Coupled Systems Mechanics, *Vol. 9, No. 3 (2020) 265-280*
DOI: https://doi.org/10.12989/csm.2020.9.3.265

Vibration analysis of FGM beam: Effect of the micromechanical models

Lazreg Hadji*[1,2]

[1]*Department of Mechanical Engineering, University of Tiaret, BP 78 Zaaroura, Tiaret, 14000, Algeria*
[2]*Laboratory of Geomatics and Sustainable Development, Ibn Khaldoun University of Tiaret, Algeria*

(Received November 29, 2019, Revised January 12, 2020, Accepted January 18, 2020)

Abstract. In this paper, a new refined hyperbolic shear deformation beam theory for the free vibration analysis of functionally graded beam is presented. The theory accounts for hyperbolic distribution of the transverse shear strains and satisfies the zero traction boundary conditions on the surfaces of the functionally graded beam without using shear correction factors. In addition, the effect of different micromechanical models on the free vibration response of these beams is studied. Various micromechanical models are used to evaluate the mechanical characteristics of the FG beams whose properties vary continuously across the thickness according to a simple power law. Based on the present theory, the equations of motion are derived from the Hamilton's principle. Navier type solution method was used to obtain frequencies, and the numerical results are compared with those available in the literature. A detailed parametric study is presented to show the effect of different micromechanical models on the free vibration response of a simply supported FG beams.

Keywords: functionally graded beam; free vibration; micromechanical models; Hamilton's principle; Navier solution

1. Introduction

Functionally graded materials (FGMs) are new materials which are designed to achieve a functional performance with gradually variable properties in one or more directions (Koizumi *et al.* 1992). This continuity prevents the material from having disadvantages of composites such as delamination due to large interlaminar stresses, initiation and propagation of cracks because of large plastic deformation at the interfaces and so on. Typically, FGMs are made of a mixture of ceramics and a combination of different metals. FGMs are regarded as one of the most promising candidates for future advanced composites in many engineering sectors such as the aerospace, aircraft, automobile, and defense industries, and most recently the electronics and biomedical sectors. Consequently, studies devoted to understand the static and dynamic behaviors of FGM beams, plates have being paid more and more attentions in recent years. Li (2008) investigated static bending and transverse vibration of FGM Timoshenko beams, in which by introducing a new function, the governing equations for bending and vibration of FGM beams were decoupled and the deflection, rotational angle and the resultant force and moment were expressed only in the terms of this new

*Corresponding author, Ph.D., E-mail: had_laz@yahoo.fr

http://www.techno-press.org/?journal=csm&subpage=8 ISSN: 2234-2184 (Print), 2234-2192 (Online)

function. Simsek (2010) investigated dynamic responses of functionally graded beams with the framework of Euler–Bernoulli, Timoshenko and the third-order shear deformation theories, in which the system of equations of motion were derived by using Lagrange's equations based Hamilton's principle. Li and Liu (2010) derived the analytical proportional relationships between the deflections, critical buckling loads and natural frequencies of a FGM beams and those of the corresponding homogenous beams based on the classical beam theory and proved that the transition relationships are the same for different load and boundary conditions. Akbaş *et al.* (2015a) studied the wave propagation of a functionally graded beam in thermal environments. Trinh *et al.* (2016) investigated an analytical method for the vibration and buckling of functionally graded beams under mechanical and thermal loads.

Recently, Sayyad and Ghugal (2015, 2017a) presented a comprehensive literature review on various higher-order beam theories for the analysis of beam and plate structures. Several research papers have been published by researchers in last decade on bending, buckling and free vibration analysis of functionally graded plates and beams (Thai and Vo 2012, Sayyad and Ghugal 2017b, Simsek 2010, Hadji *et al*. 2016, Bourada *et al*. 2015, Akbaş 2015b, Akbaş 2017a, b, Akbaş 2018a, b, Akbaş 2019a, b, c, Mahmoud *et al.* 2019, Mahmoudi *et al.* 2019, Adda Bedia *et al.* 2019, Batou *et al.* 2019, Abualnour *et al.* 2019, Draiche *et al*. 2019, Karami *et al.* 2019a, Belbachir *et al.* 2019, Semmah *et al.* 2019, Boussoula *et al.* 2020).

In order to model FGM precisely, it is essential to know the effective or bulk material properties as a function of individual material properties and geometry, in particular at micromechanics level.

In recent years, different models have been proposed to estimate the effective properties of FGMs with respect to reinforcement volume fractions (Shen and Wang 2012). Consequently, several micromechanical models have been used to study and analyze the behavior of FGM structures under different loading conditions. We cite as an example the work of Gasik (1998) in which he proposed a micromechanical model to study FGMs with a random distribution of constituents. Using an appropriate micromechanical model. Mahmoudi *et al.* (2018) studied the effect of the micromechanical models on the free vibration of rectangular FGM plate resting on elastic foundation. Hadji *et al.* (2019) developed an analytical solution for bending and free vibration responses of functionally graded beams with porosities: Effect of the micromechanical models. In addition, in recent years, many researchers have dealt with the dynamic problem (Hussain *et al*. 2019, Alimirzaei *et al.* 2019, Chaabane *et al.* 2019, Berghouti *et al.* 2019, Bourada *et al*. 2019, Karami *et al.* 2019b, Karami *et al.* 2019c, Meksi *et al.* 2019, Hellal *et al.* 2019, Boulefrakh *et al.* 2019, Medani *et al.* 2019, Draoui *et al.* 2019, Tlidji *et al.* 2019, Sahla *et al.* 2019). In addition, in recent years, many researchers have dealt the effect of stretching the thickness on FGM structures (Addou *et al.* 2019, Boutaleb *et al.* 2019, Khiloun *et al.* 2019, Zarga *et al.* 2019, Boulefrakh *et al.* 2019, Boukhlif *et al.* 2019, Mahmoudi *et al.* 2019, Zaoui *et al.* 2019).

In the present study, the free vibration of simply supported FG beams was investigated by using a new hyperbolic shear deformation beam theory. The effect of different micromechanical models on the free vibration response of these beams is studied. Various micromechanical models are used to evaluate the mechanical characteristics of the FG beams whose properties vary continuously across the thickness according to a simple power law. Then, the present theory together with Hamilton's principle, are employed to extract the motion equations of the functionally graded beams. Analytical solutions for free vibration are obtained. The effects of various variables, such as span-to-depth ratio, gradient index, and micromechanical models on free vibration of FG beam are all discussed.

2. Effective properties of FGMs

Unlike traditional microstructures, in FGMs the material properties are spatially varying, which is not trivial for a micromechanics model (Jaesang and Addis 2014).

A number of micromechanics models have been proposed for the determination of effective properties of FGMs. In what follows, we present some micromechanical models to calculate the effective properties of the FG beam.

2.1 Voigt model

The Voigt model is relatively simple; this model is frequently used in most FGM analyses estimates Young's modulus *E* of FGMs as (Mishnaevsky 2007)

$$E(z) = E_c \mathrm{V_c} + E_m(1 - V_c) \tag{1}$$

2.2 Reuss model

Reuss assumed the stress uniformity through the material and obtained the effective properties as (Mishnaevsky 2007, Zimmerman 1994)

$$E(z) = \frac{E_c E_m}{E_c(1 - V_c) + E_m V_c} \tag{2}$$

2.3 Tamura model

The Tamura model uses actually a linear rule of mixtures, introducing one empirical fitting parameter known as "stress-to-strain transfer" (Gasik 1995)

$$q = \frac{\sigma_1 - \sigma_2}{\varepsilon_1 - \varepsilon_2} \tag{3}$$

Estimate for q=0 correspond to Reuss rule and with q=100 to the Voigt rule, being invariant to the consideration of with phase is matrix and which is particulate. The effective Young's modulus is found as

$$E(z) = \frac{(1 - V_c)E_m(q - E_c) + V_c E_c(q - E_m)}{(1 - V_c)(q - E_c) + V_c E_c(q - E_m)} \tag{4}$$

2.4 Description by a representative volume element (LRVE)

The local representative volume element (LRVE) is based on a "mesoscopic" length scale which is much larger than the characteristic length scale of particles (inhomogeneities) but smaller than the characteristic length scale of a macroscopic specimen (Ju and Chen 1994). The LRVE is developed based on the assumption that the microstructure of the heterogeneous material is known. The input for the LRVE for the deterministic micromechanical framework is usually volume average or ensemble average of the descriptors of the microstructures.

Young's modulus is expressed as follows by the LRVE method (Akbarzadeh *et al.* 2015)

$$E(z) = E_m\left(1 + \frac{V_c}{FE - \sqrt[3]{V_c}}\right), \quad FE = \frac{1}{1 - \frac{E_m}{E_c}} \tag{5}$$

2.5 Mori-Tanaka model

The locally effective material properties can be provided by micromechanical models such as the Mori-Tanaka estimates. This method based on the assumption that a two-phase composite material consisting of matrix reinforced by spherical particles, randomly distributed in the plate. According to Mori-Tanaka homogenization scheme, the Young's modulus is given as

$$E(z) = E_m + (E_c - E_m)\left(\frac{V_c}{1 + (1 - V_c)(E_c / E_m - 1)(1 + \nu)/(3 - 3\nu)}\right) \tag{6}$$

where $V_c = \left(\frac{1}{2} + \frac{z}{h}\right)^p$ is the volume fraction of the ceramic and where p is the power law index. Since the effects of the variation of Poisson's ratio (ν) on the response of FGM plates are very small (Kitipornchai 2006), this material parameter is assumed to be constant for convenience.

3. Problem formulation

Consider a functionally graded beam with length L and rectangular cross section $b \times h$, with b being the width and h being the height as shown in Fig. 1.

4. Kinematics and constitutive equations

4.1 Basic assumptions

The assumptions of the present theory are as follows:

(i) The origin of the Cartesian coordinate system is taken at the median surface of the FG beam.

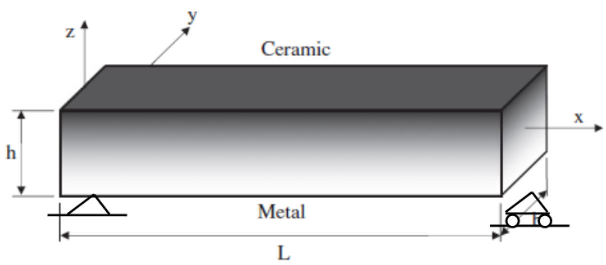

Fig. 1 Geometry and coordinate of a FG beam

(ii) The displacements are small in comparison with the height of the beam and, therefore, strains involved are infinitesimal.
(iii) The transverse displacement w includes two components of bending w_b, and shear w_s. These components are functions of coordinates x, t only.

$$w(x,z,t) = w_b(x,t) + w_s(x,t) \tag{7}$$

(iv) The transverse normal stress σ_z is negligible in comparison with in-plane stresses σ_x.
(v) The axial displacement u in x-direction, consists of extension, bending, and shear components.

$$u = u_0 + u_b + u_s \tag{8}$$

(vi) The bending component u_b is assumed to be similar to the displacements given by the classical beam theory. Therefore, the expression for u_b can be given as

$$u_b = -z\frac{\partial w_b}{\partial x} \tag{9}$$

(vii) The shear component u_s gives rise, in conjunction with w_s, to the hyperbolic variation of shear strain γ_{xz} and hence to shear stress τ_{xz} through the thickness of the beam in such a way that shear stress τ_{xz} is zero at the top and bottom faces of the beam. Consequently, the expression for u_s can be given as

$$u_s = -f(z)\frac{\partial w_s}{\partial x} \tag{10}$$

where

$$f(z) = z\left[1 + \frac{3\pi}{2}\sec h^2\left(\frac{1}{2}\right)\right] - \frac{3\pi}{2}h\tanh\left(\frac{z}{h}\right) \tag{11}$$

4.2 Kinematics and constitutive equations

Based on the assumptions made in the preceding section, the displacement field can be obtained using Eqs. (7)-(11) as

$$u(x,z,t) = u_0(x,t) - z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x} \tag{12a}$$

$$w(x,z,t) = w_b(x,t) + w_s(x,t) \tag{12b}$$

The strains associated with the displacements in Eq. (12) are

$$\varepsilon_x = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_b}{\partial x^2} - f(z)\frac{\partial^2 w_s}{\partial x^2} \tag{13a}$$

$$\gamma_{xz} = g(z)\gamma_{xz}^s \tag{13b}$$

where

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x},\ k_x^b = -\frac{\partial^2 w_b}{\partial x^2},\ k_x^s = -\frac{\partial^2 w_s}{\partial x^2},\ \gamma_{xz}^s = \frac{\partial w_s}{\partial x} \tag{14}$$

The state of stress in the beam is given by the generalized Hooke's law as follows

$$\sigma_x = Q_{11}(z)\,\varepsilon_x \quad \text{and} \quad \tau_{xz} = Q_{55}(z)\,\gamma_{xz} \tag{15a}$$

where

$$Q_{11}(z) = E(z) \quad \text{and} \quad Q_{55}(z) = \frac{E(z)}{2(1+\nu)} \tag{15b}$$

5. Equations of motion

Hamilton's principle is used herein to derive the equations of motion. The principle can be stated in analytical form as (Thai and Vo 2012)

$$\delta \int_{t_1}^{t_2} (U - T)\,dt = 0 \tag{16}$$

where t is the time; t_1 and t_2 are the initial and end time, respectively; δU is the virtual variation of the strain energy; and δT is the virtual variation of the kinetic energy. The variation of the strain energy of the beam can be stated as

$$\begin{aligned} \delta U &= \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_x \delta\,\varepsilon_x + \tau_{xz}\delta\,\gamma_{xz})\,dzdx \\ &= \int_0^L \left(N_x \frac{d\delta\,u_0}{dx} - M_b \frac{d^2\delta\,w_b}{dx^2} - M_s \frac{d^2\delta\,w_s}{dx^2} + Q_{xz}\frac{d\delta\,w_s}{dx} \right) dx \end{aligned} \tag{17}$$

where N_x, M_b, M_s and Q_{xz} are the stress resultants defined as

$$(N_x, M_b, M_s) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1, z, f)\,\sigma_x dz \quad \text{and} \quad Q_{xz} = \int_{-\frac{h}{2}}^{\frac{h}{2}} g\,\tau_{xz} dz \tag{18}$$

The variation of the kinetic energy can be expressed as

$$\begin{aligned} \delta T &= \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(z)\left[\dot{u}\delta\,\dot{u} + \dot{w}\delta\,\dot{w}\right] dzdx \\ &= \int_0^L \left\{ I_0\left[\dot{u}_0\delta\dot{u}_0 + (\dot{w}_b + \dot{w}_s)(\delta\dot{w}_b + \delta\dot{w}_s)\right] - I_1\left(\dot{u}_0\frac{d\delta\dot{w}_b}{dx} + \frac{d\dot{w}_b}{dx}\delta\,\dot{u}_0 \right) \right. \end{aligned}$$

$$+I_2\left(\frac{d\dot{w}_b}{dx}\frac{d\delta\,\dot{w}_b}{dx}\right)-J_1\left(\dot{u}_0\,\frac{d\delta\dot{w}_s}{dx}+\frac{d\,\dot{w}_s}{dx}\delta\,\dot{u}_0\right)+K_2\left(\frac{d\dot{w}_s}{dx}\frac{d\delta\,\dot{w}_s}{dx}\right)$$
$$+J_2\left(\frac{d\dot{w}_b}{dx}\frac{d\delta\,\dot{w}_s}{dx}+\frac{d\dot{w}_s}{dx}\frac{d\delta\,\dot{w}_b}{dx}\right)\bigg\}dx \tag{19}$$

where dot-superscript convention indicates the differentiation with respect to the time variable t; $\rho(z)$ is the mass density; and (I_0, I_1, J_1, I_2, J_2, K_2) are the mass inertias defined as

$$\left(I_0,I_1,J_1,I_2,J_2,K_2\right)=\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(1,z,f,z^2,zf,f^2\right)\rho(z)dz \tag{20}$$

Substituting the expressions for δU, and δT from Eqs. (17), and (19) into Eq. (16) and integrating by parts versus both space and time variables, and collecting the coefficients of δu_0, δw_b, and δw_s, the following equations of motion of the functionally graded beam are obtained

$$\delta\,u_0:\ \frac{dN_x}{dx}=I_0\ddot{u}_0-I_1\frac{d\ddot{w}_b}{dx}-J_1\frac{d\ddot{w}_s}{dx} \tag{21a}$$

$$\delta\,w_b:\ \frac{d^2M_b}{dx^2}=I_0(\ddot{w}_b+\ddot{w}_s)+I_1\frac{d\ddot{u}_0}{dx}-I_2\frac{d^2\ddot{w}_b}{dx^2}-J_2\frac{d^2\ddot{w}_s}{dx^2} \tag{21b}$$

$$\delta\,w_s:\ \frac{d^2M_s}{dx^2}+\frac{dQ_{xz}}{dx}=I_0(\ddot{w}_b+\ddot{w}_s)+J_1\frac{d\ddot{u}_0}{dx}-J_2\frac{d^2\ddot{w}_b}{dx^2}-K_2\frac{d^2\ddot{w}_s}{dx^2} \tag{21c}$$

Eq. (21) can be expressed in terms of displacements (u_0, w_b, w_s) by using Eqs. (12), (13), (15) and (18) as follows

$$A_{11}\frac{\partial^2u_0}{\partial x^2}-B_{11}^s\frac{\partial^3w_s}{\partial x^3}=I_0\ddot{u}_0-I_1\frac{d\ddot{w}_b}{dx}-J_1\frac{d\ddot{w}_s}{dx} \tag{22a}$$

$$B_{11}\frac{\partial^3u_0}{\partial x^3}-D_{11}\frac{\partial^4w_b}{\partial x^4}-D_{11}^s\frac{\partial^4w_s}{\partial x^4}$$
$$=I_0(\ddot{w}_b+\ddot{w}_s)+I_1\frac{d\ddot{u}_0}{dx}-I_2\frac{d^2\ddot{w}_b}{dx^2}-J_2\frac{d^2\ddot{w}_s}{dx^2} \tag{22b}$$

$$B_{11}^s\frac{\partial^3u_0}{\partial x^3}-D_{11}^s\frac{\partial^4w_b}{\partial x^4}-H_{11}^s\frac{\partial^4w_s}{\partial x^4}+A_{55}^s\frac{\partial^2w_s}{\partial x^2}$$
$$=I_0(\ddot{w}_b+\ddot{w}_s)+J_1\frac{d\ddot{u}_0}{dx}-J_2\frac{d^2\ddot{w}_b}{dx^2}-K_2\frac{d^2\ddot{w}_s}{dx^2} \tag{22c}$$

where A_{11}, D_{11}, etc., are the beam stiffness, defined by

$$\left(A_{11},B_{11},D_{11},B_{11}^s,D_{11}^s,H_{11}^s\right)=\int_{-\frac{h}{2}}^{\frac{h}{2}}Q_{11}\left(1,z,z^2,f(z),z\,f(z),f^2(z)\right)dz \tag{23a}$$

and

$$A_{55}^{s} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{55}\left[g(z)\right]^{2} dz \tag{23b}$$

6. Analytical solution

The equations of motion admit the Navier solutions for simply supported beams. The variables u_0, w_b, w_s can be written by assuming the following variations

$$\begin{Bmatrix} u_0 \\ w_b \\ w_s \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} U_m \cos(\lambda x)\, e^{i\omega t} \\ W_{bm} \sin(\lambda x)\, e^{i\omega t} \\ W_{sm} \sin(\lambda x)\, e^{i\omega t} \end{Bmatrix} \tag{24}$$

where U_m, W_{bm}, and W_{sm} are arbitrary parameters to be determined, ω is the eigenfrequency associated with m th eigenmode, and $\lambda = m\pi / L$.

Substituting the expansions of u_0, w_b, w_s from Eqs. (24) into the equations of motion Eq. (22), the analytical solutions can be obtained from the following equations

$$\left(\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} \right) \begin{Bmatrix} U_m \\ W_{bm} \\ W_{sm} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{25}$$

where

$$\begin{aligned} S_{11} &= A_{11}\lambda^2 \\ S_{12} &= -B_{11}\lambda^3 \\ S_{13} &= -B_{11}^{s}\lambda^3 \\ S_{22} &= D_{11}\lambda^4 \\ S_{23} &= D_{11}^{s}\lambda^4 \\ S_{33} &= H_{11}^{s}\lambda^4 + A_{55}^{s}\lambda^2 \end{aligned} \tag{26a}$$

and

$$\begin{aligned} m_{11} &= I_0 \\ m_{12} &= -I_1\lambda \\ m_{13} &= -J_1\lambda \\ m_{22} &= I_0 + I_2\lambda^2 \\ m_{23} &= I_0 + J_2\lambda^2 \\ m_{33} &= I_0 + K_2\lambda^2 \end{aligned} \tag{26b}$$

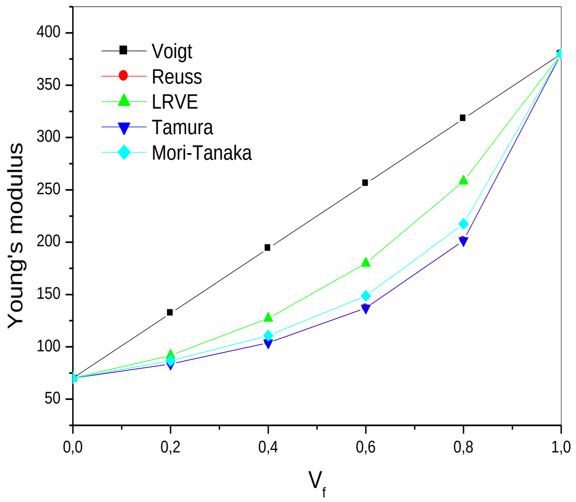

Fig. 2 Effective Young's modulus as function of volume fraction of ceramic for several micromechanical models

7. Results and discussion

In the present section, the effect of micromechanical models on the free vibration analysis of FG beams using a refined hyperbolic shear deformation beam theory is presented for investigation. In order to verify the accuracy of the present analysis, the results of this study were verified by comparing them with the various existing beam theories. The material properties used in the present study are:

Ceramic (Alumina, Al_2O_3): E_c=380 GPa; ν=0.3; ρ_c=3960 kg/m^3.

Metal (Aluminium, Al): E_m=70 GPa; ν=0.3; ρ_m=2702 kg/m^3.

For simplicity, the following non-dimensional parameter is used in the numerical examples

$$\bar{\omega} = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$$

7.1 Comparison between different micromechanical models

A comparison between the Young's modulus values calculated from the various micromechanical models is shown in Fig. 2. The estimated results are depicted as a function of volume fraction of inclusions (ceramic). The first observation emerging from this figure is that the models of Voigt and Reuss give the values max and min of the Young's modulus respectively.

The second observation is that the models of Tamura and Reuss give practically the same result in term of Young's modulus and this whatever the value of the volume fraction. These Young modulus values are slightly lower than those calculated by the Mori-Tanaka model.

7.2 Comparison studies

Firstly, the example is performed in Table 1 for FG beams with power law index n=0,0.5,1,2,5

Table 1 Variation of fundamental frequency $\bar{\omega}$ with the power-law index n for FG beam

L/h	Theory			n = 0	0.5	1	2	5	10
5	Ould Larbi *et al.* (2013)			5.1529	4.4108	3.9905	3.6263	3.4001	3.2812
	TBT (Simsek 2010)			5.1527	4.4111	3.9904	3.6264	3.4012	3.2816
	Present	Voigt		5.1529	4.4108	3.9905	3.6263	3.4001	3.2812
		Reuss		5.1529	3.6232	3.3819	3.2381	3.1072	2.9952
		LRVE		5.1529	3.9094	3.5730	3.3732	3.2309	3.1071
		Tamura	(q=0)	5.1529	3.6232	3.3819	3.2381	3.1072	2.9952
			(q=100)	5.1529	3.9142	3.5887	3.3794	3.2278	3.1069
		Mori-Tanaka		5.1529	3.7112	3.4441	3.2825	3.1460	3.0301
20	Ould Larbi *et al.* (2013)			5.4603	4.6511	4.2051	3.8361	3.6484	3.5389
	TBT (Simsek 2010)			5.4603	4.6511	4.2051	3.8361	3.6485	3.5390
	Present	Voigt		5.4603	4.6511	4.2051	3.8361	3.6484	3.5389
		Reuss		5.4603	3.8364	3.5956	3.4626	3.3350	3.2063
		LRVE		5.4603	4.1256	3.7815	3.5970	3.4765	3.3374
		Tamura	(q=0)	5.4603	3.8364	3.5956	3.4626	3.3350	3.2063
			(q=100)	5.4603	4.1318	3.7998	3.6013	3.4696	3.3364
		Mori-Tanaka		5.4603	3.9258	3.6568	3.5070	3.3789	3.2468

and 10 two span-to-depth ratio L/h. Effective Young's modulus is calculated using the aforementioned five micromechanical models. The obtained results are compared with with those given by Simsek (2010) and the theory of Ould Larbi *et al.* (2013).

From this table two observations can be made. First, the results obtained from the present hyperbolic shear theory for the Voigt model are very close to those of Oul Larbi *et al.* (2013) and Simsek (2010). Secondly, the results from the present theory and calculated with the four other models, namely LRVE, Tamura, Mori-Tanaka and Reuss, are slightly different. This can be explained by the way who the Young's modulus is calculated.

The first three nondimensional frequencies $\bar{\omega}$ of FG beams with the various micromechanical models are presented in the Table 2. The comparison is made between the results of the present theory and those of Ould Larbi *et al*. (2013) and the classical beam theory. Here again we note the same observation that the results are very close for the model of Voigt and a slight difference is noticed compared to the others. Also, the results of the frequency are increasing with increasing mode number.

7.3 Parametric studies

In the present paragraph some results and considerations about the effect of the micromechanical models on the free vibration problem of functionally beams are presented. The analysis has been carried out by means of numerical procedures illustrated above.

In Fig. 3, the variations of the non-dimensional fundamental natural frequency $\bar{\omega}$ versus the power law index n for the value of span-to-depth ratio L/h=5 are given for different micromechanical models. It is seen from the figure that the increase of the power law index n produces a decrease

Table 2 First three nondimensional frequencies $\bar{\omega}$ of FG beams

L/h	Mode	Theory		*n* = 0	0.5	1	2	5	10
5	1	CBT		5.3953	4.5931	4.1484	3.7793	3.5949	3.4921
		Ould Larbi *et al.* (2013)		5.1529	4.4108	3.9905	3.6263	3.4001	3.2812
		Present	Voigt	5.1529	4.4108	3.9905	3.6263	3.4001	3.2812
			Reuss	5.1529	3.6232	3.3819	3.2381	3.1072	2.9952
			LRVE	5.1529	3.9094	3.5730	3.3732	3.2309	3.1071
			Tamura (*q*=0)	5.1529	3.6232	3.3819	3.2381	3.1072	2.9952
			Mori-Tanaka	5.1529	3.7112	3.4441	3.2825	3.1460	3.0301
	2	CBT		20.6187	17.5415	15.7982	14.3260	13.5876	13.2376
		Ould Larbi *et al.* (2013)		17.8844	15.4613	14.0121	12.6404	11.5349	11.0216
		Present	Voigt	17.8844	15.4613	14.0121	12.6404	11.5349	11.0216
			Reuss	17.8844	12.6006	11.6697	11.0507	10.5280	10.1981
			LRVE	17.8844	13.6078	12.4405	11.5753	10.9000	10.5134
			Tamura (*q*=0)	17.8844	12.6006	11.6697	11.0507	10.5280	10.1981
			Mori-Tanaka	17.8844	12.9315	11.9149	11.2218	10.6478	10.2978
	3	CBT		43.3483	36.8308	33.0278	29.7458	28.0850	27.4752
		Ould Larbi *et al.* (2013)		34.2248	29.8496	27.1085	24.3196	21.6987	20.5555
		Present	Voigt	34.2248	29.8496	27.1085	24.3196	21.6987	20.5555
			Reuss	34.2248	24.1793	22.2481	20.8737	19.7644	19.2174
			LRVE	34.2248	26.4043	23.9089	21.9742	20.3977	19.7143
			Tamura (*q*=0)	34.2248	24.1799	22.2481	20.8738	19.7644	19.2174
			Mori-Tanaka	34.2248	24.8534	22.7662	21.2297	19.9731	19.3769
20	1	CBT		5.4777	4.6641	4.2163	3.8472	3.6628	3.5547
		Ould Larbi *et al.* (2013)		5.4603	4.6511	4.2051	3.8361	3.6484	3.5389
		Present	Voigt	5.4603	4.6511	4.2051	3.8361	3.6484	3.5389
			Reuss	5.4603	3.8364	3.5956	3.4626	3.3350	3.2063
			LRVE	5.4603	4.1256	3.7815	3.5970	3.4765	3.3374
			Tamura (*q*=0)	5.4603	3.8364	3.5956	3.4626	3.3350	3.2063
			Mori-Tanaka	5.4603	3.9258	3.6568	3.5070	3.3789	3.2468
	2	CBT		21.8438	18.5987	16.8100	15.3334	14.5959	14.1676
		Ould Larbi *et al.* (2013)		21.5734	18.3964	16.6345	15.1617	14.3732	13.9257
		Present	Voigt	21.5734	18.3964	16.6345	15.1617	14.3732	13.9257
			Reuss	21.5734	15.1598	14.1951	13.6523	13.1379	12.6386
			LRVE	21.5734	16.3091	14.9444	14.1911	13.6874	13.1452
			Tamura (*q*=0)	21.5734	15.1598	14.1950	13.6523	13.1379	12.6386
			Mori-Tanaka	21.5734	15.5166	14.4411	13.8300	13.3088	12.7955
	3	CBT		48.8999	41.6328	37.6173	34.2954	32.6357	31.6883
		Ould Larbi *et al.* (2013)		47.5940	40.6534	36.7686	33.4681	31.5719	30.5342
		Present	Voigt	47.5940	40.6534	36.7686	33.4681	31.5719	30.5342
			Reuss	47.5940	33.4531	31.2802	30.0242	28.8557	27.7846
			LRVE	47.5940	36.0462	32.9829	31.2389	30.0368	28.8644
			Tamura (*q*=0)	47.5940	33.4531	31.2802	30.0242	28.8557	27.7846
			Mori-Tanaka	47.5940	34.2519	31.8369	30.4244	29.2246	28.1199

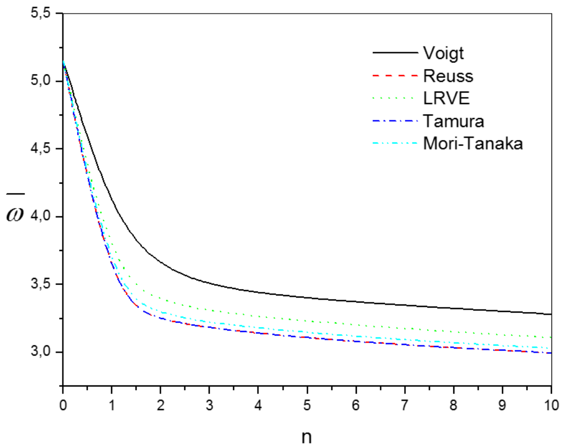

Fig. 3 Variation of the nondimensional fundamental frequency $\overline{\omega}$ of FG beam with power law index n for different micromechanical models (L/h=5)

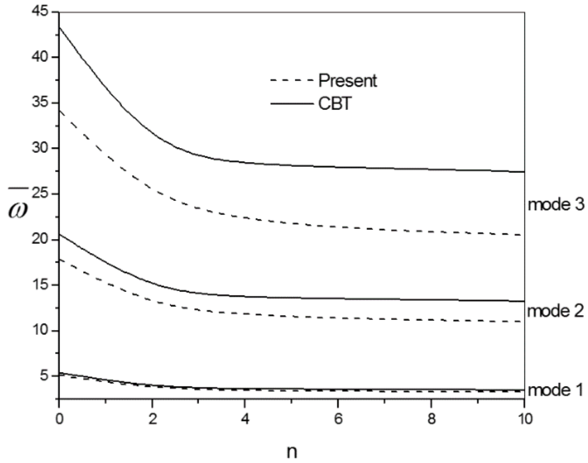

Fig. 4 Variation of first three frequency parameter $\overline{\omega}$ with n (L/h=5) - Voigt model

in the values of the frequencies and this whatever the model used. The full ceramic beams (n=0) lead to a highest frequency for all models. However, the lowest frequency values are obtained for full metal beams ($n \rightarrow \infty$). In addition, the Voigt model has the highest frequencies values compared to other models. While that of Reuss has the lowest values. The Tamura and Reuss models have the practically same results.

The effects of parameter n and shear deformation are shown in Fig. 4 for the first three frequencies using the Voigt model. The difference between CBT and the present theory is increasing with increasing mode number and all frequencies are decreasing with increasing n.

8. Conclusions

In this paper, we have developed a new refined hyperbolic shear deformation beam theory for the solutions of free vibration of FG beam. The theory accounts for hyperbolic distribution of the transverse shear strains and satisfies the zero traction boundary conditions on the surfaces of the functionally graded beam without using shear correction factors. Different micromechanical models were used to determine the effective properties of the FG beams. The Navier method is used for the analytical solutions of the FG beam with simply supported boundary conditions. The results obtained using this new theory, are in a good agreement with reference solutions available in literature.

From these results and comparisons between different micromechanical models, it has been found significant differences between some models. This proves the need for a proper micromechanical modeling of FGMs to accurately estimate the frequenies.

References

Abualnour, M., Chikh, A., Hebali, H., Kaci, A., Tounsi, A., Bousahla, A.A. and Tounsi, A. (2019), "Thermomechanical analysis of antisymmetric laminated reinforced composite plates using a new four variable trigonometric refined plate theory", *Comput. Concrete*, **24**(6), 489-498. https://doi.org/10.12989/cac.2019.24.6.489.

Adda Bedia, W., Houari, M.S.A., Bessaim, A., Bousahla, A.A., Tounsi, A., Saeed, T. and Alhodaly, M.S. (2019), "A new hyperbolic two-unknown beam model for bending and buckling analysis of a nonlocal strain gradient nanobeams", *J. Nano Res.*, **57**, 175-191. https://doi.org/10.4028/www.scientific.net/JNanoR.57.175.

Addou, F.Y., Meradjah, M., Bousahla, A.A., Benachour, A., Bourada, F., Tounsi, A. and Mahmoud, S.R. (2019), "Influences of porosity on dynamic response of FG plates resting on Winkler/Pasternak/Kerr foundation using quasi 3D HSDT", *Comput. Concrete*, **24**(4), 347-367. https://doi.org/10.12989/cac.2019.24.4.347.

Akbarzadeh, A.H., Abedini, A. and Chen, Z.T. (2015), "Effect of micromechanical models on structural responses of functionally graded plates", *Compos. Struct.*, **119**, 598-609. https://doi.org/10.1016/j.compstruct.2014.09.031.

Akbaş, S.D. (2015a), "Wave propagation of a functionally graded beam in thermal environments", *Steel Compos. Struct.*, **19**(6), 1421-1447. http://dx.doi.org/10.12989/scs.2015.19.6.1421.

Akbaş, S.D. (2015b), "Free vibration and bending of functionally graded beams resting on elastic foundation", *Res. Eng. Struct. Mater.*, **1**(1), 25-37. http://dx.doi.org/10.17515/resm2015.03st0107.

Akbaş, S.D. (2017a), "Forced vibration analysis of functionally graded nanobeams", *Int. J. Appl. Mech.*, **9**(7), 1750100. https://doi.org/10.1142/S1758825117501009.

Akbaş, S.D. (2017b), "Nonlinear static analysis of functionally graded porous beams under thermal effect", *Coupl. Syst. Mech.*, **6**(4), 399-415. https://doi.org/10.12989/csm.2017.6.4.399.

Akbaş, S.D. (2018a), "Forced vibration analysis of functionally graded porous deep beams", *Compos. Struct.*, **186**, 293-302. https://doi.org/10.1016/j.compstruct.2017.12.013.

Akbaş, S.D. (2018b), "Forced vibration analysis of cracked functionally graded microbeams", *Adv. Nano Res.*, **6**(1), 39-55. https://doi.org/10.12989/anr.2018.6.1.039.

Akbaş, S.D. (2019a), "Forced vibration analysis of functionally graded sandwich deep beams", *Coupl. Syst. Mech.*, **8**(3), 259-271. https://doi.org/10.12989/csm.2019.8.3.259.

Akbaş, S.D. (2019b), "Hygro-thermal nonlinear analysis of a functionally graded beam", *J. Appl. Comput. Mech.*, **5**(2), 477-485. https://doi.org/10.22055/JACM.2018.26819.1360.

Akbaş, S.D. (2019c), "Hygro-thermal post-buckling analysis of a functionally graded beam", *Coupl. Syst. Mech.*, **8**(5), 459-471. https://doi.org/10.12989/csm.2019.8.5.459.

Alimirzaei, S., Mohammadimehr, M. and Tounsi, A. (2019), "Nonlinear analysis of viscoelastic micro-composite beam with geometrical imperfection using FEM: MSGT electro-magneto-elastic bending, buckling and vibration solutions", *Struct. Eng. Mech.*, **71**(5), 485-502.

https://doi.org/10.12989/sem.2019.71.5.485.

Batou, B., Nebab, M., Bennai, R., Ait Atmane, H., Tounsi, A. and Bouremana, M. (2019), “Wave dispersion properties in imperfect sigmoid plates using various HSDTs”, *Steel Compos. Struct.*, **33**(5), 699-716. https://doi.org/10.12989/scs.2019.33.5.699.

Belbachir, N., Draiche, K., Bousahla, A.A., Bourada, M., Tounsi, A. and Mahmoud, S.R. (2019), “Bending analysis of anti-symmetric cross-ply laminated plates under nonlinear thermal and mechanical loadings”, *Steel Compos. Struct.*, **33**(1), 913-924. https://doi.org/10.12989/scs.2019.33.1.081.

Berghouti, H., Adda Bedia, E.A., Benkhedda, A. and Tounsi, A. (2019), “Vibration analysis of nonlocal porous nanobeams made of functionally graded material”, *Adv. Nano Res.*, **7**(5), 351-364. https://doi.org/10.12989/anr.2019.7.5.351.

Boukhlif, Z., Bouremana, M., Bourada, F., Bousahla, A.A., Bourada, M., Tounsi, A. and Al-Osta, M.A. (2019), “A simple quasi-3D HSDT for the dynamics analysis of FG thick plate on elastic foundation”, *Steel Compos. Struct.*, **31**(5), 503-516. https://doi.org/10.12989/scs.2019.31.5.503.

Boulefrakh, L., Hebali, H., Chikh, A., Bousahla, A.A., Tounsi, A. and Mahmoud, S.R. (2019), “The effect of parameters of visco-Pasternak foundation on the bending and vibration properties of a thick FG plate”, *Geomech. Eng.*, **18**(2), 161-178. https://doi.org/10.12989/gae.2019.18.2.161.

Bourada, F., Bousahla, A.A., Bourada, M., Azzaz, A., Zinata, A. and Tounsi, A. (2019), “Dynamic investigation of porous functionally graded beam using a sinusoidal shear deformation theory”, *Wind Struct.*, **28**(1), 19-30. https://doi.org/10.12989/was.2019.28.1.019.

Bourada, M., Kaci, A., Houari, M.S.A. and Tounsi, A. (2015), “A new simple shear and normal deformations theory for functionally graded beams”, *Steel Compos. Struct.*, **18**(2), 409-423. https://doi.org/10.12989/scs.2015.18.2.409.

Boussoula *et al.* (2020), “A simple nth-order shear deformation theory for thermomechanical bending analysis of different configurations of FG sandwich plates”, *Smart Structures Systems.* (Accepted)

Boutaleb, S., Benrahou, K.H., Bakora, A., Algarni, A., Bousahla, A.A., Tounsi, A., Tounsi, A. and Mahmoud, S.R. (2019), “Dynamic Analysis of nanosize FG rectangular plates based on simple nonlocal quasi 3D HSDT”, *Adv. Nano Res.*, **7**(3), 189-206. https://doi.org/10.12989/anr.2019.7.3.191.

Chaabane, L.A., Bourada, F., Sekkal, M., Zerouati, S., Zaoui, F.Z., Tounsi, A., Derras, A., Bousahla, A.A, and Tounsi, A. (2019), “Analytical study of bending and free vibration responses of functionally graded beams resting on elastic foundation”, *Struct. Eng. Mech.*, **71**(2), 185-196. https://doi.org/10.12989/sem.2019.71.2.185.

Draiche, K., Bousahla, A.A., Tounsi, A., Alwabli, A.S., Tounsi, A. and Mahmoud, S.R. (2019), “Static analysis of laminated reinforced composite plates using a simple first-order shear deformation theory”, *Comput. Concrete*, **24**(4), 369-378. https://doi.org/10.12989/cac.2019.24.4.369.

Draoui, A., Zidour, M., Tounsi, A. and Adim, B. (2019), “Static and dynamic behavior of nanotubes-reinforced sandwich plates using (FSDT)”, *J. Nano Res.*, **57**, 117-135. https://doi.org/10.4028/www.scientific.net/JNanoR.57.117.

Gasik, M. (1995), “Scand. Ch226”, *Acta Polytech*, **72**.

Gasik, M.M. (1998), “Micromechanical modeling of functionally graded materials”, *Comput. Mater. Sci.*, **13**, 42-55. https://doi.org/10.1007/0-306-46935-9_13.

Hadji, L., Khelifa, Z. and Bedia, E.A.A. (2016), “A new higher order shear deformation model for functionally graded beams”, *KSCE J. Civil Eng.*, **20**(5), 1835-1841. https://doi.org/10.1007/s12205-015-0252-0.

Hadji, L., Zouatnia, N. and Bernard, F. (2019), “An analytical solution for bending and free vibration responses of functionally graded beams with porosities: Effect of the micromechanical models”, *Struct. Eng. Mech.*, **69**(2), 231-241. https://doi.org/10.12989/sem.2019.69.2.231.

Hellal, H., Bourada, M., Hebali, H., Bourda, F., Tounsi, A., Bousahla, A.A. and Mahmour, S.R. (2019), “Dynamic and stability analysis of functionally graded material sandwich plates in hygro-thermal environment using a simple higher shear deformation theory”, *J. Sandw. Struct. Mater.*, 1099636219845841. https://doi.org/10.1177/1099636219845841.

Hussain, M., Naeem, M.N., Tounsi, A. and Taj, M. (2019), “Nonlocal effect on the vibration of armchair and zigzag SWCNTs with bending rigidity”, *Adv. Nano Res.*, **7**(6), 431-442.

https://doi.org/10.12989/anr.2019.7.6.431.

Jaesang, Y. and Addis, K. (2014), "Modeling functionally graded materials containing multiple heterogeneities", *Acta Mechanica*, **225**(7), 1931-1943. https://doi.org/10.1007/s00707-013-1033-9.

Ju, J. and Chen, T.M. (1994), "Micromechanics and effective moduli of elastic composites containing randomly dispersed ellipsoidal inhomogeneities", *Acta Mechanica*, **103**(1-4), 103-121. https://doi.org/10.1007/BF01180221.

Karami, B., Janghorban, M. and Tounsi, A. (2019a), "Galerkin's approach for buckling analysis of functionally graded anisotropic nanoplates/different boundary conditions", *Eng. Comput.*, **35**, 1297-1316. https://doi.org/10.1007/s00366-018-0664-9.

Karami, B., Janghorban, M. and Tounsi, A. (2019c), "On exact wave propagation analysis of triclinic material using three dimensional bi-Helmholtz gradient plate model", *Struct. Eng. Mech.*, **69**(5), 487-497. https://doi.org/10.12989/sem.2019.69.5.487.

Karami, B., Shahsavari, D., Janghorban, M. and Tounsi, A. (2019b), "Resonance behavior of functionally graded polymer composite nanoplates reinforced with grapheme nanoplatelets", *Int. J. Mech. Sci.*, **156**, 94-105. https://doi.org/10.1016/j.ijmecsci.2019.03.036.

Khiloun, M., Bousahla, A.A., Kaci, A., Bessaim, A., Tounsi, A. and Mahmoud, S.R. (2019), "Analytical modeling of bending and vibration of thick advanced composite plates using a four-variable quasi 3D HSDT", *Eng. Comput.*, **36**, 807-821. https://doi.org/10.1007/s00366-019-00732-1.

Kitipornchai, S., Yang, J. and Liew, K.M. (2006), "Random vibration of the functionally graded laminates in thermal environments", *Comput. Meth. Appl. Mech. Eng.*, **195**, 1075-1095. https://doi.org/10.1016/j.cma.2005.01.016.

Koizumi, M. (1992), "The concept of FGM", *Proceedings of the Second International Symposium on Functionally Gradient Materials at the Third International Ceramic Science and Technology Congress*.

Li, S.R. and Liu, P. (2010), "Analogous transformation of static and dynamic solutions between functionally graded material and uniform beams", *Mech. Eng*., **32**(5), 45-49. (in Chinese)

Li, X.F. (2008), "A unified approach for analyzing static and dynamic behaviors of functionally graded Timoshenko and Euler–Bernoulli beams", *J. Sound Vib.*, **318**, 1210-1229. https://doi.org/10.1016/j.jsv.2008.04.056.

Mahmoud, S.R. and Tounsi, A. (2019), "On the stability of isotropic and composite thick plates", *Steel Compos. Struct.*, **33**(4), 551-568. https://doi.org/10.12989/scs.2019.33.4.551.

Mahmoudi, A., Benyoucef, S., Tounsi, A., Benachour, A. and Adda Bedia, E.A. (2018), "On the effect of the micromechanical models on the free vibration of rectangular FGM plate resting on elastic foundation", *Earthq. Struct.*, **14**(2), 117-128. https://doi.org/10.12989/eas.2018.14.2.117.

Mahmoudi, A., Benyoucef, S., Tounsi, A., Benachour, A., Adda Bedia, E.A. and Mahmoud, S.R. (2019), "A refined quasi-3D shear deformation theory for thermo-mechanical behavior of functionally graded sandwich plates on elastic foundations", *J. Sandw. Struct. Mater.*, **21**(6), 1906-1926. https://doi.org/10.1177/1099636217727577.

Medani, M., Benahmed, A., Zidour, M., Heireche, H., Tounsi, A., Bousahla, A.A., Tounsi, A. and Mahmoud, S.R. (2019), "Static and dynamic behavior of (FG-CNT) reinforced porous sandwich plate", *Steel Compos. Struct*., **32**(5), 595-610. https://doi.org/10.12989/scs.2019.32.5.595.

Meksi, R., Benyoucef, S., Mahmoudi, A., Tounsi, A., Adda Bedia, E.A. and Mahmoud, S.R. (2019), "An analytical solution for bending, buckling and vibration responses of FGM sandwich plates", *J. Sandw. Struct. Mater*., **21**(2), 727-757. https://doi.org/10.1177/1099636217698443.

Mishnaevsky, J.L. (2007), *Computational Mesomechanics of Composites*, John Wiley & Sons, U.K.

Ould Larbi, L., Kaci, A., Houari, M.S.A. and Tounsi, A. (2013), "An efficient shear deformation beam theory based on neutral surface position for bending and free vibration of functionally graded beams", *Mech. Bas. Des. Struct. Mach*., **41**, 421-433. https://doi.org/10.1080/15397734.2013.763713.

Sahla, M., Saidi, H., Draiche, K., Bousahla, A.A., Bourada, F. and Tounsi, A. (2019), "Free vibration analysis of angle-ply laminated composite and soft core sandwich plates", *Steel Compos. Struct*., **33**(5), 663-679. https://doi.org/10.12989/scs.2019.33.5.663.

Sayyad, A.S. and Ghugal, Y.M. (2015), "On the free vibration analysis of laminated composite and sandwich plates: A review of recent literature with some numerical results", *Compos. Struct.*, **129**, 177-201. https://doi.org/10.1016/j.compstruct.2015.04.007.

Sayyad, A.S. and Ghugal, Y.M. (2017a), "Bending, buckling and free vibration of laminated composite and sandwich beams: A critical review of literature", *Compos. Struct.*, **171**, 484-504. https://doi.org/10.1016/j.compstruct.2017.03.053.

Sayyad, A.S. and Ghugal, Y.M. (2017b), "A unified shear deformation theory for the bending of isotropic, functionally graded, laminated and sandwich beams and plates", *Int. J. Appl. Mech.*, **9**(1), 1-36. https://doi.org/10.1142/S1758825117500077.

Semmah, A., Heireche, H., Bousahla, A.A. and Tounsi, A. (2019), "Thermal buckling analysis of SWBNNT on Winkler foundation by non local FSDT", *Adv. Nano Res.*, **7**(2), 89-98. ttps://doi.org/10.12989/anr.2019.7.2.089.

Shen, H.S. and Wang, Z.X. (2012), "Assessment of Voigt and Mori–Tanaka models for vibration analysis of functionally graded plates", *Compos. Struct.*, **94**(7), 2197-2208. https://doi.org/10.1016/j.compstruct.2012.02.018.

Simsek, M. (2010), "Fundamental frequency analysis of functionally graded beams by using different higher-order beam theories", *Nucl. Eng. Des.*, **240**, 697-705. https://doi.org/10.1016/j.nucengdes.2009.12.013.

Thai, H.T. and Vo, T.P. (2012), "Bending and free vibration of functionally graded beams using various higher-order shear deformation beam theories", *Int. J. Mech. Sci.*, **62**(1), 57-66. https://doi.org/10.1016/j.ijmecsci.2012.05.014.

Tlidji, Y., Zidour, M., Draiche, K., Safa, A., Bourada, M., Tounsi, A., Bousahla, A.A. and Mahmoud, S.R. (2019), "Vibration analysis of different material distributions of functionally graded microbeam", *Struct. Eng. Mech.*, **69**(6), 637-649. https://doi.org/10.12989/sem.2019.69.6.637.

Trinh, L.C., Vo, T.P., Thai, H.T. and Nguyen, T.K. (2016), "An analytical method for the vibration and buckling of functionally graded beams under mechanical and thermal loads", *Compos. Part B*, **100**, 152-163. https://doi.org/10.1016/j.compositesb.2016.06.067.

Zaoui, F.Z., Ouinas, D. and Tounsi, A. (2019), "New 2D and quasi-3D shear deformation theories for free vibration of functionally graded plates on elastic foundations", *Compos. Part B*, **159**, 231-247. https://doi.org/10.1016/j.compositesb.2018.09.051.

Zarga, D., Tounsi, A., Bousahla, A.A., Bourada, F. and Mahmoud, S.R. (2019), "Thermomechanical bending study for functionally graded sandwich plates using a simple quasi-3D shear deformation theory", *Steel Compos. Struct.*, **32**(3), 389-410. https://doi.org/10.12989/scs.2019.32.3.389.

Zimmerman, R.W. (1994), "Behavior of the Poisson ratio of a two-phase composite material in the high-concentration limit", *Appl. Mech. Rev.*, **47**(1), 38-44. https://doi.org/10.1115/1.3122819.

CC