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Time harmonic interactions in fractional thermoelastic diffusive thick circular plate

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Abstract. Here in this investigation, a two-dimensional thermoelastic problem of thick circular plate of finite thickness under fractional order theory of thermoelastic diffusion has been considered in frequency domain. The effect of frequency in the axisymmetric thick circular plate has been depicted. The upper and lower surfaces of the thick plate are traction free and subjected to an axisymmetric heat supply. The solution is found by using Hankel transform techniques. The analytical expressions of displacements, stresses and chemical potential, temperature change and mass concentration are computed in transformed domain. Numerical inversion technique has been applied to obtain the results in the physical domain. Numerically simulated results are depicted graphically. The effect frequency has been shown on the various components.

Keywords: fractional order; isotropic thermoelastic; frequency; hankel transform; plane axisymmetric; diffusion

1. Introduction

The use of fractional order derivatives and integrals leads to the formulation of certain physical problems which is more economical and useful than the classical approach. There exist many material and physical situations like amorphous media, colloids, glassy and porous materials, manmade and biological materials/polymers, transient loading etc., where the classical thermoelasticity based on Fourier type heat conduction breaks down. In such cases, one needs to use a generalized thermoelasticity theory based on an anomalous heat conduction model involving time fractional (non- integer order) derivatives.

Diffusion is defined as the spontaneous movement of the particles from high concentration region to the low concentration region, and it occurs in response to a concentration gradient expressed as the change in concentration due to change in position. Thermal diffusion utilizes the transfer of heat across a thin liquid or gas to accomplish isotope separation. The thermodiffusion in elastic solids is due to coupling of fields of temperature, mass diffusion and that of strain in addition to heat and mass exchange with the environment.

Povstenko (2005) proposed a quasi-static uncoupled theory of thermoelasticity based on the heat conduction equation with a time-fractional derivative of order α. Because the heat conduction equation in the case $1 \le \alpha \le 2$ interpolates the parabolic equation (α=1) and the wave equation (α=2),

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this theory interpolates a classical thermoelasticity and a thermoelasticity without energy dissipation. He also obtained the stresses corresponding to the fundamental solutions of a Cauchy problem for the fractional heat conduction equation for one-dimensional and two-dimensional cases.

Povstenko (2009) investigated the nonlocal generalizations of the Fourier law and heat conduction by using time and space fractional derivatives. Youssef (2010) proposed a new model of thermoelasticity theory in the context of a new consideration of heat conduction with fractional order and proved the uniqueness theorem. Jiang and Xu (2010) obtained a fractional heat conduction equation with a time fractional derivative in the general orthogonal curvilinear coordinate and also in other orthogonal coordinate system. Povstenko (2010) investigated the fractional radial heat conduction in an infinite medium with a cylindrical cavity and associated thermal stresses.

Ezzat (2011a) constructed a new model of the magneto-thermoelasticity theory in the context of a new consideration of heat conduction with fractional derivative. Ezzat (2011b) studied the problem of state space approach to thermoelectric fluid with fractional order heat transfer. The Laplace transform and state-space techniques were used to solve a one-dimensional application for a conducting half space of thermoelectric elastic material. Povstenko (2011) investigated the generalized Cattaneo-type equations with time fractional derivatives and formulated the theory of thermal stresses. Biswas and Sen (2011) proposed a scheme for optimal control and a pseudo state space representation for a particular type of fractional dynamical equation. Ezzat and Ezzat (2016) constructed fractional thermoelasticity applications for porous asphaltic materials. Several researchers (Ezzat and Bary 2016, Marin and Oechsner 2017, Marin 2013, 1997, Marin *et al.* 2013, Kumar *et al*. 2016b, 2016a, 2017), Lata 2018, 2018a, Mahmoud 2016) presented modelling of magneto-thermoelasticity for perfect conducting materials.

Ying and Yun (2015) built a fractional dual-phase-lag model and the corresponding bio-heat transfer equation. Tripathi *et al*. (2015) analysed generalized thermoelastic diffusion problem in a thick circular plate with axisymmetric heat supply. Many active researchers worked and contributed in this area Abbas and Kumar (2015), Abbas *et al.* (2015), Abbas *et al*. (2015), Zenkour and Abbas (2014). Xiong and Niu (2017) established fractional order generalized thermoelastic diffusion theory for anisotropic and linearly thermoelastic diffusive media. Kumar and Sharma (2017) studied the effect of fractional order on energy ratios at the boundary surface of piezothermoelastic medium. Tripathi *et al.* (2018) studied fractional order generalized thermoelastic response in a half space due to a periodically varying heat source.

Here in this investigation, an axisymmetric thick circular plate under fractional order theory of thermoelastic diffusion has been examined in frequency domain. The upper and lower surfaces of the thick plate are traction free and subjected to an axisymmetric heat supply. The solution is found by using Hankel transform techniques. The components of displacements, stresses and chemical potential, temperature change and mass concentration are computed numerically. Numerically computed results are depicted graphically. The effect of frequency has been shown on the various components.

2. Basic equations

Following Ezzat and Fayik (2014), the basic equations of motion, heat conduction and mass diffusion using the fractional order theory of thermoelastic diffusion in a homogeneous isotropic thermoelastic solid in the absence of body forces, heat sources and mass diffusion sources are

$$(\lambda+\mu)\nabla(\nabla.u)+\mu\nabla^2 u-\beta_1\nabla T-\beta_2\nabla C=\rho\ddot{u}, \tag{1}$$

$$KT_{,ii}=\left(1+\frac{(\tau_0)^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)[\rho C_E\dot{T}+\beta_1 T_0\dot{e}_{kk}+aT_0\dot{C}], \tag{2}$$

$$(D\beta_2\nabla^2(\nabla.u)+Da\nabla^2 T-Db\nabla^2 C)+\frac{\partial}{\partial t}\left(1+\frac{(\tau)^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)C=0, \tag{3}$$

and the constitutive relations are

$$\sigma_{ij}=2\mu e_{ij}+\delta_{ij}(\lambda e_{kk}-\beta_1 T-\beta_2 C), \tag{4}$$

$$\rho T_0 S=\left(1+\frac{(\tau_0)^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)(\rho C_E T+\beta_1 T_0 e_{kk}+aT_0 C), \tag{5}$$

$$P=-\beta_2 e_{kk}-aT-bC. \tag{6}$$

Following Caputo (1967), the fractional derivative of order $\alpha\in(0,1]$ of the absolutely continuous function *f*(*t*) is

$$\frac{d^\alpha}{dt^\alpha}f(t)=I^{1-\alpha}f'(t), \tag{7}$$

and the fractional integral

$$I^\alpha f(t)=\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau)d\tau,\ \alpha>0. \tag{8}$$

where I^α is the fractional integral of the function $f(t)$ of order α defined by Miller and Ross (1993), $\frac{d^\alpha}{dt^\alpha}$ represents the derivative of order α, $f(t)$ is any well defined continuous function of variable t, $\beta_1=(3\lambda+2\mu)\alpha_t$, $\beta_2=(3\lambda+2\mu)\alpha_c$, α_c is the coefficient of linear diffusion expansion and α_t is the coefficient of thermal linear expansion. In above equations, a comma followed by suffix denotes spatial derivative and a superposed dot denotes derivative with respect to time.

3. Formulation and solution of the problem

Consider a thick circular plate of thickness $2d$ occupying the space D defined by $0\le r\le\infty$, $-d\le z\le d$. Let the plate be subjected to an axisymmetric heat supply and chemical potential source with stress free boundary depending on the radial and axial directions of the cylindrical co-ordinate system. The initial temperature in the thick plate is given by a constant temperature T_0. The heat flux and chemical potential sources of unit magnitude are prescribed along with vanishing of stress components on the upper and lower boundary surfaces along with traction free boundary $z=\pm d$. We take a cylindrical polar co-ordinate system (r,θ,z) with symmetry about z –axis. As the problem considered is plane axisymmetric, the field component $u_\theta=0$, and u_r,u_z,T and C are independent of θ. The components of displacement vector $\vec{u}$ for the two-dimensional axisymmetric problem take the form

$$\vec{u}=(u_r,0,u_z), \tag{9}$$

Eqs. (1)-(6) with the aid of (9) take the form

$$(\lambda+\mu)\frac{\partial e}{\partial r}+\mu\left(\nabla^2-\frac{1}{r^2}\right)u_r-\beta_1\frac{\partial T}{\partial r}-\beta_2\frac{\partial C}{\partial r}=\rho\frac{\partial^2 u_r}{\partial t^2}\,, \tag{10}$$

$$(\lambda+\mu)\frac{\partial e}{\partial z}+\mu\nabla^2 u_z-\beta_1\frac{\partial T}{\partial z}-\beta_2\frac{\partial C}{\partial z}=\rho\frac{\partial^2 u_z}{\partial t^2}\,, \tag{11}$$

$$\mathrm{K}\nabla^2 T=\left(1+\frac{(\tau_0)^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)[\rho C_E\dot{T}+\beta_1 T_0\frac{\partial}{\partial t}div\, u+aT_0\frac{\partial C}{\partial t}]\,, \tag{12}$$

$$(D\beta_2\nabla^2 div\, u+Da\nabla^2 T-Db\nabla^2 C)+\frac{\partial}{\partial t}\left(1+\frac{(\tau)^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)C=0\,. \tag{13}$$

We define the dimensionless quantities

$$r'=\frac{\omega_1}{c_1}r\,,\ z'=\frac{\omega_1}{c_1}z\,,\ (u_r',u_z')=\frac{\omega_1}{c_1}(u_r,u_z),\ t'=\omega_1 t\,,\ (\sigma_{rr}',\sigma_{\theta\theta}',\sigma_{zz}',\sigma_{rz}')=\frac{1}{\beta_1 T_0}(\sigma_{rr},\sigma_{\theta\theta},\sigma_{zz},\sigma_{rz}),\ T'=\frac{\beta_1}{\rho c_1^2}T,\ C'=\frac{\beta_2}{\rho c_1^2}C,\ \tau_0'=\omega_1\tau_0\,,\ \tau'=\omega_1\tau\,,\ P'=\frac{P}{\beta_2}. \tag{14}$$

where

$$\omega_1=\frac{\rho C_E c_1^2}{K}\,,\ c_1^2=\frac{\lambda+2\mu}{\rho}\,.$$

Using the dimensionless quantities defined by (14) in the Eqs. (10)-(13) and suppressing the primes for convenience yield

$$\frac{(\lambda+\mu)}{\rho c_1^2}\frac{\partial e}{\partial r}+\frac{\mu}{\rho c_1^2}\left(\nabla^2-\frac{1}{r^2}\right)u_r-\frac{\partial T}{\partial r}-\frac{\partial C}{\partial r}=\frac{\partial^2 u_r}{\partial t^2}, \tag{15}$$

$$\frac{\mu}{\rho c_1^2}\frac{\partial e}{\partial z}+\frac{\mu}{\rho c_1^2}\nabla^2 u_z-\frac{\partial T}{\partial z}-\frac{\partial C}{\partial z}=\frac{\partial^2 u_z}{\partial t^2}\,, \tag{16}$$

$$K\nabla^2 T=\left(1+\tau_0\frac{\partial}{\partial t}\right)\frac{1}{\omega_1}\left[\rho C_E c_1^2\dot{T}+\frac{{\beta_1}^2 T_0}{\rho}\frac{\partial}{\partial t}div\, u+\frac{aT_0\beta_1 c_1^2}{\beta_2}\frac{\partial C}{\partial t}\right], \tag{17}$$

$$\left(D\beta_2\nabla^2 div\, u+Da\nabla^2 T\frac{\rho c_1^2}{\beta_1}-Db\nabla^2 C\frac{\rho c_1^2}{\beta_1}\right)+\frac{\partial}{\partial t}\left(1+\tau\frac{\partial}{\partial t}\right)\frac{\rho c_1^4}{\beta_1\omega_1}C=0\,. \tag{18}$$

Using (4), (6) and (14), the stress components and Chemical potential source in dimensionless form are

$$\sigma_{rr}=\mu^1\frac{\partial u_r}{\partial r}+\lambda^1 e-\frac{\rho c_1^2}{\beta_1^2 T_0}T-\frac{\beta_2\rho c_1^2}{\beta_1 T_0}C, \tag{19}$$

$$\sigma_{\theta\theta}=\mu^1\frac{u_r}{r}+\lambda^1 e-\frac{\rho c_1^2}{\beta_1^2 T_0}T-\frac{\beta_2\rho c_1^2}{\beta_1 T_0}C\,, \tag{20}$$

$$\sigma_{zz}=\mu^1\frac{\partial u_z}{\partial z}+\lambda^1 e-\frac{\rho c_1^2}{\beta_1^2 T_0}T-\frac{\beta_2\rho c_1^2}{\beta_1 T_0}C, \tag{21}$$

$$\sigma_{rz} = \frac{\mu^1}{4}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), \tag{22}$$

$$\sigma_{r\theta} = 0 = \sigma_{z\theta} \ , \tag{23}$$

$$P = -e - \frac{a\rho c_1^2}{\beta_2\beta_1}T - \frac{b\rho c_1^2}{\beta_2^2}C, \tag{24}$$

where

$$\mu^1 = \frac{2\mu}{\beta_1 T_0}, \lambda^1 = \frac{\lambda}{\beta_1 T_0}.$$

Assuming the harmonic behaviour as

$$(u_r, u_z, \varphi, T, e)(r, z, t) = (u_r, u_z, \varphi, T, e)(r, z)e^{i\omega t}. \tag{25}$$

where ω is the angular frequency

Following Debnath (1995), the Hankel transform of order n of $\bar{f}\ (r, z, \omega)$ with respect to the variable r is defined by

$$H(\bar{f}(r, z, \omega)) = \bar{f}^*\ (\xi, z, s) = \int_0^\infty \bar{f}(r, z, \omega) r J_n(r\xi)\, dr, \tag{26}$$

Using (25)-(26) with application on the Eqs. (15)-(18) and eliminating $\bar{T}^*$, $\bar{C}^*$ and $\bar{e}^*$, we obtain

$$(M\frac{d^6}{dz^6} + Q\frac{d^4}{dz^4} + R\frac{d^2}{dz^2} + S)(\ \bar{T}\ , \bar{C}, \bar{e}) = 0\ , \tag{27}$$

where

$$M = \frac{Db\rho c_1^2}{\beta_1} - D\beta_2\ ,$$

$$Q = Q' - 3\xi^2\ , Q' = \tau_0^1\left(\frac{KaT_0D\beta_2}{\rho C_E\beta_1} + \frac{KbT_0D\beta_1}{\rho C_E} + Ki\omega D\beta_2 - \frac{Db\rho c_1^2}{\beta_1}\right) - \frac{i\omega K\tau}{C_E}\ ,$$

$$R = 3P\xi^4 - 2Q'\xi^2 + R'\ , R' = K\tau_0^1\left(\frac{-K\beta_1^2T_0\tau}{\rho^2 {C_E}^2} + \frac{Db\rho c_1^2(i\omega)^3}{\beta_1} - \frac{D\beta_2 aT_0c_1^2\omega^2}{C_E{\beta_1}^2}\right) - \frac{K\tau c_1^2\omega^2(K+i\omega)}{C_E}\ ,$$

$$S = -P^{\xi^6} + Q'^{\xi^4} - R'^{\xi^2} - S'\ , S' = \frac{-i\omega\ K\ \tau^1 c_1^4 D\beta_2 a\rho}{C_E\beta_1},$$

$${\tau_0}^1 = 1 + (i\omega)^\alpha\tau_0\ , \tau^1 = 1 + (i\omega)^\alpha\tau.$$

The solution of Eq. (27) is assumed of the form

$$\bar{T}^* = \sum_{i=1}^{3} A_i\cosh(q_i z), \tag{28}$$

$$\bar{C}^* = \sum_{i=1}^{3} d_i A_i \cosh(q_i z)\,, \tag{29}$$

$$\bar{e}^* = \sum_{i=1}^{3} f_i A_i\cosh(q_i z), \tag{30}$$

where q_i (i=1,2,3) are the roots of (27)
and the coupling constants d_i and f_i are given by

$$d_i = \frac{\zeta_{10}q_i^4 + q_i^2\zeta_{10}(-2\xi^2 - \zeta_{14} + \zeta_{13}) + \zeta_{10}(\xi^4 + \zeta_{14}i\omega\xi^2 - \zeta_{15})}{(-q_i^2 + \xi^2)(\zeta_{11} + \zeta_{12}) + \zeta_{12}},$$

$$f_i = \frac{\zeta_{16}q_i^4 + (q_i^2 - \xi^2)(-\zeta_{16}\zeta_{14} + \zeta_{13}\zeta_{17} + \zeta_{18}) - \zeta_{14}\zeta_{18}}{(-q_i^2 + \xi^2)(\zeta_{11} + \zeta_{12}) + \zeta_{12}},$$

where

$$\zeta_{11} = \frac{K a T_0 \tau_0{}^1\, D\beta_2}{\rho C_E \beta_1}, \zeta_{12} = \frac{K {\beta_1}^2 T_0}{\rho^2 c_E^2}\frac{\tau_0{}^1 i\omega K \tau^1}{}s\,,\ \zeta_{13} = \frac{K T_0 \tau_0{}^1 a\rho}{\rho^2 C_E}, \zeta_{14} = \tau_0{}^1 K i\omega\,,$$

$$\zeta_{15} = \frac{K\beta_1 T_0 \tau_0{}^1 \xi^2}{\rho C_E}\ \zeta_{16} = \frac{-Db\rho c_1^2}{\beta_1}\ , \zeta_{17} = \frac{D\beta_2 c_1^2}{{\beta_1}^2}, \zeta_{18} = \frac{i\omega\tau^1\, k c_1^2}{\beta_1 C_E}, \zeta_{10} = D\beta_2\,.$$

4. Boundary conditions

We consider a thermal source and chemical potential source along with vanishing of stress components at the stress free surface at $z = \pm d$. Mathematically, these can be written as

$$\frac{\partial T}{\partial z} = \pm g_0 F(r,z), \tag{31}$$

$$\sigma_{zz} = 0, \tag{32}$$

$$\sigma_{rz} = 0, \tag{33}$$

$$P = f(r,t), \tag{34}$$

Using the dimensionless quantities defined by (12) in the boundary conditions (31)-(34), and using (25)-(26) on the resulting quantities, and substituting the values of $\bar{T},\ \overline{\sigma_{zz}}, \overline{\sigma_{rz}}$ and $\bar{P}$, yields

$$\textstyle\sum_{i=1}^{3} A_i \cosh(q_i z) = g_0 \bar{F}(\xi, d), \tag{35}$$

$$\mu^1 A q sinh(qz) + \textstyle\sum_{i=1}^{3} \gamma_i A_i \cosh(q_i z) = 0, \tag{36}$$

$$\frac{\mu^1}{2} A sinh(qz) + \textstyle\sum_{i=1}^{3} \alpha_i A_i \sinh(q_i z) = 0, \tag{37}$$

$$\textstyle\sum_{i=1}^{3} \nu_i A_i \cosh(q_i z) = \bar{f}(\xi), \tag{38}$$

Solving the system of Eqs. (35)-(38) to obtain the values of A, A_i, i=1,2,3 with the help of (31)-(34) and substituting the values A, A_i, i=1,2,3 in (28)-(30) and upon simplification of (15)-(18) yield the components of displacement, stress components, chemical potential function, temperature change, mass concentration and cubic dilatation as

$$\begin{aligned}\bar{u}_r^* = {} & \frac{g_0 \bar{F}(\xi\ , d)}{\Delta}\left(\frac{\eta_1}{m_1}\Lambda_1\vartheta_1 - \frac{\eta_2}{m_2}\Lambda_2\vartheta_2 + \frac{\eta_3}{m_3}\Lambda_3\vartheta_3 - \Lambda_4\vartheta\right) \\ & - \frac{\bar{f}(\xi)}{\Delta}\left(\frac{\eta_1}{m_1}\Lambda^1\vartheta_1 - \frac{\eta_2}{m_2}\Lambda^2\vartheta_2 + \frac{\eta_3}{m_3}\Lambda^3\vartheta_3 - \Lambda^4\vartheta\right),\end{aligned} \tag{39}$$

$$\bar{u}_z^* = \frac{g_0\bar{F}(\xi\,,d)}{\Delta}\left(\frac{q_1\mu_1}{m_1}\Lambda_1\theta_1 - \frac{q_2\mu_2}{m_2}\Lambda_2\theta_2 + \frac{q_3\mu_3}{m_3}\Lambda_3\theta_3 - \Lambda_4\theta\right) - \frac{\bar{f}(\xi,\omega)}{\Delta}\left(\frac{q_1\mu_1}{m_1}\Lambda^1\theta_1 - \frac{q_2\mu_2}{m_2}\Lambda^2\theta_2 + \frac{q_3\mu_3}{m_3}\Lambda^3\theta_3 - \Lambda^4\theta\right), \tag{40}$$

$$\bar{\sigma}_{zz}^* = \frac{g_0\bar{F}(\xi\,,d)}{\Delta}(\gamma_1\Lambda_1\vartheta_1 - \gamma_2\Lambda_2\vartheta_2 + \gamma_3\Lambda_3\vartheta_3 - 2\mu q\Lambda_4\theta) - \frac{\bar{f}(\xi,\omega)}{\Delta}(\gamma_1\Lambda^1\vartheta_1 - \gamma_2\Lambda^2\vartheta_2 + \gamma_3\Lambda^3\vartheta_3 - 2\mu q\Lambda^4\vartheta), \tag{41}$$

$$\bar{\sigma}_{rz}^* = \frac{g_0\bar{F}(\xi\,,d)}{\Delta}\left(\alpha_1\Lambda_1\theta_1 - \alpha_2\Lambda_2\theta_2 + \alpha_3\Lambda_3\theta_3 - \mu(q\;\xi)\,\Lambda_4\frac{\theta}{2}\right) - \frac{\bar{f}(\xi,\omega)}{\Delta}\left(\alpha_1\Lambda^1\theta_1 - \alpha_2\Lambda^2\theta_2 + \alpha_3\Lambda^3\theta_3 - \mu(q+\xi)\Lambda^4\frac{\theta}{2}\right), \tag{42}$$

$$\bar{\sigma}_{\theta\theta}^* = \frac{g_0\bar{F}(\xi\,,d)}{\Delta}(\zeta_1\Lambda_1\vartheta_1 - \zeta_2\Lambda_2\vartheta_2 + \zeta_3\Lambda_3\vartheta_3 - 2\mu\xi\Lambda_4\vartheta) - \frac{\bar{f}(\xi,\omega)}{\Delta}(\zeta_1\Lambda^1\vartheta_1 - \zeta_2\Lambda^2\vartheta_2 + \zeta_3\Lambda^3\vartheta_3 - 2\mu\xi\Lambda^4\vartheta), \tag{43}$$

$$\bar{P}^* = \frac{g_0\bar{F}(\xi\,,d)}{\Delta}(\nu_1\Lambda_1\vartheta_1 - \nu_2\Lambda_2\vartheta_2 + \nu_3\Lambda_3\vartheta_3) - \frac{\bar{f}(\xi,\omega)}{\Delta}(\nu_1\Lambda^1\vartheta_1 - \nu_2\Lambda^2\vartheta_1 + \nu_3\Lambda^3\vartheta_3), \tag{44}$$

$$\bar{T}^* = \frac{g_0\bar{F}(\xi\,,d)}{\Delta}(\Lambda_1\vartheta_1 - \Lambda_2\vartheta_2 + \Lambda_3\vartheta_3) - \frac{\bar{f}(\xi,\omega)}{\Delta}(\Lambda^1\vartheta_1 - \Lambda^2\vartheta_2 + \Lambda^3\vartheta_3), \tag{45}$$

$$\bar{C}^* = \frac{g_0\bar{F}(\xi\,,d)}{\Delta}(d_1\Lambda_1\vartheta_1 - d_2\Lambda_2\vartheta_2 + d_3\Lambda_3\vartheta_3) - \frac{\bar{f}(\xi,\omega)}{\Delta}(d_1\Lambda^1\vartheta_1 - d_2\Lambda^2\vartheta_2 + d_3\Lambda^3\vartheta_3), \tag{46}$$

$$\bar{e}^* = \frac{g_0\bar{F}(\xi\,,d)}{\Delta}(f_1\Lambda_1\vartheta_1 - f_2\Lambda_2\vartheta_2 + f_3\Lambda_3\vartheta_3) - \frac{\bar{f}(\xi,\omega)}{\Delta}(f_1\Lambda^1\vartheta_1 - f_2\Lambda^2\vartheta_2 + f_3\Lambda^3\vartheta_3), \tag{47}$$

where

$$\Delta = \Delta_{24}\Delta_{11}(\Delta_{43}\Delta_{32} - \Delta_{33}\Delta_{42}) + \Delta_{24}\Delta_{12}(\Delta_{43}\Delta_{31} - \Delta_{41}\Delta_{33}) - \Delta_{13}\Delta_{24}(\Delta_{31}\Delta_{42} - \Delta_{32}\Delta_{41}) + \Delta_{11}\Delta_{34}(\Delta_{22}\Delta_{43} - \Delta_{23}\Delta_{42}) - \Delta_{34}\Delta_{12}(\Delta_{43}\Delta_{21} - \Delta_{41}\Delta_{23}) + \Delta_{34}\Delta_{13}(\Delta_{21}\Delta_{42} - \Delta_{22}\Delta_{41}),$$

$$\Lambda_1 = \Delta_{43}(\Delta_{24}\Delta_{32} - \Delta_{34}\Delta_{22}) + \Delta_{42}(\Delta_{23}\Delta_{34} - \Delta_{24}\Delta_{33})\,,$$

$$\Lambda^1 = \Delta_{12}(\Delta_{23}\Delta_{34} - \Delta_{24}\Delta_{33}) - \Delta_{13}(\Delta_{22}\Delta_{34} - \Delta_{24}\Delta_{32}),$$

$$\Lambda_2 = \Delta_{24}(\Delta_{31}\Delta_{43} - \Delta_{23}\Delta_{41}) - \Delta_{34}(\Delta_{21}\Delta_{43} - \Delta_{23}\Delta_{41}),$$

$$\Lambda^2 = -\Delta_{24}(\Delta_{11}\Delta_{33} - \Delta_{13}\Delta_{31}) + \Delta_{34}(\Delta_{11}\Delta_{23} - \Delta_{13}\Delta_{21}),$$

$$\Lambda_3 = \Delta_{24}(\Delta_{31}\Delta_{43} - \Delta_{32}\Delta_{41}) - \Delta_{34}(\Delta_{21}\Delta_{43} - \Delta_{22}\Delta_{41}),$$

$$\Lambda^3 = \Delta_{24}(\Delta_{11}\Delta_{32} - \Delta_{12}\Delta_{31}) - \Delta_{34}(\Delta_{11}\Delta_{22} - \Delta_{12}\Delta_{21}),$$

$$\Lambda_4 = \Delta_{21}(\Delta_{43}\Delta_{32} - \Delta_{33}\Delta_{42}) - \Delta_{22}(\Delta_{43}\Delta_{31} - \Delta_{41}\Delta_{33}) + \Delta_{23}(\Delta_{31}\Delta_{42} - \Delta_{32}\Delta_{41}),$$

$$\Lambda^4 = \Delta_{11}(\Delta_{22}\Delta_{33} - \Delta_{23}\Delta_{32}) - \Delta_{12}(\Delta_{21}\Delta_{33} - \Delta_{23}\Delta_{31}) + \Delta_{13}(\Delta_{21}\Delta_{32} - \Delta_{31}\Delta_{22}),$$
$$\Delta_{1i} = q_i \sinh(q_i d),\ i\text{=1,2,3},\ \Delta_{14} = 0,\ \Delta_{2i} = \gamma_i \cosh(q_i d)\ ,\ i\text{=1,2,3},$$
$$\Delta_{24} = 2\mu q \sinh(\mathrm{qd}),\ \Delta_{3i} = \alpha_i \sinh(q_i d)\ ,\ i\text{=1,2,3},$$
$$\Delta_{34} = \mu(q+\xi)\sinh(\mathrm{qd})/2,\ \Delta_{4i} = \nu_i \cosh(q_i d),\ i\text{=1,2,3},\ \Delta_{44} = 0.$$
$$\eta_i = \xi(-\frac{\lambda+\mu}{\rho c_1^2} f_i + 1 + d_i),$$
$$\mu_i = 1 + d_i + \mu f_i/\rho c_1^2,$$
$$m_i = \frac{\mu}{\rho c_1^2}\left(q_i^2 - \xi^2\right) - \omega^2,$$
$$\gamma_i = \frac{q_i^2 \mu_i}{m_i \beta_1 T_0} + \frac{\lambda}{\beta_1 T_0} f_i - \frac{\rho c_1^2}{\beta_1 T_0} - \frac{\rho c_1^2}{\beta_1 T_0} d_i,$$
$$\alpha_i = \frac{\eta_i q_i}{m_i} + \frac{q_i \mu_i \xi}{m_i},\ \nu_i = -f_i - a\frac{\rho c_1^2}{\beta_2 \beta_1} + \frac{b\rho c_1^2}{\beta_2^2} d_i,$$
$$\zeta_i = \frac{2\mu\xi\eta_i}{\beta_1 T_0 m_i} + \frac{\lambda f_i}{\beta_1 T_0} - \rho c_1^2 - \frac{\rho c_1^2}{\beta_1 T_0} d_i,\ q = sqrt(\xi^2 - \frac{\omega^2 \rho c_1^2}{\mu}).$$

5. Applications

As an application of the problem, we take the source functions as

$$F(r,z) = z^2 e^{-\omega r}, \tag{48}$$

$$f(r,t) = H(\propto - r)e^{i\omega t}, \tag{49}$$

where $H(\propto - r)$ is the Dirac delta function.

Applying Hankel Transform on the Eqs. (48)-(49), gives

$$\bar{f}(\xi,\omega) = \frac{\propto J_1(\xi\propto)}{\xi} e^{i\omega t}, \tag{50}$$

$$\bar{F}^*(\xi,z) = \frac{z^2\omega}{(\xi^2+\omega^2)^{3/2}}\ , \tag{51}$$

Here J_1 is the Bessel's function of first kind of order 1, the expressions of components of displacement, stress components, chemical potential function, temperature change, mass concentration and cubic dilatation can be obtained from the Eqs. (39)-(47), by substituting the value of $\bar{F}(\xi\ , d)$ and $\bar{f}(\xi,\omega)$ from (50)-(51).

6. Particular cases

(i). If we neglect the diffusion effect (i.e., $\beta_2, a, b = 0$) in the Eqs. (39)-(47), we obtain the expressions for components of displacement, stress, chemical potential functions, temperature change, mass concentration and cubic dilatation for thermoelastic isotropic half space.

(ii) If $\alpha = 0$ in the fractional heat equation and putting in Eqs. (39)-(47), the resulting expressions reduce for thermoelastic interactions in a thick circular plate frequency domain with diffusion

7. Inversion of double transform

To obtain the solution of the problem in physical domain, we must invert the transforms in Eqs. (39)-(47) These expressions are functions of ξ *and* z ,and hence are of the form $\tilde{f}(\xi, z, \omega)$. To get the function $f(r, z, \omega)$ in the physical domain, we invert the Hankel transform using

$$f(r, z, \omega) = \int_0^\infty \xi \, \bar{f}\xi, z, \omega) J_n(\xi r) d\xi \, , \tag{52}$$

The last step is to calculate the integral in Eq. (52). The method for evaluating this integral is described in Press *et al.* (1986). It involves the use of Romberg's integration with adaptive step size. This also uses the results from successive refinements of the extended trapezoidal rule followed by extrapolation of the results to the limit when the step size tends to zero.

8. Numerical results and discussion

The mathematical model is prepared using Matlab 8.4.0 with copper material for purposes of numerical computation. The material constants for the problem are taken from Youssef (2006) and are given by

$$\lambda = 7.76 \times 10^{10} Nm^{-2},\ \mu = 3.86 \times 10^{10} Nm^{-2}, K = 386 JK^{-1}m^{-1}s^{-1}, \rho = 8954\ Kgm^{-3},$$
$$\beta_1 = 5.518 \times 10^6\ Nm^{-2}deg^{-1}, \beta_2 = 61.38 \times 10^7\ Nm^{-2}deg^{-1}, a = 1.2\ \times 10^4 m^2/s^2k, b =$$
$$0.9\ \times 10^6 m^5/kgs^2, D = 0.88\ \times 10^{-8} kgs/m^3,\ , T_0 = 293\text{K}, C_E = 383.1\, Jkg^{-1}K^{-1}\ .$$

An investigation has been conducted to compare the effect of frequency and the graphs have been plotted in the range $0 \le r \le 3$, frequency values are taken as

$$\omega = .25\ , \omega = .5 \text{ and } \omega = .75$$

- Solid line with centre symbol circle corresponds to $\omega = .25$
- Small dashed line corresponds to $\omega = .5$
- Small dashed line with centre symbol diamond corresponds to $\omega = .75$.

Fig. 1 represents the variations of axial displacement u_z with respect to distance *r*. Here, in the range $0 \le r \le 1$, the values are decreasing whereas increase in the rest corresponding to three values of ω with change of amplitude.

Fig. 2 exhibits the variations of temperature change T with distance r. Here corresponding to $\omega = .5$, the variations increase in the whole range whereas for $\omega = .25$ and $\omega = .75$, the variations increase in the range $0 \le r \le 2$ and decrease in the rest.

Fig. 3 exhibits the variations of chemical potential P with distance r. Here, we notice a continuous decrease in the whole range corresponding to $\omega = .5$ whereas the pattern is oscillatory for $\omega = .25$ and $\omega = .75$ with change of amplitude.

Fig. 4 shows variations of mass concentration C with distance r. Here corresponding to $\omega = .25$ and $\omega = .75$, the values decrease in the range $0 \le r \le 2$ and increase in the rest whereas continuous decrease is noticed for $\omega = .5$.

Fig. 5 expresses the variations of vertical stress component σ_{zz} with distance r . Here we find that corresponding to $\omega = .25, \omega = .5$ and $\omega = .75$, the values decay in the whole range.

Fig. 6 shows variations of radial stress component σ_{rr} with displacement r. Here the pattern of variations is same as discussed in Fig. 4.

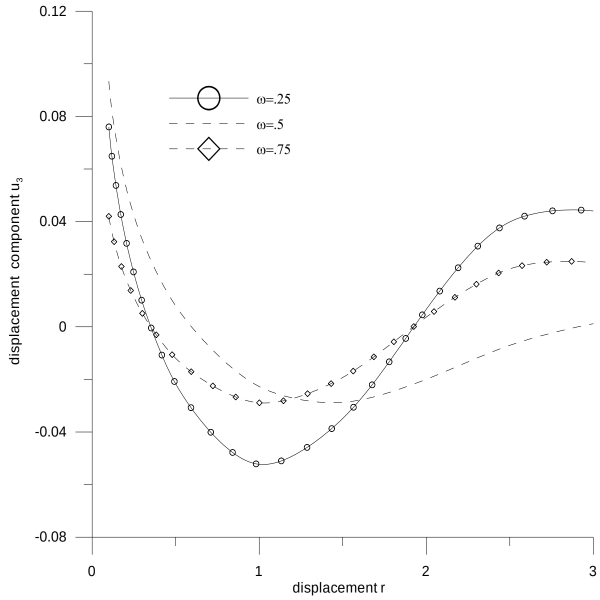

Fig. 1 Variations of axial displacement u_z with distance r

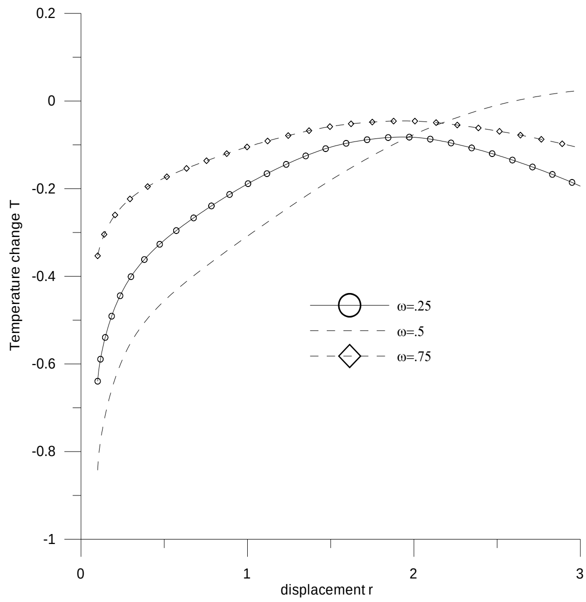

Fig. 2 Variations of temperature change T with distance r

9. Conclusions

In this paper, we depicted the effect of time harmonic sources due to axisymmetric heat supply in a thick circular plate.

• We discussed the problem within the context fractional theory of thermoelastic diffusion. The upper and lower surfaces of the plate are taken to be traction free.

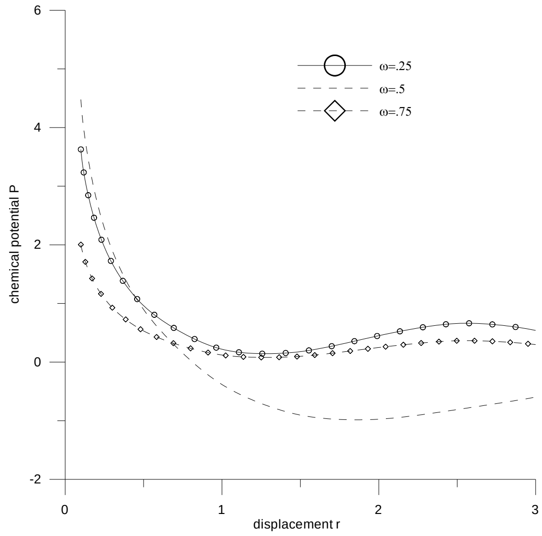

Fig. 3 Variations of chemical potential function P with distance r

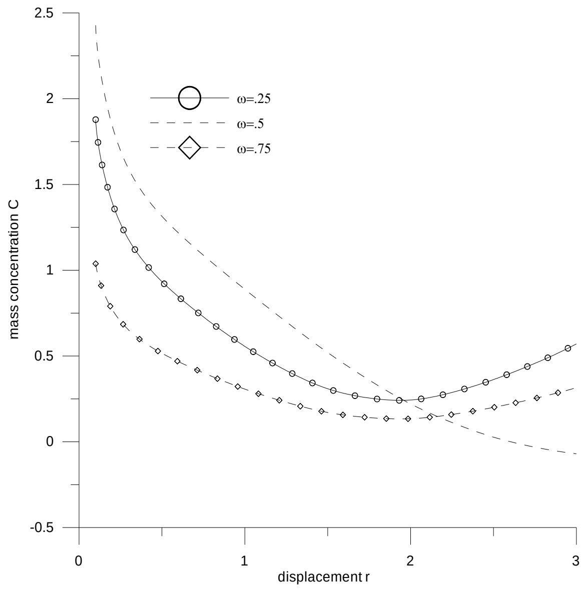

Fig. 4 Variations of mass concentration C with distance r

• We find that change in frequency changes the behaviour of deformations of the various components of stresses, displacement, chemical potential function, temperature change and mass concentration.

• Though variations being oscillatory, a big difference in the magnitudes is noticed.

• The use of fractional theory of thermoelastic diffusion gives a more realistic model of thermoelastic media as it allows a delayed response between the relative mass flux vector and the potential gradient.

• The result of the problem is useful in the two-dimensional problem of dynamic response due

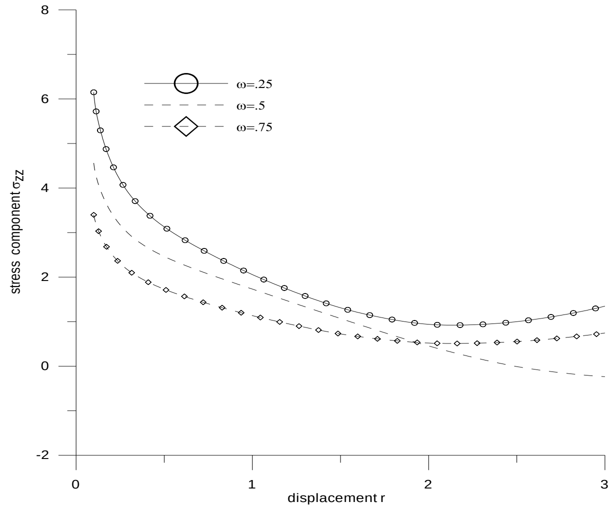

Fig. 5 Variations of vertical stress component σ_{zz} with distance r

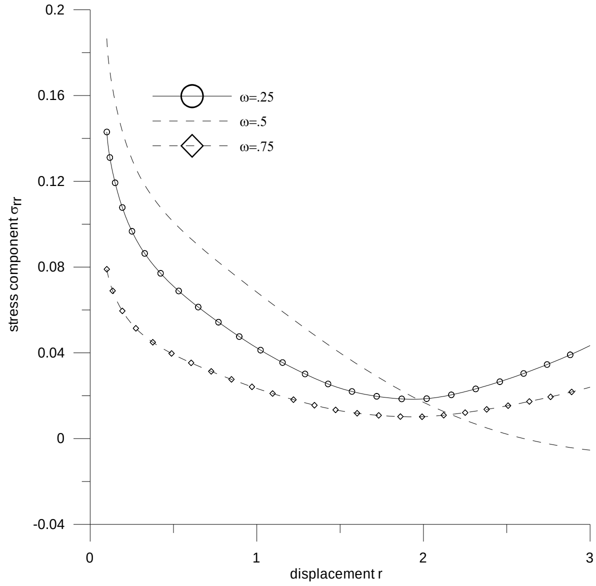

Fig. 6 Variations of radial stress component σ_{rr} with displacement r

to various sources of thermodiffusion which has various Geophysical and industrial applications.

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CC

Nomenclature

α	is the order of fractional integral,
λ, μ	are Lame's constants,
ρ	is the density assumed to be independent of time,
D	is the diffusivity,
P	is the chemical potential per unit mass,
C	is the concentration,
u_i	are components of displacement vector u,
K	is the coefficient of thermal conductivity,
C_E	is the specific heat at constant strain,
$T = \vartheta - T_0$	is small temperature increment,
ϑ	is the absolute temperature of the medium,
T_0	is the reference temperature of the body such that $\left\lvert \frac{T}{T_0} \right\rvert \ll 1$,

a is the coefficient describing the measure of thermodiffusion effect,

b is the coefficients describing the measure of mass diffusion effect,

σ_{ij} are the components of stress,

e_{ij} are the components of strain,

e_{kk} is dilatation,

S is the entropy per unit mass,

$\beta_1 = (3\lambda + 2\mu)\alpha_t$,

$\beta_2 = (3\lambda + 2\mu)\alpha_c$,

α_c is the coefficient of linear diffusion expansion,

α_t is the coefficient of thermal linear expansion,

τ_0 is the thermal relaxation time,

τ is the diffusion relaxation time,

ω is the angular frequency,

J_1 is the Bessel's function of first kind of order 1,

$2d$ is the thickness of the plate.