through the thickness of the beam. Therefore, from Eqs. (1)-(2), the effective material properties of the FG nanobeam can be expressed as follows

$$E(z) = \left(E_c - E_m\right)\left(\frac{z}{h} + \frac{1}{2}\right)^p + E_m \tag{3a}$$

$$\nu(z) = \left(\nu_c - \nu_m\right)\left(\frac{z}{h} + \frac{1}{2}\right)^p + \nu_m \tag{3b}$$

Additionally, in this study, Mori-Tanaka homogenization technique is also employed to model the effective material properties of the FG nanobeam. According to Mori-Tanaka homogenization technique the local effective material properties of the FG nanobeam such as effective local bulk modulus K_e and shear modulus μ_e can be calculated (Simsek and Reddy 2013)

$$\frac{K_e - K_m}{K_c - K_m} = \frac{V_c}{1 + V_m (K_c - K_m)/(K_m + 4\mu_m/3)} \tag{4a}$$

$$\frac{\mu_e - \mu_m}{\mu_c - \mu_m} = \frac{V_c}{1 + V_m(\mu_c - \mu_m)/[(\mu_m + \mu_m(9K_m + 8\mu_m)/(6(K_m + 2\mu_m))]} \tag{4b}$$

Therefore from Eq. (4), the effective Young's modulus (E), Poisson's ratio (v) based on Mori-Tanaka scheme can be expressed by

$$E(z) = \frac{9K_e\mu_e}{3K_e + \mu_e} \tag{5a}$$

$$\nu(z) = \frac{3K_e - 2\mu_e}{6K_e + 2\mu_e} \tag{5b}$$

The shear modulus $G(z)$ of FG nanobeam with respect to both classical rule of mixture and Mori–Tanaka homogenization is defined as

$$G(z) = \frac{E(z)}{2(1+\nu(z))} \tag{6}$$

The material composition of FG nanobeam at the upper surface (z=+h/2) is supposed to be the pure ceramic and it changes continuously to the opposite side surface (z=-h/2) which is pure metal.

2.2 Kinematic relations

Based on the third order shear deformation (Reddy) beam theory, the displacement field at any point of the beam can be written as

$$u_x(x,z) = u(x) + z\varphi(x) - \alpha z^3(\varphi + \frac{\partial w}{\partial x}) \tag{7}$$

$$u_z(x,z) = w(x) \tag{8}$$

where $\alpha = \frac{4}{3h^2}$ and u and w are the longitudinal and the transverse displacements, φ is the rotation of the cross section at each point of the neutral axis. Nonzero strains of the Reddy beam model are expressed as follows

$$\varepsilon_{xx} = \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)} + z^3\varepsilon_{xx}^{(3)} \tag{9}$$

$$\gamma_{xz} = \gamma_{xz}^{(0)} + z^2\gamma_{xz}^{(2)} \tag{10}$$

where

$$\varepsilon_{xx}^{(0)} = \frac{\partial u}{\partial x},\ \varepsilon_{xx}^{(1)} = \frac{\partial \varphi}{\partial x},\ \varepsilon_{xx}^{(3)} = -\alpha(\frac{\partial \varphi}{\partial x} + \frac{\partial^2 w}{\partial x^2}) \tag{11}$$

$$\gamma_{xz}^{(0)} = \frac{\partial w}{\partial x} + \varphi,\ \gamma_{xz}^{(2)} = -\beta(\frac{\partial w}{\partial x} + \varphi) \tag{12}$$

And $\beta = \dfrac{4}{h^2}$.

By using the Hamilton's principle, in which the motion of an elastic structure in the time interval $t1<t<t2$ is so that the integral with respect to time of the total potential energy is extremum

$$\int_0^t \delta(U+V)\,dt = 0 \tag{13}$$

Here U is strain energy, and V is work done by external forces. The virtual strain energy can be calculated as

$$\delta U = \int_v \sigma_{ij}\,\delta\,\varepsilon_{ij}\,dV = \int_v (\sigma_{xx}\,\delta\,\varepsilon_{xx} + \sigma_{xz}\,\delta\,\gamma_{xz})\,dV \tag{14}$$

Substituting Eqs. (7)-(10) into Eq. (13) yields

$$\delta U = \int_0^L (N\delta\varepsilon_{\mathrm{xx}}{}^{(0)} + M\delta\varepsilon_{\mathrm{xx}}{}^{(1)} + P\delta\varepsilon_{\mathrm{xx}}{}^{(3)} + Q\delta\gamma_{xz}{}^{(0)} + R\delta\gamma_{xz}{}^{(2)})\,dx \tag{15}$$

In which the variables introduced in arriving at the last expression are defined as follows

$$\begin{aligned} &N = \int_A \sigma_{xx}\,dA,\ M = \int_A \sigma_{xx} z\,dA,\ P = \int_A \sigma_{xx} z^3 dA \\ &Q = \int_A \sigma_{xz}\,dA,\ R = \int_A \sigma_{xz} z^2\,dA \end{aligned} \tag{16}$$

The first variation of the work done by applied forces can be written in the form

$$\begin{aligned} \delta V = \int_0^L (&\overline{N}\frac{\partial w}{\partial x}\frac{\partial\delta\,w}{\partial x} + q\,\delta\,w + f\,\delta\,u - k_w\delta\,w + k_p\,\frac{\partial^2\delta w}{\partial x^2} - N\delta\varepsilon_{\mathrm{xx}}{}^{(0)} \\ &- M\,\frac{\partial\delta\varphi}{\partial x} + \alpha P\frac{\partial^2\delta w}{\partial x^2} - Q\delta\gamma_{xz}{}^{(0)})dx \end{aligned} \tag{17}$$

Where $M=M-\alpha P$, $Q=Q-\beta R$ and $\overline{N}$ is the applied axial compressive load and $q(x)$ and $f(x)$ are the transverse and axial distributed loads and k_w and k_p are linear and shear coefficient of elastic foundation.

By Substituting Eqs. (15) and (17) into Eq. (13) and setting the coefficients of δu, δw and $\delta\varphi$ to zero, the following Euler–Lagrange equation can be obtained

$$\frac{\partial N}{\partial x} + f = 0 \tag{18a}$$

$$\frac{\partial\hat{M}}{\partial x} - \hat{Q} = 0 \tag{18b}$$

$$\frac{\partial\hat{Q}}{\partial x} + q - \overline{N}\,\frac{\partial^2 w}{\partial x^2} + \alpha\,\frac{\partial^2 P}{\partial x^2} - k_w\,w + k_p\,\frac{\partial^2 w}{\partial x^2} = 0 \tag{18c}$$

2.3 The nonlocal elasticity model for FG nanobeam

According to Eringen nonlocal elasticity model (Eringen and Edelen 1972), the stress state at a point inside a body is regarded to be function of strains of all points in the neighbor regions. For homogeneous elastic solids the nonlocal stress-tensor components σ_{ij} at each point x in the solid can be defined as

$$\sigma_{ij}(x) = \int_{\Omega} \alpha(|x' - x|, \tau)\, t_{ij}(x')\, d\Omega(x') \tag{19}$$

where $t_{ij}(x')$ are the components available in local stress tensor at point x which are associated to the strain tensor components ε_{kl} as

$$t_{ij} = C_{ijkl}\varepsilon_{kl} \tag{20}$$

The concept of Eq. (19) is that the nonlocal stress at any point is weighting average of local stress of all points in the near region that point, the size that is related to the nonlocal kernel $\alpha(|x'-x|,\tau)$. Also $|x'-x|$ is Euclidean distance and τ is a constant as follows

$$\tau = \frac{e_0 a}{l} \tag{21}$$

which indicates the relation of a characteristic internal length, (for instance lattice parameter, *C–C* bond length and granular distance) and a characteristic external length, l (for instance crack length and wavelength) using a constant, e_0, dependent on each material. The value of e_0 is experimentally estimated by comparing the scattering curves of plane waves and atomistic dynamics. According to (Eringen and Edelen 1972) for a class of physically admissible kernel $\alpha(|x'-x|,\tau)$ it is possible to represent the integral constitutive relations given by Eq. (19) in an equivalent differential form as

$$(1-(e_0a)\nabla^2)\sigma_{kl} = t_{kl} \tag{22}$$

where ∇^2 is the Laplacian operator. Thus, the scale length e_0a considers the influences of small scale on the response of nano-structures. The magnitude of the small scale parameter relies on several parameters including mode shapes, boundary conditions, chirality and the essence of motion. The parameter $e_0=(\pi^2-4)^{1/2}/2\pi \cong 0.39$ was given by Eringen (1983). Also, Zhang *et al*. (2005) found the value of 0.82 nm for nonlocal parameter when they compared the vibrational results of simply supported single-walled carbon nanotubes with molecular dynamics simulations. The nonlocal parameter, μ, is experimentally obtained for various materials; for instance, a conservative estimate of μ<4 (nm)^2 for a single-walled carbon nanotube is proposed (Wang 2005). It is worth mentioning that this magnitude is dependent of size and chirality, because the properties of carbon nanotubes are extensively confirmed to be dependent of chirality. There is no serious study conducted to determining the value of small scale to simulate mechanical behavior of FG micro/nanobeams (Eltaher *et al*. 2012). Hence all researchers who worked on size-dependent mechanical behavior of functionally graded nanobeams on the basis the nonlocal elasticity method investigated the influence of small scale parameter on mechanical behavior of FG nanobeams by changing the value of the small scale parameter. So, for a material in the one-dimension case, the constitutive relations of nonlocal theory can be expressed as

$$\sigma_{xx} - (e_0a)^2 \frac{\partial^2\sigma_{xx}}{\partial x^2} = E\,\varepsilon_{xx} \tag{23}$$

$$\sigma_{xz} - (e_0 a)^2 \frac{\partial^2 \sigma_{xz}}{\partial x^2} = G\gamma_{xz} \tag{24}$$

where σ and ε are the nonlocal stress and strain, respectively. E is the Young's modulus, $G(z)=E(z)/2(1+v(z))$ is the shear modulus (where v is the poisson's ratio). For a nonlocal FG beam, Eqs. (23) and (24) can be written as

$$\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E(z)\varepsilon_{xx} \tag{25}$$

$$\sigma_{xz} - \mu \frac{\partial^2 \sigma_{xz}}{\partial x^2} = G(z)\gamma_{xz} \tag{26}$$

where $\mu=(e_0 a)^2$. Integrating Eqs. (25) and (26) over the beam's cross-section area, we obtain the force-strain and the moment-strain of the nonlocal Reddy FG beam theory can be obtained as follows

$$N - \mu \frac{\partial^2 N}{\partial x^2} = A_{xx} \frac{\partial u}{\partial x} + (B_{xx} - \alpha E_{xx}) \frac{\partial \varphi}{\partial x} - \alpha E_{xx} \frac{\partial^2 w}{\partial x^2} \tag{27}$$

$$M - \mu \frac{\partial^2 M}{\partial x^2} = B_{xx} \frac{\partial u}{\partial x} + (D_{xx} - \alpha F_{xx}) \frac{\partial \varphi}{\partial x} - \alpha F_{xx} \frac{\partial^2 w}{\partial x^2} \tag{28}$$

$$P - \mu \frac{\partial^2 P}{\partial x^2} = E_{xx} \frac{\partial u}{\partial x} + (F_{xx} - \alpha H_{xx}) \frac{\partial \varphi}{\partial x} - \alpha H_{xx} \frac{\partial^2 w}{\partial x^2} \tag{29}$$

$$Q - \mu \frac{\partial^2 Q}{\partial x^2} = (A_{xz} - \beta D_{xz})(\frac{\partial w}{\partial x} + \varphi) \tag{30}$$

$$R - \mu \frac{\partial^2 R}{\partial x^2} = (D_{xz} - \beta F_{xz})(\frac{\partial w}{\partial x} + \varphi) \tag{31}$$

In which the cross-sectional rigidities are defined as follows

$$(A_{xx}, B_{xx}, D_{xx}, E_{xx}, F_{xx}, H_{xx}) = \int_A E(z)(1, z, z^2, z^3, z^4, z^6)\, dA \tag{32}$$

$$(A_{xz}, D_{xz}, F_{xz}) = \int_A G(z)(1, z^2, z^4) dA \tag{33}$$

The explicit relation of the nonlocal normal force can be derived by substituting for the second derivative of N from Eq. (18a) into Eq. (27) as follows

$$N = A_{xx} \frac{\partial u}{\partial x} + K_{xx} \frac{\partial \varphi}{\partial x} - \alpha E_{xx} \frac{\partial^2 w}{\partial x^2} - \mu \frac{\partial f}{\partial x} \tag{34}$$

Eliminating $\hat{Q}$ from Eqs. (18b) and (18c), we obtain the following equation

$$\frac{\partial^2 \hat{M}}{\partial x^2} = -\alpha \frac{\partial^2 P}{\partial x^2} - q + \overline{N} \frac{\partial^2 w}{\partial x^2} + k_w w - k_p \frac{\partial^2 w}{\partial x^2} \tag{35}$$

Also the explicit relation of the nonlocal bending moment can be derived by substituting for the second derivative of *M* from Eq. (18b) into Eq. (28) and using Eqs. (28) and (29) as follows

$$\hat{M} = K_{xx}\frac{\partial u}{\partial x} + I_{xx}\frac{\partial \varphi}{\partial x} - \alpha J_{xx}(\frac{\partial \varphi}{\partial x} + \frac{\partial^2 w}{\partial x^2}) + \mu(-\alpha\frac{\partial^2 P}{\partial x^2} - q + \overline{N}\frac{\partial^2 w}{\partial x^2} + k_w w - k_p\frac{\partial^2 w}{\partial x^2}) \tag{36}$$

where

$$K_{xx} = B_{xx} - \alpha E_{xx},\ I_{xx} = D_{xx} - \alpha F_{xx},\ J_{xx} = F_{xx} - \alpha H_{xx} \tag{37}$$

By substituting for the second derivative of *Q* from Eq. (18c) into Eq. (30), and using Eqs. (30) and (31) the following expression for the nonlocal shear force is derived

$$Q = \overline{A}_{xz}(\frac{\partial w}{\partial x} + \varphi) + \mu(\overline{N}\frac{\partial^3 w}{\partial x^3} - \alpha\frac{\partial^3 P}{\partial x^3} - \frac{\partial q}{\partial x} + k_w\frac{\partial w}{\partial x} - k_p\frac{\partial^3 w}{\partial x^3}) \tag{38}$$

where

$$\overline{A}_{xz} = A^*_{xz} - \beta I^*_{xz},\ A^*_{xz} = A_{xz} - \beta D_{xz},\ I^*_{xz} = D_{xz} - \beta F_{xz} \tag{39}$$

Now we use *M* and *Q* from Eqs. (36) and (38) and the identity

$$\alpha\frac{\partial^2}{\partial x^2}(P - \mu\frac{\partial^2 P}{\partial x^2}) = \alpha(E_{xx}\frac{\partial^3 u}{\partial x^3} + F_{xx}\frac{\partial^3 \varphi}{\partial x^3} - \alpha H_{xx}(\frac{\partial^3 \varphi}{\partial x^3} + \frac{\partial^4 w}{\partial x^4})) \tag{40}$$

The nonlocal governing equations of third order shear deformation FG nanobeam in terms of the displacement can be derived by substituting for *N*, *M* and *Q* from Eqs. (34), (36) and (38), respectively, into Eq. (18) as follows

$$A_{xx}\frac{\partial^2 u}{\partial x^2} + K_{xx}\frac{\partial^2 \varphi}{\partial x^2} - \alpha E_{xx}\frac{\partial^3 w}{\partial x^3} - \mu\frac{\partial^2 f}{\partial x^2} + f = 0 \tag{41}$$

$$K_{xx}\frac{\partial^2 u}{\partial x^2} + I_{xx}\frac{\partial^2 \varphi}{\partial x^2} - \alpha J_{xx}(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^3 w}{\partial x^3}) - \overline{A}_{xz}\left(\varphi + \frac{\partial w}{\partial x}\right) = 0 \tag{42}$$

$$\begin{aligned} &\overline{A}_{xz}\left(\frac{\partial \varphi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + \mu(\overline{N}\frac{\partial^4 w}{\partial x^4} - \frac{\partial^2 q}{\partial x^2} + k_w\frac{\partial^2 w}{\partial x^2} - k_p\frac{\partial^4 w}{\partial x^4}) + q - \overline{N}\frac{\partial^2 w}{\partial x^2} \\ &-k_w w + k_p\frac{\partial^2 w}{\partial x^2} + \alpha(E_{xx}\frac{\partial^3 u}{\partial x^3} + J_{xx}\frac{\partial^3 \varphi}{\partial x^3} - \alpha H_{xx}\frac{\partial^4 w}{\partial x^4}) = 0 \end{aligned} \tag{43}$$

3. Solution procedure

Here, on the basis the Navier method, an analytical solution of the governing equations for buckling of a simply supported FG nanobeam is presented. To satisfy governing equations of motion and the simply supported boundary condition, the displacement variables are adopted to be of the form

$$u(x)=\sum_{n=1}^{\infty}U_n \cos\left(\frac{n\pi}{L}x\right) \tag{44}$$

$$w(x)=\sum_{n=1}^{\infty}W_n \sin\left(\frac{n\pi}{L}x\right) \tag{45}$$

$$\varphi(x)=\sum_{n=1}^{\infty}\varphi_n \cos\left(\frac{n\pi}{L}x\right) \tag{46}$$

where (U_n, W_n, φ_n) are the unknown Fourier coefficients to be determined for each n value. Boundary conditions for simply supported beam are as Eq. (46)

$$u(0)=0\ ,\ \frac{\partial u}{\partial x}(L)=0$$

$$w(0)=w(L)=0\ ,\ \frac{\partial \varphi}{\partial x}(0)=\frac{\partial \varphi}{\partial x}(L)=0 \tag{47}$$

Substituting Eqs. (44) - (46) into Eqs. (41)-(43) respectively, leads to Eqs.(48)-(50)

$$(-A_{\mathrm{xx}}(\frac{n\pi}{l})^2)U_n+(-K_{\mathrm{xx}}(\frac{n\pi}{l})^2)\phi_n+(\alpha E_{\mathrm{xx}}(\frac{n\pi}{l})^3)W_n=0 \tag{48}$$

$$\begin{aligned}&(-K_{xx}(\frac{n\pi}{l})^2)U_n+(-I_{xx}(\frac{n\pi}{l})^2+\alpha J_{xx}(\frac{n\pi}{l})^2-\bar{A}_{xz})\phi_n+(\alpha J_{xx}(\frac{n\pi}{l})^3\\&-\bar{A}_{xz}(\frac{n\pi}{l}))W_n=0\end{aligned} \tag{49}$$

$$\begin{aligned}&(\alpha E_{\mathrm{xx}}(\frac{n\pi}{l})^3)U_n+(-\bar{A}_{xz}(\frac{n\pi}{l})+J_{\mathrm{xx}}(\frac{n\pi}{l})^3)\phi_n+[-\bar{A}_{xz}(\frac{n\pi}{l})^2\\&+\bar{N}(\frac{n\pi}{l})^2(1+\mu(\frac{n\pi}{l})^2)-k_w(1+\mu(\frac{n\pi}{l})^2)-k_p(\frac{n\pi}{l})^2(1+\mu(\frac{n\pi}{l})^2)\\&-\alpha^2(\frac{n\pi}{l})^4]W_n=0\end{aligned} \tag{50}$$

By setting the determinant of the coefficient matrix of the above equations, the analytical solutions can be obtained from the following equations

$$[K]\begin{bmatrix}U_n\\W_n\\\phi_n\end{bmatrix}=0 \tag{51}$$

where [K] is stiffness matrix. By setting this polynomial to zero, we can find buckling loads.

Table 1 Material properties of FGM constituents.

Properties	Steel	Alumina (Al_2O_3)
E	210 (GPa)	390 (GPa)
v	0.3	0.3

Table 2 Comparison of the nondimensional buckling load for a S-S FG nanobeam with various gradient indexes without elastic foundation (L/h=20)

		μ=1			μ=2	
p	EBT (Eltaher *et al*. 2013b)	RBT (Rahmani and Jandaghian 2015)	Present	EBT(Eltaher *et al*. 2013b)	RBT (Rahmani and Jandaghian 2015)	Present
0	8.9843	8.9258	8.925759	8.2431	8.1900	8.190046
0.1	10.1431	9.7778	9.777865	9.2356	8.9719	8.971916
0.2	10.2614	10.3898	10.389845	9.7741	9.5334	9.533453
0.5	11.6760	11.4944	11.494448	10.6585	10.5470	10.547009
1	12.4581	12.3709	12.370918	12.0652	11.3512	11.351234
2	13.1254	13.1748	13.174885	12.4757	12.0889	12.088934
5	13.5711	14.2363	14.236343	13.2140	13.0629	13.062900
		μ=3			μ=4	
p	EBT(Eltaher *et al*. 2013b)	RBT (Rahmani and Jandaghian 2015)	Present	EBT(Eltaher *et al*. 2013b)	RBT (Rahmani and Jandaghian 2015)	Present
0	7.6149	7.5663	7.566381	7.0765	7.0309	7.030978
0.1	8.5786	8.2887	8.288712	8.0416	7.7021	7.702196
0.2	9.3545	8.8074	8.807489	8.3176	8.1842	8.184264
0.5	9.8093	9.7438	9.743863	9.0585	9.0543	9.054379
1	10.9776	10.4869	10.486847	9.9816	9.7447	9.744790
2	11.7415	11.1683	11.168372	10.4649	10.3781	10.378089
5	12.2786	12.0682	12.068171	11.5231	11.2142	11.214218

4. Numerical results and discussions

Through this section, the effects of FG distribution, nonlocality effect and mode number on the buckling loads of the FG nanobeam will be figured out. The FG nanobeam is a combination of Steel and Alumina (Al_2O_3) where their properties are given in Table 1. The following dimensions for the beam geometry is considered: L (length)=10000 nm, b (width)=1000 nm (Eltaher *et al*. 2012, Rahmani and Pedram 2014). Also, for better presentation of the results the following dimensionless quantities are adopted (Simsek and Reddy 2013)

$$N_{cr} = \overline{N}\frac{L^2}{E_m I}\ ,\ K_w = k_w\frac{L^4}{E_m I}\ ,\ K_p = k_p\frac{L^2}{E_m I} \tag{52}$$

where I=bh^3/12 is the moment inertia of the beam’s cross section. For the verification purpose, the

Table 3 The variation of the nondimensional buckling loads of S-S FG nanobeam with various gradient indexes and nonlocal parameters (K_p=0, *L/h*=20)

K_w	μ	Gradient index (p)							
		0		0.5		1		5	
		PL-FGM	MT-FGM	PL-FGM	MT-FGM	PL-FGM	MT-FGM	PL-FGM	MT-FGM
0	0	9.80670	9.8067	12.6289	12.3794	13.5919	13.3665	15.6414	15.4096
	1	8.92576	8.92576	11.4944	11.2673	12.3709	12.1657	14.2363	14.0254
	2	8.19005	8.19005	10.5470	10.3386	11.3512	11.1630	13.0629	12.8693
	3	7.56638	7.56638	9.74386	9.55132	10.4868	10.3129	12.0682	11.8894
	4	7.03098	7.03098	9.05438	8.87546	9.74479	9.58317	11.2142	11.0481
25	0	12.3397	12.3397	15.1619	14.9124	16.1249	15.8995	18.1744	17.9427
	1	11.4588	11.4588	14.0275	13.8003	14.9039	14.6988	16.7694	16.5584
	2	10.7231	10.7231	13.0800	12.8716	13.8843	13.6960	15.5959	15.4024
	3	10.0994	10.0994	12.2769	12.0844	13.0199	12.8460	14.6012	14.4224
	4	9.56401	9.56401	11.5874	11.4085	12.2778	12.1162	13.7472	13.5811
50	0	14.8728	14.8728	17.6950	17.4454	18.6579	18.4325	20.7075	20.4757
	1	13.9918	13.9918	16.5605	16.3334	17.4370	17.2318	19.3024	19.0915
	2	13.2561	13.2561	15.6131	15.4047	16.4173	16.2290	18.1290	17.9354
	3	12.6324	12.6324	14.8099	14.6174	15.5529	15.3790	17.1342	16.9554
	4	12.0970	12.0970	14.1204	13.9415	14.8108	14.6492	16.2803	16.1141
100	0	19.9388	19.9388	22.7610	22.5115	23.7240	23.4986	25.7735	25.5418
	1	19.0579	19.0579	21.6266	21.3994	22.5030	22.2979	24.3685	24.1575
	2	18.3222	18.3222	20.6791	20.4707	21.4834	21.2951	23.1950	23.0015
	3	17.6985	17.6985	19.8760	19.6834	20.6190	20.4450	22.2003	22.0215
	4	17.1631	17.1631	19.1865	19.0076	19.8769	19.7153	21.3463	21.1802

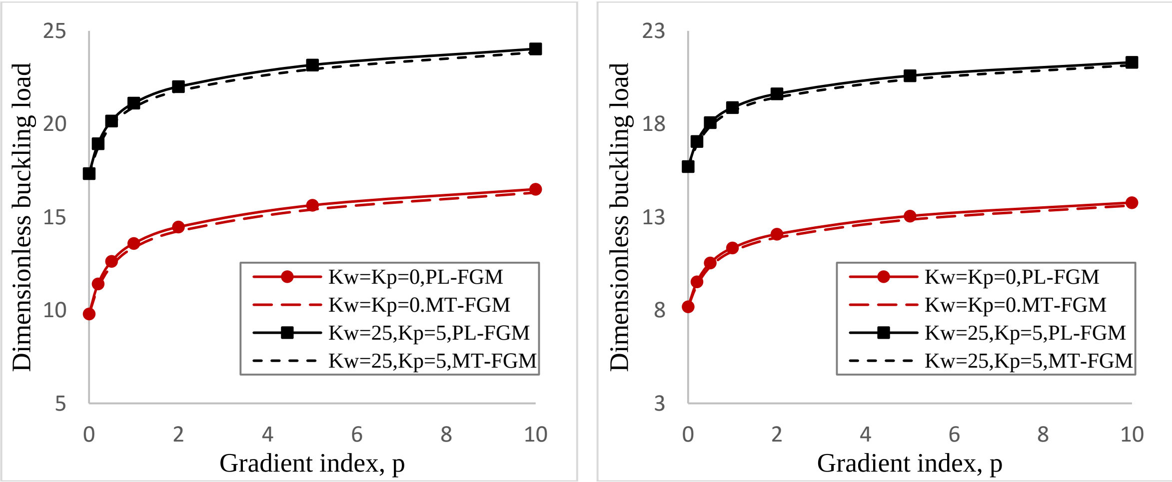
non-dimensional buckling loads of simply supported FG nanobeam with various nonlocal parameters and gradient indexes are compared with the results presented by Eltaher *et al*. (2013) and Rahmani and Jandaghian (2015) for nonlocal Euler-Bernoulli and nonlocal Reddy beam theory, respectively. In these work, the variation of poisson ratio (v) along the thickness of beam is not considered and the value of it is constant equal to 0.3.The reliability of the presented method and procedure for FG nanobeam may be concluded from Table 2; where the results are in an excellent agreement as values of non-dimensional buckling load are consistent with presented analytical solution. It can be observed from Table 2 that the result of nonlocal Reddy beam theory are smaller than those of nonlocal Euler beam theory. This is attributed to the fact that Euler–Bernoulli beam model is unable to capture the influence of shear deformation.

The variation of the dimensionless buckling loads of FG nanobeam for both power-law and Mori–Tanaka models with different gradient indexes (p=0.0.5,1,5), nonlocal parameters, foundation parameters and slenderness ratios is presented in Tables 3-5. The present results for Mori–Tanaka model and power-law model are referred to as MT-FGM and PL-FGM, respectively. It can be noticed from the tables that the non-dimensional buckling loads predicted with respect to power-law model are larger than that of Mori–Tanaka homogenization scheme, related to the fact that FG nanobeam becomes more flexible according to Mori–Tanaka homogenization scheme than with respect to power-law model for a constant gradient index. The obtained results using Mori–Tanaka and power-law models are exactly same at p=0 since the nanobeam is full ceramic. From this point of view, the difference between the results of these two models is significant when the gradient index value is more than p=0. Considering aforementioned explanations and according to Tables 3-5, it must be noted that, as the gradient index increases the dimensionless buckling load increases (constant nonlocal parameter). In addition, at a fixed gradient index the dimensionless buckling load decreases as the nonlocal parameter increases. Furthermore, it should be stated that when the foundation parameters (Winkler and Pasternak parameter) increases the non-dimensional buckling load increases which indicates the stiffening effect of foundation parameters on the FG nanobeam.

The effect of presence of elastic foundation on the non-dimensional buckling load of FG nanobeam with varying of gradient index at L/h=20 is presented in Fig. 2 and the variation of the non-dimensional buckling load with and without elastic foundation based on both power-law and Mori–Tanaka models are compared with each other. It is seen from the results of the figure that the dimensionless buckling loads of FG nanobeam embedded in elastic medium are larger than that of

Table 4 The variation of the nondimensional buckling loads of S-S FG nanobeam with various gradient indexes and nonlocal parameters (K_p=5, L/h=20)

K_w	μ	Gradient index (p)							
		0		0.5		1		5	
		PL-FGM	MT-FGM	PL-FGM	MT-FGM	PL-FGM	MT-FGM	PL-FGM	MT-FGM
0	0	14.8067	14.8067	17.6289	17.3794	18.5919	18.3665	20.6414	20.4096
	1	13.9258	13.9258	16.4944	16.2673	17.3709	17.1657	19.2363	19.0254
	2	13.1900	13.1900	15.5470	15.3386	16.3512	16.1630	18.0629	17.8693
	3	12.5664	12.5664	14.7439	14.5513	15.4868	15.3129	17.0682	16.8894
	4	12.0310	12.0310	14.0544	13.8755	14.7448	14.5832	16.2142	16.0481
25	0	17.3397	17.3397	20.1619	19.9124	21.1249	20.8995	23.1744	22.9427
	1	16.4588	16.4588	19.0275	18.8003	19.9039	19.6988	21.7694	21.5584
	2	15.7231	15.7231	18.0800	17.8716	18.8843	18.6960	20.5959	20.4024
	3	15.0994	15.0994	17.2769	17.0844	18.0199	17.8460	19.6012	19.4224
	4	14.5640	14.564	16.5874	16.4085	17.2778	17.1162	18.7472	18.5811
50	0	19.8728	19.8728	22.6950	22.4454	23.6579	23.4325	25.7075	25.4757
	1	18.9918	18.9918	21.5605	21.3334	22.4370	22.2318	24.3024	24.0915
	2	18.2561	18.2561	20.6131	20.4047	21.4173	21.2290	23.129	22.9354
	3	17.6324	17.6324	19.8099	19.6174	20.5529	20.3790	22.1342	21.9554
	4	17.0970	17.0970	19.1204	18.9415	19.8108	19.6492	21.2803	21.1141
100	0	24.9388	24.9388	27.7610	27.5115	28.7240	28.4986	30.7735	30.5418
	1	24.0579	24.0579	26.6266	26.3994	27.5030	27.2979	29.3685	29.1575
	2	23.3222	23.3222	25.6791	25.4707	26.4834	26.2951	28.1950	28.0015
	3	22.6985	22.6985	24.8760	24.6834	25.6190	25.4450	27.2003	27.0215
	4	22.1631	22.1631	24.1865	24.0076	24.8769	24.7153	26.3463	26.1802

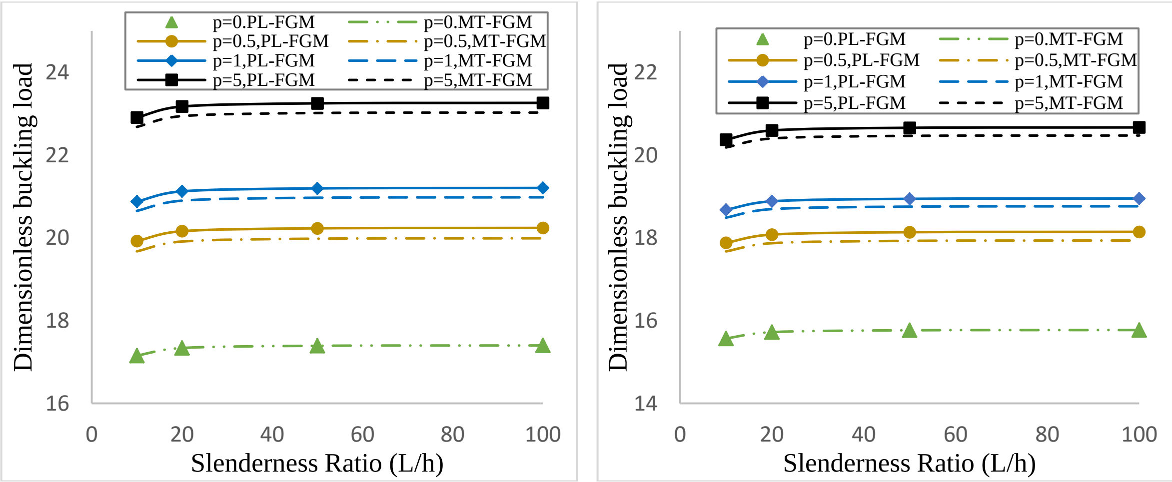


(a) (b)

Fig. 2 The effect of presence of elastic foundation on the dimensionless buckling load of S-S FG nanobeam based on power-law and Mori-Tanaka models with gradient index when L/h=20; (a) μ=0, (b) μ=2

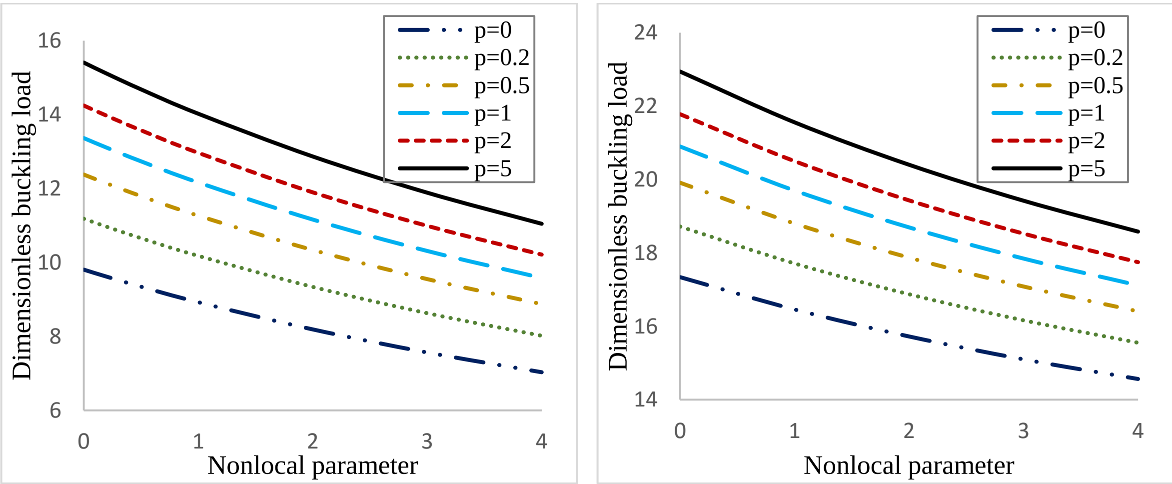
Table 5 The variation of the nondimensional buckling loads of S-S FG nanobeam with various gradient indexes and nonlocal parameters (K_p=10, L/h=20)

K_w	μ	Gradient index (p)							
		0		0.5		1		5	
		PL-FGM	MT-FGM	PL-FGM	MT-FGM	PL-FGM	MT-FGM	PL-FGM	MT-FGM
0	0	19.8067	19.8067	22.6289	22.3794	23.5919	23.3665	25.6414	25.4096
	1	18.9258	18.9258	21.4944	21.2673	22.3709	22.1657	24.2363	24.0254
	2	18.1900	18.1900	20.5470	20.3386	21.3512	21.1630	23.0629	22.8693
	3	17.5664	17.5664	19.7439	19.5513	20.4868	20.3129	22.0682	21.8894
	4	17.0310	17.0310	19.0544	18.8755	19.7448	19.5832	21.2142	21.0481
25	0	22.3397	22.3397	25.1619	24.9124	26.1249	25.8995	28.1744	27.9427
	1	21.4588	21.4588	24.0275	23.8003	24.9039	24.6988	26.7694	26.5584
	2	20.7231	20.7231	23.0800	22.8716	23.8843	23.6960	25.5959	25.4024
	3	20.0994	20.0994	22.2769	22.0844	23.0199	22.8460	24.6012	24.4224
	4	19.5640	19.5640	21.5874	21.4085	22.2778	22.1162	23.7472	23.5811
50	0	24.8728	24.8728	27.6950	27.4454	28.6579	28.4325	30.7075	30.4757
	1	23.9918	23.9918	26.5605	26.3334	27.4370	27.2318	29.3024	29.0915
	2	23.2561	23.2561	25.6131	25.4047	26.4173	26.2290	28.1290	27.9354
	3	22.6324	22.6324	24.8099	24.6174	25.5529	25.3790	27.1342	26.9554
	4	22.0970	22.0970	24.1204	23.9415	24.8108	24.6492	26.2803	26.1141
100	0	29.9388	29.9388	32.7610	32.5115	33.7240	33.4986	35.7735	35.5418
	1	29.0579	29.0579	31.6266	31.3994	32.5030	32.2979	34.3685	34.1575
	2	28.3222	28.3222	30.6791	30.4707	31.4834	31.2951	33.1950	33.0015
	3	27.6985	27.6985	29.8760	29.6834	30.6190	30.4450	32.2003	32.0215
	4	27.1631	27.1631	29.1865	29.0076	29.8769	29.7153	31.3463	31.1802



(a) (b)

Fig. 3 The comparison of the dimensionless buckling load versus slenderness ratio for different gradient indexes when $K_p = 5$, $K_w = 25$; (a) classical beam theory μ=0. (b) Nonlocal beam theory μ=2



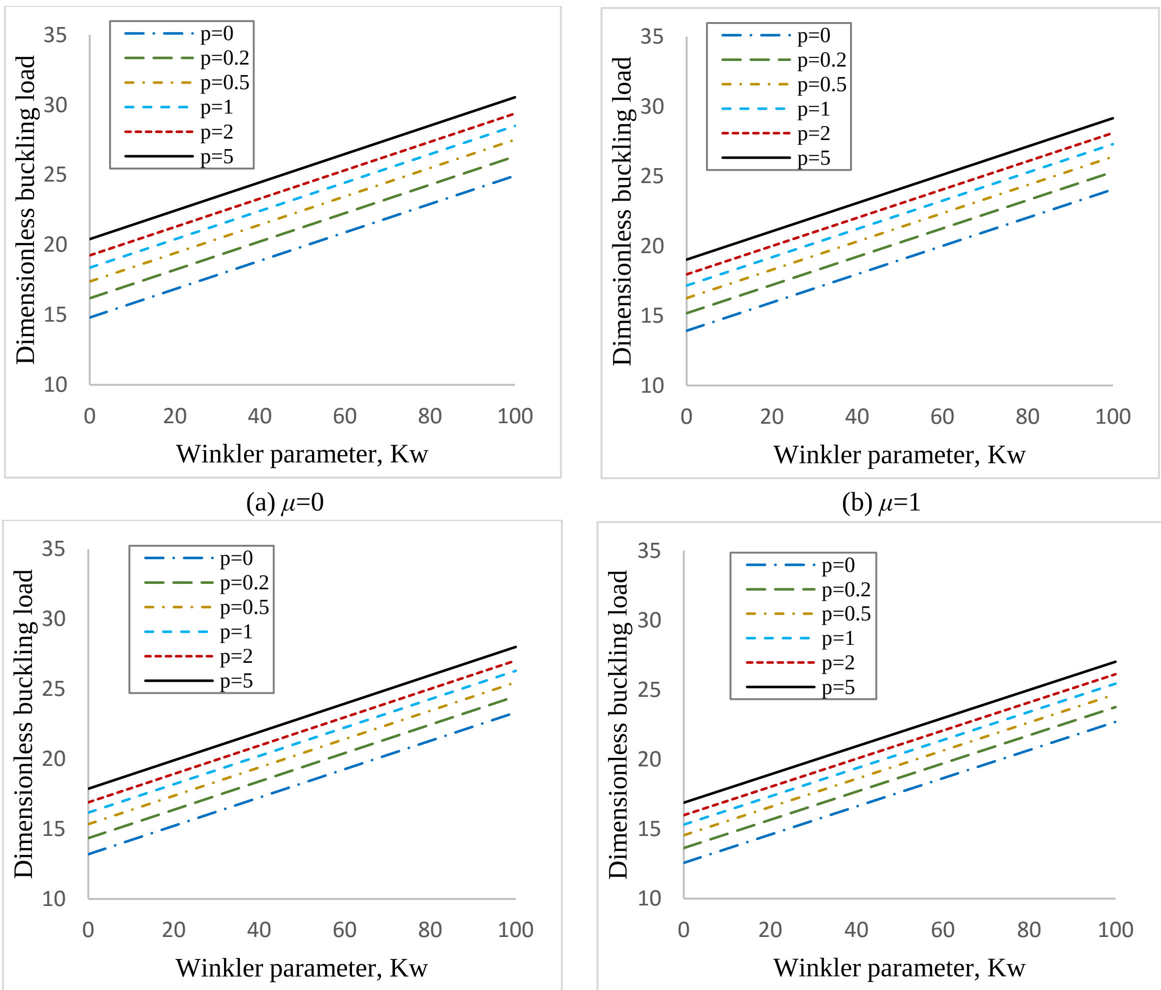
(a) (b)

Fig. 4 The variation of the dimensionless buckling load of S-S FG nanobeam with nonlocal parameter and gradient index at L/h=20; (a) $K_w = K_p = 0$. (b) $K_w = 25, K_p = 5$

FG nanobeam without elastic foundation. This is due to the fact that when the both foundation parameters increase the nanobeam becomes stiffer. Also, the Mori–Tanaka scheme estimates lower values for the non-dimensional buckling loads with comparing to the power-law model. The reason is that, Mori–Tanaka model provides smaller values for Young's modulus than the power-law model, and that gives rise to a more flexible structure. Also, it is noticed from the figures that, the dimensionless buckling load increases with high rate where the gradient index changes from 0 to 2 than that where gradient index changes from 2 to 10. Also it can be seen that increasing nonlocal parameter shows a decreasing effect on the dimensionless buckling load. So, as a general consequence, the presence of nonlocality and elastic foundation softens and stiffens the structure, respectively.

Fig. 3, demonstrates the variation of the dimensionless buckling load of S-S FG nanobeam with respect to slenderness ratio (at K_p=5, K_w=25) for various values of gradient indexes used in Mori–Tanaka model as well as power-law model. It is seen from the figure that, the dimensionless buckling load increases with increase in slenderness ratio. But this observation is accurate when slenderness ratio is in the range L/h<20. Therefore, it can be deduced that the effect of slenderness ratio on dimensionless buckling load is approximately diminishes for the values greater than L/h>20.

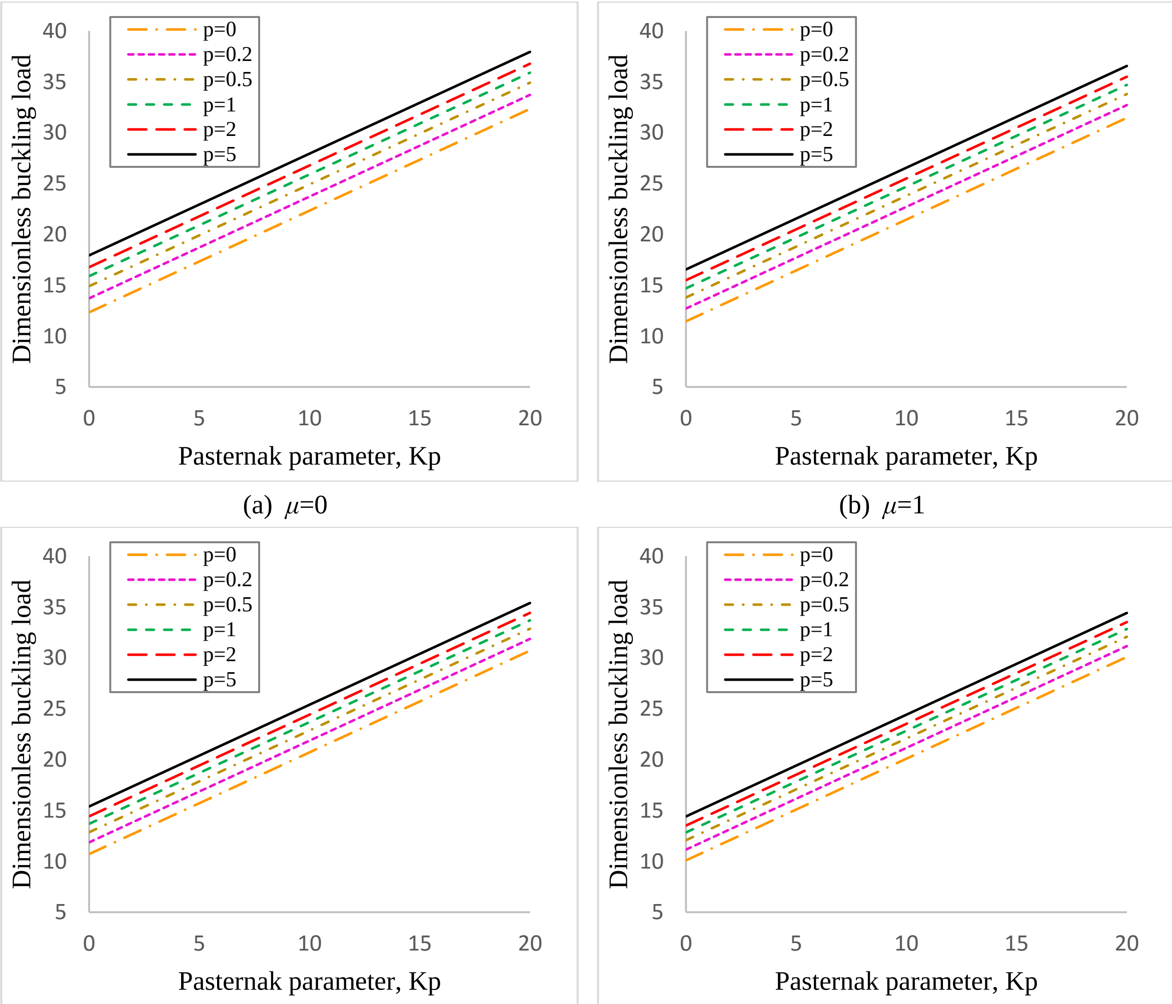
The softening effect of nonlocal parameter on the dimensionless buckling load of S-S FG nanobeams for various gradient index at L/h=20 with and without elastic foundation is shown in Fig. 4, so what as the nonlocal parameter growths, the dimensionless buckling load reduces for all



(a) μ=0 (b) μ=1

(c) μ=2 (d) μ=3

Fig. 5 The variation of the dimensionless buckling load of S-S FG nanobeam with Winkler parameter and gradient index for different nonlocal parameters at L/h=20 and K_p=5



(a) μ=0 (b) μ=1

(c) μ=2 (d) μ=3

Fig. 6 The variation of the dimensionless buckling load of S-S FG nanobeam with Pasternak parameter and gradient index for different nonlocal parameters at L/h=20 and $K_w = 25$

gradient indexes.

The variation of the dimensionless buckling load of S-S FG nanobeam with Winkler parameter for different nonlocal parameters and gradient indexes is presented in Fig. 5. In this figure, the Mori–Tanaka model is adopted. It is seen that with increase of the Winkler parameter the dimensionless buckling load increases linearly for all values of gradient index. Also, it is observed that increasing the gradient index yields the increment in dimensionless buckling load at constant Winkler and nonlocal parameters.

The variation of the dimensionless buckling load of S-S FG nanobeam with respect to Pasternak parameter K_p and different gradient indexes and nonlocal parameters is presented in Fig. 6. It is observed that with increase of the Pasternak parameter the dimensionless buckling load increases with a linear manner for all values of gradient index and nonlocal parameter. Also, it is seen that increasing the gradient index results in increase of dimensionless buckling load at constant Pasternak parameter. Comparing this figure with Fig. 5 specifies that the influence of the Pasternak parameter (K_p) on the non-dimensional buckling load is more significant than that of the Winkler parameter (K_w).

5. Conclusions

In the present work, buckling analysis of size-dependent FG nanobeams embedded in two-parameter elastic foundation is performed based on nonlocal third order shear deformation beam theory in conjunction with Navier analytical method. Two kinds of mathematical models, namely, power law and Mori-Tanaka models are considered. The nonlocal governing differential equations in elastic medium are derived by implementing Hamilton's principle and using nonlocal constitutive equations of Eringen. Accuracy of the results is examined using available date in the literature. The effects of small scale parameter, material graduation, foundation parameters and slenderness ratio on buckling behavior of FG nanobeams are investigated.

It is observed that, with an increase of Winkler or Pasternak parameter, the beam becomes more rigid and the dimensionless buckling load of FG nanobeams increases. Also, it is found that presence of nonlocality has a notable decreasing effect on the dimensionless buckling load of FG nanobeams, which shows the prominence of the nonlocal effect. So, it should be noted that reasonable selection of the value of the nonlocal parameter is also vital to ensure the accuracy of the nonlocal beam models. It must be pointed out that the power-law and Mori-Tanaka indexes have a remarkable effect on the buckling responses of FG nanobeam. Moreover, often the differences of the buckling loads between PL and MT models is very small, specifically at the range of lower gradient indexes. Thus, considering both material models, it is deduced that with the increase of gradient index the buckling loads increase.

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