Coupled Systems Mechanics, *Vol. 3, No. 4 (2014) 385-403*
DOI: http://dx.doi.org/10.12989/csm.2014.3.4.385

Forced vibration analysis of a dam-reservoir interaction problem in frequency domain

Amirhossein Keivani[1a], Ahmad Shooshtari[*1] and Ahmad Aftabi Sani[2b]

[1]*Civil Engineering Faculty, Ferdowsi University of Mashhad, Mashhad, Iran*
[2]*Civil Engineering Department, Mashhad Branch, Islamic Azad University, Mashhad, Iran*

(Received September 18, 2013, Revised November 2, 2013, Accepted November 12, 2013)

Abstract. In this paper, the forced vibration problem of an Euler-Bernoulli beam that is joined with a semi-infinite field of a compressible fluid is considered as a boundary value problem (BVP). This BVP includes two partial differential equations (PDE) and some boundary conditions (BC), which are introduced comprehensively. After that, the closed-form solution of this fluid-structure interaction problem is obtained in the frequency domain. Some mathematical techniques are utilized, and two unknown functions of the BVP, including the beam displacement at each section and the fluid dynamic pressure at all points, are attained. These functions are expressed as an infinite series and evaluated quantitatively for a real example in the results section. In addition, finite element analysis is carried out for comparison.

Keywords: fluid-structure interaction; closed-form solution; dynamic analysis; frequency domain; finite element method; compressible fluid

1. Introduction

In recent years, many alternatives have been proposed for the dynamic analysis of the interaction between elastic structures and compressible fluids (Panza 2004, Pekau and Yuzhu 2004, Tong *et al*. 2007). In general, these procedures may treat the problem in time or frequency domain, according to the type of system (Ghaemian and Ghobarah 1999, Neild *et al*. 2003, Zilian *et al*. 2009). For example, in the dam-reservoir problem, in which the structure is placed in the vicinity of a semi-infinite reservoir, researchers have most often preferred the latter alternative (i.e., analysis in frequency domain) due to existence of a semi-infinite region in the system (Lotfi and Samii 2012). In this case, the rigorous modeling of the reservoir far-field can be easily performed in comparison to the analysis in time domain. On the other hand, the absence of an unbounded region in the other well-known fluid-structure problem, i.e., the fulfilled fluid container, requires time domain analysis to obtain the dynamic responses of the problem (Biswal *et al*. 2004).

Moreover, the fluid-structure interaction (FSI) problems can be formulated at different levels of complexity and completeness of the physical representation (Maity and Bhattacharyya 2003, Baudille and Biancolini 2008, Hariri-Ardebili *et al*. 2013). At present, some accurate and robust

*Corresponding author, Assistant Professor, E-mail: ashoosht@um.ac.ir
[a] Ph.D. Candidate, E-mail: amir.keivani@gmail.com
[b] Assistant Professor, E-mail: ahmad_aftabi@yahoo.com

http://www.techno-press.org/?journal=csm&subpage=7 ISSN: 2234-2184 (Print), 2234-2192 (Online)

formulations are available that can rigorously model a system with all precise assumptions (Aznárez *et al*. 2006). However, the complexity of the problem and its related formulas oftenrender the analytical solution impossible, and consequently, the numerical approaches may be the only methods to solve the problem. Greater progress in the numerical solutions for different FSI problems and, conversely, the lack of closed-form solutions (except in some simple cases with uncomplicated assumptions such as rigid structures (Bouaanani *et al*. 2003) and incompressible fluids (Jeong 2003)) further indicate that the analytical method is impractical. Herein, the latter motivates work on the closed-form solution of one of the FSI problems, i.e., the forced vibration of a flexible slender beam in the vicinity of a semi-infinite fluid field as a two-dimensional region.

From a mathematical point of view, this problem is a boundary value problem (BVP) with two partial differential equations (PDE) and some boundary conditions (BC), which are all expressed in the frequency domain. In the first part of this article, the problem and its assumptions are defined comprehensively. Then, the BVP is gradually assembled by introducing the PDEs and BCs. In the third part, the BVP constructed is analytically solved using some mathematical techniques, such as separation of variables, implementation of the particular solution and orthogonal functions. Thus, both unknown functions of the problem, i.e., the beam displacement and the fluid dynamic pressure, are obtained as an infinite series. These series converge witha few considered first terms, as shown in the results of this paper.

Furthermore, to challenge the analytical model, a numerical analysis was also carried out, utilizing a robust and modern technique (Lotfi 2004). The modeling concept is based on the FE-(FE-HE) approach, i.e., Finite Element-(Finite Element-Hyper Element). The structure is discretized by the finite elements, and the fluid field is divided into two parts, a near field region (usually an irregular shape) in the vicinity of the structure and a far field part (assuming a uniform channel), which extends to infinity. The former region is discretized by fluid finite elements and the latter part is modeled by a two-dimensional fluid hyper-element. This method allows for the inclusion of fluid-structure interaction and other assumptions of the problem.

In the last stage, the analysis of a real model is presented using both the analytical and numerical methods, and the results are presented as frequency response functions (FRF) and the contours of the dynamic pressure in the fluid field. Thus, both the “exact” and “FEM” responses are provided and discussed.

2. Definition of the problem

In this section, the boundary value problem (BVP) for a two-dimensional compressible fluid-slender structure interaction is explained briefly. Like other BVPs, this problem contains some partial differential equations (PDEs) and several boundary conditions (BCs).

The following assumptions are made for the theoretical formulation:

1- The fluid is linearly compressible and inviscid, with small amplitude and irrotational motions.

2- Gravity surface waves are neglected.

3- The reservoir has regular rectangular boundaries with an exactly horizontal floor.

4- The bottom of the reservoir is assumed to be impermeable and rigid.

5- The exciting ground acceleration is assumed to be harmonic and horizontal.

6- The flexible uniform beam is made of linearly elastic, homogeneous, and isotropic material.

7- The shear deformation and material damping in structure are negligible.

8- The beam is prismatic and exactly vertical.

Fig. 1 illustrates a two-dimensional flexible slender structure–fluid interacting system in which the fluid domain extends to infinity in the far-field. The slender structure width is assumed to be negligibly small compared to the infinite fluid domain and the length of the structure. Here, x-y represents a two-dimensional Cartesian coordinate system with its origin located at the intersection of the central line of the slender beam and the horizontal floor of the reservoir. In addition, the height of fluid is equal to the length of beam, and the freeboard is assumed to be zero.

This system may be considered as a model of dam–water interaction system in which the dam and water domain are assumed to be infinitely long in the z direction perpendicular to the x-y plane. Therefore, the strain of the dam in the z direction vanishes, which leads to a classical plane strain problem. Herein, the thickness of the beam and the width of the fluid domain in the z direction are assumed to be 1. As a result, the system can be analyzed using a two-dimensional sheet for the dam and a water thickness equal to 1. The height H of the dam is significantly larger than its width and thickness, so that the sheet of the dam is considered to be a slender structure. Thus, it can be modeled by the classical Euler-Bernoulli beam theory, which implies that only the deflection in the x direction is considered as a variable in the analysis of the deformation of the dam sheet, which is denoted by $\overline{\mathrm{u}}(y,t)$. This transverse displacement, together with the dynamic pressure in the fluid, i.e., $\overline{\mathrm{P}}(x,y,t)$, are the unknown multivariable functions of the BVP, which is found below.

2.1 Fluid domain

2.1.1 Differential equations

Under the aforementioned assumptions, the hydrodynamic pressure in the reservoir obeys the following equations

$$\frac{\partial^2\overline{\mathrm{P}}}{\partial x^2}+\frac{\partial^2\overline{\mathrm{P}}}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2\overline{\mathrm{P}}}{\partial t^2} \tag{1}$$

where $\overline{\mathrm{P}}(x,y,t)$ is the amplitude of the dynamic pressure in the time domain, c is the velocity of pressure waves in the fluid, x and y are the space coordinates as shown in Fig. 1, and t is the time. For harmonic ground motions, the pressure in the reservoir can be expressed in the frequency domain as $\mathrm{P}(x,y,\omega)=\overline{\mathrm{P}}(x,y,t)e^{i\omega t}$, where ω is the exciting frequency, and $\mathrm{P}(x,y,\omega)$ is the complex-valued frequency response function for the fluid dynamic pressure. Using this transformation in Eq. (1) yields the classical Helmholtz equation

$$\frac{\partial^2\mathrm{P}}{\partial x^2}+\frac{\partial^2\mathrm{P}}{\partial y^2}+\frac{\omega^2}{c^2}\mathrm{P}=0 \tag{2}$$

2.1.2 Boundary conditions

The boundary conditions to be satisfied by Eq. (1) are as follows

(1) The zero pressure condition at the free surface

$$\mathrm{P}(x,\mathrm{H},\omega)=0 \tag{3}$$

(2) The acceleration boundary condition at the horizontal floor of the reservoir

$$\partial \mathrm{P}/\partial(-y)\big|_{(x,0,\omega)} = 0 \tag{4}$$

(3) The radiation condition at infinity

$$\lim_{x\to-\infty} \mathrm{P}(x,\mathrm{y},\omega) = 0 \tag{5}$$

The fourth boundary condition of the fluid domain depends on the structure, and it will be illustrated in Section 2.3.

2.2 Slender structure

2.2.1 Differential equations

According to the assumptions above, the equation of free motion for the dynamic transverse displacement of the beam can be expressed with the well-known Euler-Bernoulli equation (Failla and Santini 2008)

$$\mathrm{EI}\frac{\partial^4 \bar{\mathrm{u}}}{\partial y^4} = -\rho\frac{\partial^2 \bar{\mathrm{u}}_{\mathrm{T}}}{\partial t^2} \tag{6}$$

where $\bar{\mathrm{u}}(y,t)$ is the amplitude of the relative displacement in the time domain, and $\bar{\mathrm{u}}_{\mathrm{T}}(y,t)$ denotes the total displacement, which is the summation of the unknown $\bar{\mathrm{u}}(y,t)$ and the known ground displacement, i.e., $\bar{\mathrm{u}}_{\mathrm{T}} = \bar{\mathrm{u}} + \bar{\mathrm{u}}_{\mathrm{g}}$. In addition, '*E*' is the modulus of elasticity, '*I*' is the moment of inertia, is the product of the mass density with the cross-sectional area, y is the axial coordinate as shown in Fig. 1, and t is the time.

Similar to the fluid domain, for harmonic ground motion, the beam displacement can be expressed in the frequency domain as $\mathrm{u}(y,\omega) = \bar{\mathrm{u}}(y,t)e^{i\omega t}$, where ω is the exciting frequency, and $\mathrm{u}(y,\omega)$ is the complex-valued frequency response function for the transverse displacement. Using this transformation in Eq. (7) and imposing the rigid body displacement as $\mathrm{u}_{\mathrm{g}}(y,\omega) = 1\, e^{i\omega t}$ gives

$$\mathrm{EI}\frac{\partial^4 \mathrm{u}}{\partial y^4} = \rho\omega^2(\mathrm{u}+1) \tag{7}$$

In this problem, the beam is forced with the dynamic pressure of the fluid. Therefore, the external distributed load should be added to the right-hand side of Eq. (9). This load is the product of two terms: the beam width, which was assumed to be 1, and the fluid pressure on the interface boundary at $x = 0$. Fig. 2 and Eq. (8) describe the external load.

$$\mathrm{EI}\frac{\partial^4 \mathrm{u}}{\partial y^4} = \rho\omega^2(\mathrm{u}+1) - \mathrm{P}\big|_{(0,y,\omega)} \tag{8}$$

The minus sign before P shows that its direction is opposite to the u direction.

Fig. 1 A two-dimensional slender beam–water interaction system subjected to a horizontal ground acceleration

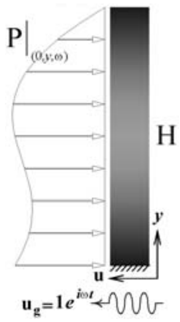

Fig. 2 The external load on the beam

2.2.2 Boundary conditions

The structure is assumed to be a cantilever beam with a fixed base and a free end. Therefore

$$\begin{cases} \mathrm{u}(0,\omega)=0 \\ \dfrac{\partial \mathrm{u}}{\partial y}(0,\omega)=0 \\ \mathrm{EI}\dfrac{\partial^2 \mathrm{u}}{\partial y^2}(\mathrm{H},\omega)=0 \\ \mathrm{EI}\dfrac{\partial^3 \mathrm{u}}{\partial y^3}(\mathrm{H},\omega)=0 \end{cases} \tag{9}$$

2.3 Fluid-structure interaction interface

At the fluid-structure interaction interface, an impermeable and motion consistent boundary condition is assumed, which implies that the fluid cannot flow into the structure and has the same displacement, velocity and acceleration in the x direction as on the wet interface of the structure. Therefore, the following relation governs both unknown functions for the pressure ‘P’ in the water and the horizontal displacement ‘u’ of the wet structure section (Xing 2007)

$$\frac{\partial \mathrm{P}}{\partial x}(0, y, \omega) = -\rho_{\mathrm{F}}\left(-\frac{\partial^2 \mathrm{u_T}}{\partial t^2}\right) \tag{10}$$

By incorporating all differential equations and their boundary conditions, which were explained in Sections 2.1, 2.2 and 2.3, the complete form of the BVP can be constructed as

$$\text{Unknown Functions:} \begin{cases} \mathrm{P}(x, y, \omega) \\ \mathrm{u}(y, \omega) \end{cases}$$

$$\text{PDEs:} \begin{cases} \dfrac{\partial^2 \mathrm{P}}{\partial x^2} + \dfrac{\partial^2 \mathrm{P}}{\partial y^2} + \dfrac{\omega^2}{c^2}\mathrm{P} = 0 \\ \mathrm{EI}\dfrac{\partial^4 \mathrm{u}}{\partial y^4} - \rho\omega^2 \mathrm{u} = \rho\omega^2 - \mathrm{P}\,|_{(0,y,\omega)} \end{cases}$$

$$\text{BCs:} \begin{cases} \mathrm{P}(x, \mathrm{H}, \omega) = 0 \\ \left.\dfrac{\partial \mathrm{P}}{\partial y}\right|_{(x,0,\omega)} = 0 \\ \lim\limits_{x \to -\infty} \mathrm{P}(x, y, \omega) = 0 \\ \left.\dfrac{\partial \mathrm{P}}{\partial x}\right|_{(0,y,\omega)} = -\rho_{\mathrm{F}}\,\omega^2(\mathrm{u}+1) \end{cases} \quad \& \quad \begin{cases} \mathrm{u}(0, \omega) = 0 \\ \dfrac{\partial \mathrm{u}}{\partial y}(0, \omega) = 0 \\ \mathrm{EI}\dfrac{\partial^2 \mathrm{u}}{\partial y^2}(\mathrm{H}, \omega) = 0 \\ \mathrm{EI}\dfrac{\partial^3 \mathrm{u}}{\partial y^3}(\mathrm{H}, \omega) = 0 \end{cases}$$

The above BVP expresses the dynamic behavior of the compressible fluid-flexible beam interaction subjected to the unit ground motion in the frequency domain. In the next section, the above-mentioned problem is solved analytically, utilizing some mathematical techniques.

3. Closed-form solution

Fortunately, all PDEs and BCs of the BVP are homogenous. Therefore, the 'separation of variables method' can be used, and the dynamic pressure in the fluid field may be assumed to be $\mathrm{P}(x, y) = \mathrm{X}(x)\mathrm{Y}(y)$. Substituting this relation into Eq. (2) yields

$$\frac{\mathrm{X}''}{\mathrm{X}} + \frac{\omega^2}{c^2} = -\frac{\mathrm{Y}''}{\mathrm{Y}} \tag{11}$$

The left- and right-hand sides of this equation are only functions of *x* and *y*, respectively, and, thus, both of them must be equal to a constant number. Then

$$\begin{cases} \dfrac{\mathrm{X}''}{\mathrm{X}} + \dfrac{\omega^2}{c^2} = \lambda^2 \\ \dfrac{\mathrm{Y}''}{\mathrm{Y}} = -\lambda^2 \end{cases} \Rightarrow \begin{cases} \mathrm{X}'' - \left(\lambda^2 - \dfrac{\omega^2}{c^2}\right)\mathrm{X} = 0 \\ \mathrm{Y}'' + \lambda^2\,\mathrm{Y} = 0 \end{cases} \tag{12}$$

By solving these ordinary differential equations (ODE) and introducing a new parameter '*k*' as $\mathrm{k}^2 = \lambda^2 - \omega^2 / c^2$, the spatial x and y functions are obtained as

$$\begin{cases} \mathrm{X}(x) = \mathrm{A}e^{-\mathrm{k}x} + \mathrm{B}e^{\mathrm{k}x} \\ \mathrm{Y}(y) = \mathrm{A}'\mathrm{Sin}(\lambda y) + \mathrm{B}'\mathrm{Cos}(\lambda y) \end{cases} \tag{13}$$

The three unknown constants of this relation can be found easily by utilizing Eq. (3)-(5)

$$\begin{cases} \lim\limits_{t\to\infty} \mathrm{P}(-\infty, \mathrm{y}, t) = 0 & \Rightarrow \quad \mathrm{A} = 0 \\ \left.\dfrac{\partial \mathrm{P}}{\partial y}\right|_{(x,0,\omega)} = 0 \;\equiv\; \mathrm{Y}'(0) = 0 & \Rightarrow \quad \mathrm{A}' = 0 \\ \mathrm{P}(x, \mathrm{H}, \omega) = 0 \;\equiv\; \mathrm{Y}(\mathrm{H}) = 0 & \Rightarrow \quad \mathrm{B}' \neq 0 \;\;\&\;\; \lambda_{\mathrm{j}} = (2\mathrm{j}-1)\pi/2\mathrm{H} : \; \mathrm{j} = 1,2,... \end{cases} \tag{14}$$

Therefore, the function of the dynamic pressure in the fluid becomes

$$\mathrm{P}(x, y, \omega) = \sum_{\mathrm{j}=1}^{\infty} \mathrm{B}_{\mathrm{j}}\, e^{\mathrm{k}_{\mathrm{j}}x} \mathrm{Cos}(\lambda_{\mathrm{j}} y) \tag{15}$$

To obtain the last constant, the fluid-structure interaction boundary condition should be applied. Using Eq. (10) yields

$$\sum_{\mathrm{j}=1}^{\infty} \mathrm{B}_{\mathrm{j}}\, \mathrm{k}_{\mathrm{j}}\, \mathrm{Cos}(\lambda_{\mathrm{j}} y) = -\rho_{\mathrm{F}}\, \omega^2 (\mathrm{u}+1) \tag{16}$$

On the other hand, the fluid dynamic pressure is entered on the right-hand side of the beam Eq. (8). Substituting Eq. (15) with the assumption of $x=0$ into Eq. (8) gives

$$\mathrm{EI}\,\mathrm{u}^{\mathrm{IV}} - \rho\,\omega^2 \mathrm{u} = \rho\,\omega^2 - \sum_{\mathrm{j}=1}^{\infty} \mathrm{B}_{\mathrm{j}}\, \mathrm{Cos}(\lambda_{\mathrm{j}} y) \tag{17}$$

This linear fourth-order non-homogenous ODE has one general solution with four unknown constants and two particular solutions with respect to both terms of the right-hand expression. The homogenous form of the equation is

$$\mathrm{u}^{\mathrm{IV}} - \alpha^4\, \mathrm{u} = 0 \tag{18}$$

where

$$\alpha = \left(\frac{\rho\,\omega^2}{\mathrm{EI}} \right)^{\frac{1}{4}} \tag{19}$$

The general solution should satisfy the homogenous Eq. (18), which is

$$\mathrm{u}_{\mathrm{G}} = c_1 \mathrm{Sin}(\alpha\, y) + c_2 \mathrm{Cos}(\alpha\, y) + c_3 \mathrm{Sinh}(\alpha\, y) + c_4 \mathrm{Cosh}(\alpha\, y) \tag{20}$$

In addition, the first particular solution with regard with the first right-hand term of Eq. (17) is

$$\mathrm{EI}\,\mathrm{u}^{\mathrm{IV}} - \rho\,\omega^2 \mathrm{u} = \rho\,\omega^2 \quad \Rightarrow \; \mathrm{u}_{\mathrm{p}_1} = -1 \tag{21}$$

The second particular solution is obtained for the single general term of the summation on the right-hand of Eq. (17), and, then, the summation of solution of this form is utilized as the second particular solution, according to the superposition principle. In fact, each particular solution should satisfy the following equation

$$\mathrm{EI}\,\mathrm{u}^{\mathrm{IV}} - \rho\,\omega^2 \mathrm{u} = -\mathrm{B}_\mathrm{j}\,\mathrm{Cos}\,(\lambda_\mathrm{j} y) \tag{22}$$

By considering the hypothesis that $\lambda_\mathrm{j} \neq \alpha$, the solution of the equation above (the 'j'th expression of the second particular solution) should be

$$\mathrm{u}_{\mathrm{p}_{2\,\mathrm{j}}} = \mathrm{D}_\mathrm{j}\,\mathrm{Cos}(\lambda_\mathrm{j} y) \tag{23}$$

Substituting this expression into Eq. (22) yields

$$\mathrm{D}_\mathrm{j} = -\frac{\mathrm{B}_\mathrm{j}}{\mathrm{EI}(\lambda_\mathrm{j}^{\;4} - \alpha^4)} \tag{24}$$

Consequently, the second particular solution is

$$\mathrm{u}_{\mathrm{p}_2} = \sum_{\mathrm{j=1}}^{\infty} \mathrm{D}_\mathrm{j}\,\mathrm{Cos}(\lambda_\mathrm{j} y) \tag{25}$$

Thus, the dynamic transverse displacement of the beam is

$$\mathrm{u}\,(y) = c_1\,\mathrm{Sin}\,(\alpha\, y) + c_2\,\mathrm{Cos}\,(\alpha\, y) + c_3\,\mathrm{Sinh}\,(\alpha\, y) + c_4\,\mathrm{Cosh}\,(\alpha\, y) - 1 + \sum_{\mathrm{j=1}}^{\infty} \mathrm{D}_\mathrm{j}\,\mathrm{Cos}\,(\lambda_\mathrm{j} y) \tag{26}$$

Four unknown constants, c_1 through c_4, can be obtained by applying the four boundary condition of the beam,as presented in Eq. (9). Herein, to avoid the details, only the final result of these substitutions is presented below

$$\begin{cases} c_1 = \dfrac{\beta}{\gamma} - \dfrac{1}{\gamma}\displaystyle\sum_{\mathrm{j=1}}^{\infty} \delta_\mathrm{j}\,\mathrm{B}_\mathrm{j} \\ c_2 = \eta\, c_1 + \xi + \displaystyle\sum_{\mathrm{j=1}}^{\infty} \varepsilon_\mathrm{j}\,\mathrm{B}_\mathrm{j} \\ c_3 = -\dfrac{\beta}{\gamma} + \dfrac{1}{\gamma}\displaystyle\sum_{\mathrm{j=1}}^{\infty} \delta_\mathrm{j}\,\mathrm{B}_\mathrm{j} \\ c_4 = 1 - c_2 + \displaystyle\sum_{\mathrm{j=1}}^{\infty} \sigma_\mathrm{j}\,\mathrm{B}_\mathrm{j} \end{cases} \tag{27}$$

where

$$\begin{cases} \beta = \dfrac{\mathrm{Sin}(\alpha \mathrm{H})\mathrm{Cosh}(\alpha \mathrm{H}) + \mathrm{Cos}(\alpha \mathrm{H})\mathrm{Sinh}(\alpha \mathrm{H})}{\mathrm{Cos}(\alpha \mathrm{H}) + \mathrm{Cosh}(\alpha \mathrm{H})} \\ \gamma = \dfrac{2\,\mathrm{Cos}(\alpha \mathrm{H})\mathrm{Cosh}(\alpha \mathrm{H}) + 2}{\mathrm{Cos}(\alpha \mathrm{H}) + \mathrm{Cosh}(\alpha \mathrm{H})} \\ \xi = \dfrac{\mathrm{Cosh}(\alpha \mathrm{H})}{\mathrm{Cos}(\alpha \mathrm{H}) + \mathrm{Cosh}(\alpha \mathrm{H})} \\ \eta = -\dfrac{\mathrm{Sin}(\alpha \mathrm{H}) + \mathrm{Sinh}(\alpha \mathrm{H})}{\mathrm{Cos}(\alpha \mathrm{H}) + \mathrm{Cosh}(\alpha \mathrm{H})} \end{cases} \tag{28}$$

Substituting the constants c_1 through c_4 into Eq. (26) yields

$$\begin{aligned} \mathrm{u}(y) = & \left(\frac{\beta}{\gamma} - \frac{1}{\gamma}\sum_{\mathrm{j}=1}^{\infty}\delta_{\mathrm{j}}\,\mathrm{B}_{\mathrm{j}}\right)\left(\mathrm{Sin}(\alpha y) - \mathrm{Sinh}(\alpha y)\right) \\ & + \left(\frac{\eta\beta + \xi\gamma}{\gamma} + \sum_{\mathrm{j}=1}^{\infty}\varepsilon_{\mathrm{j}}\,\mathrm{B}_{\mathrm{j}} - \frac{\eta}{\gamma}\sum_{\mathrm{j}=1}^{\infty}\delta_{\mathrm{j}}\,\mathrm{B}_{\mathrm{j}}\right)\left(\mathrm{Cos}(\alpha y) - \mathrm{Cosh}(\alpha y)\right) \\ & + \left(1 + \sum_{\mathrm{j}=1}^{\infty}\sigma_{\mathrm{j}}\,\mathrm{B}_{\mathrm{j}}\right)\mathrm{Cosh}(\alpha y) - \left(1 + \sum_{\mathrm{j}=1}^{\infty}\sigma_{\mathrm{j}}\,\mathrm{B}_{\mathrm{j}}\,\mathrm{Cos}(\lambda_{\mathrm{j}} y)\right) \end{aligned} \tag{29}$$

The last unknown parameter in this function is B_{j}, which should be found by imposing the boundary condition of the fluid-structure interface. Thus, 'u' is obtained with Eq. (29), and Eq. (16) has only one set of unknown variables, i.e., B_{j}

$$\sum_{\mathrm{j}=1}^{\infty}\mathrm{B}_{\mathrm{j}}\,\mathrm{k}_{\mathrm{j}}\,\mathrm{Cos}(\lambda_{\mathrm{j}} y) = -\rho_{\mathrm{F}}\,\omega^2(\mathrm{u}+1) \qquad \text{Repeated (16)}$$

Clearly, B_{j} is not entered into the above equation 'alone' but together with the other unknown 'B_{j}'s. In fact, both the left- and right-hand sides of Eq. (16) have infinite unknowns. To remedy this problem, the orthogonal property of the cosine function can be used. The following relations are well-known.

$$\int_0^{\mathrm{H}} Cos(\lambda_{\mathrm{i}} y)\,Cos(\lambda_{\mathrm{j}} y)\,\mathrm{d}y = \begin{cases} 0 & :\mathrm{i} \neq \mathrm{j} \\ \dfrac{\mathrm{H}}{2} & :\mathrm{i} = \mathrm{j} \end{cases} \tag{30}$$

Subsequently, multiplying both sides of Eq. (16) by $\mathrm{Cos}(\lambda_{\mathrm{i}} y)$ and integrating the result from 0 to *H* gives

$$\mathrm{B}_{\mathrm{i}}\,\mathrm{k}_{\mathrm{i}}\,\frac{\mathrm{H}}{2} = -\rho_{\mathrm{F}}\,\omega^2\int_0^{\mathrm{H}}(\mathrm{u}+1)\,\mathrm{Cos}(\lambda_{\mathrm{i}} y)\,\mathrm{d}y \tag{31}$$

By using this technique, B_{i} becomes the only term on the left-hand side of Eq. (31). However, the right-hand side of this equation contains countless B_{j} terms. These B_{j}s are in the '*u*' function according to Eq. (29). Before remedying this problem, some integrals must be analytically calculated, which are created after multiplying the 'u' function by $\mathrm{Cos}(\lambda_i y)$ in Eq. (31). The '*i*' in the following relations is only a simple counter

$$
\begin{aligned}
\mathrm{I}_1^i &= \int_0^H \mathrm{Sin}(\alpha\, y)\mathrm{Cos}(\lambda_i y)\; dy = \frac{2\mathrm{H}[2\alpha\,\mathrm{H} - (1-2\mathrm{i})\,\pi(-1)^{\mathrm{i}}\mathrm{Sin}(\alpha\,\mathrm{H})]}{(2\alpha\,\mathrm{H})^2 - (1-2\mathrm{i})^2\pi^2} \\
\mathrm{I}_2^i &= \int_0^H \mathrm{Cos}(\alpha\, y)\mathrm{Cos}(\lambda_i y)\; dy = \frac{\mathrm{Cos}(\mathrm{i}\pi - \alpha\,\mathrm{H})}{2\alpha\,\mathrm{H} + (1-2\mathrm{i})\,\pi} - \frac{\mathrm{Cos}(\mathrm{i}\pi + \alpha\,\mathrm{H})}{2\alpha\,\mathrm{H} - (1-2\mathrm{i})\,\pi} \\
\mathrm{I}_3^i &= \int_0^H \mathrm{Sinh}(\alpha\, y)\mathrm{Cos}(\lambda_i y)\; dy = -\frac{2\mathrm{H}[2\alpha\,\mathrm{H} - (1-2\mathrm{i})\,\pi(-1)^{\mathrm{i}}\mathrm{Sinh}(\alpha\,\mathrm{H})]}{(2\alpha\,\mathrm{H})^2 + (1-2\mathrm{i})^2\pi^2} \\
\mathrm{I}_4^i &= \int_0^H \mathrm{Cosh}(\alpha\, y)\mathrm{Cos}(\lambda_i y)\; dy = \frac{2\,\mathrm{H}\,(1-2\mathrm{i})\,\pi(-1)^{\mathrm{i}}\mathrm{Cosh}(\alpha\,\mathrm{H})}{(2\alpha\,\mathrm{H})^2 + (1-2\mathrm{i})^2\pi^2}
\end{aligned}
\tag{32}
$$

Substituting this integral into Eq. (31), and introducing some new parameters and some mathematical operations, yield

$$
\left.
\begin{aligned}
a_{ij} &= \frac{1}{\gamma}(\mathrm{I}_3^i - \mathrm{I}_1^i)\,\delta_j + \frac{\gamma\,\varepsilon_j - \eta\,\delta_j}{\gamma}(\mathrm{I}_2^i - \mathrm{I}_4^i) + \mathrm{I}_4^i\,\sigma_j \\
d_i &= \frac{\mathrm{H}}{2}\left(\frac{\mathrm{k}_i}{\rho_F\,\omega^2} - \sigma_i\right) \\
b_i &= \frac{\beta}{\gamma}(\mathrm{I}_3^i - \mathrm{I}_1^i) + \frac{\eta\beta + \xi\gamma}{\gamma}(\mathrm{I}_4^i - \mathrm{I}_2^i) - \mathrm{I}_4^i
\end{aligned}
\right\} \Rightarrow\; d_i \mathrm{B}_i + \sum_{j=1}^{\infty} a_{ij}\,\mathrm{B}_j = b_i
\tag{33}
$$

By writing Eq. (33) for a finite number of 'i' (for example, $\mathrm{i} = 1,2,\ldots,\mathrm{N}$) and expanding the summation to *N*, the *N* × *N* linear system is found, and its solution is the unknowns B_1 to B_N. Subsequently, the function of fluid dynamic pressure and the beam transverse displacement are attained by using Eqs. (15) and (29), respectively. As shown in Section 5, a low number of N can treat the problem well.

4. Finite element analysis

In this section, the numerical approach to solve the "compressible fluid-flexible structure interaction" problem by the well-known Finite Element Method (FEM) is reviewed briefly. This numerical approximate solution is compared with the analytical exact solution, which as proposed in the previous section.

To analyze the system by FEM, three types of elements are utilized:

1- Bending beam elements with 2 nodes and 4 degrees of freedom (both displacement and rotation)

2- Fluid elements with 4 nodes and 4 pressure degrees of freedom

3- Fluid hyper-elements with 2 nodes in each sub-layer

The afore-mentioned elements are utilized, respectively, for modeling (1) the structure, (2) the fluid near-field region, and (3) the far-field region of the fluid domain, which extends to infinity. Fig. 3 shows the whole FE-(FE-HE) model of the system. As mentioned before, the structure is discretized by finite elements, while the reservoir is divided into two parts, a near field region

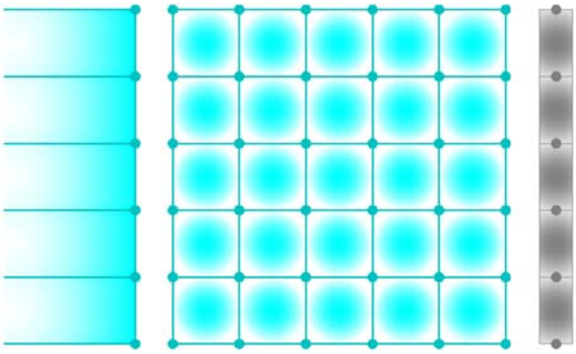

Fig. 3 FE-(FE-HE) model of the system

(usually an irregular shape) in the vicinity of the structure and a far field part (assuming uniform channel with horizontal floor), which extends to infinity. The former region is discretized by fluid finite elements, and the latter part is modeled by a two-dimensional fluid hyper-element. Also, the numerical analyses are carried out in the frequency domain.

If the mass and stiffness matrices of the structure are denoted as $\mathbf{M}$ and $\mathbf{K}$, the characteristic matrices of the fluid domain are denoted as $\mathbf{G}$ and $\mathbf{H}$, the impedance matrix of the fluid hyper-element as $\mathbf{H}_\mathrm{h}$, the interaction matrix as $\mathbf{B}$, the matrix of rigid body motion as $\mathbf{J}$, the vector of ground accelerations as $\mathbf{a}_\mathrm{g}$, the vector of the unknown nodal displacements of the structure as $\mathbf{D}$, the vector of fluid dynamic pressure as $\mathbf{P}$, the exciting frequency as ω, the velocity of pressure waves in fluid as c, and the fluid mass density as ρ_F, then the coupled equation of the system can be written as

$$\begin{bmatrix} \mathbf{K}-\omega^2\mathbf{M} & -\mathbf{B}^\mathrm{T} \\ -\rho_\mathrm{F}\,\omega^2\mathbf{B} & \mathbf{H}+\mathbf{H}_\mathrm{h}-(\omega^2/c^2)\mathbf{G} \end{bmatrix} \begin{pmatrix} \mathbf{D} \\ \mathbf{P} \end{pmatrix} = \begin{pmatrix} -\mathbf{M}\,\mathbf{J}\,\mathbf{a}_\mathrm{g} \\ -\rho_\mathrm{F}\,\mathbf{B}\,\mathbf{J}\,\mathbf{a}_\mathrm{g} \end{pmatrix} \tag{34}$$

In Eq. (34), the material damping in the fluid and structure are negligible. However, the radiation damping is taken into account by the complex-valued impedance matrix of the fluid hyper-element $\mathbf{H}_\mathrm{h}$ (Lotfi 2004).

By solving the linear system (34), the nodal displacements of the beam and the nodal pressure of the fluid domain are attained. These discrete results can be directly used for comparison with the continuous exact solution, i.e., Eqs. (29) and (15), respectively.

5. Results of analyses

The proposed exact approach and the finite element method were both implemented to obtain the dynamic response of the fluid-structure system to the ground motion. The beam width and height were 20 and 200 meters, respectively. The structure was made of concrete, and the fluid was assumed to be water. Hence, the basic parameters of the system are

$$\text{Structure}:\begin{cases}\mathrm{E}=2\times10^{10} & \mathrm{N/m^2}\\ \mathrm{I}=\dfrac{1\times20^3}{12} & \mathrm{m^4}\\ \rho=2500\times20\times1 & \mathrm{Kg/m}\end{cases}\quad,\quad \text{Fluid}:\begin{cases}\rho_{\mathrm{F}}=1000 & \mathrm{Kg/m^3}\\ c=1440 & \mathrm{m/sec.}\\ \mathrm{H}=200 & \mathrm{m}\\ \mathrm{L/H}=1 & \end{cases}$$

The first output is the frequency response function (FRF) for the free end displacement of the beam, i.e., the point with $y=\mathrm{H}$. This response is composed of the amplitudes of the complex-valued acceleration for the point when the system is subjected to a unit ground acceleration of $\mathbf{a}_g(t)=1e^{i\omega t}$. Moreover, the exact result was obtained with the assumption that $\mathrm{N}=10$ and the FE analysis was carried out with 10 beam elements, 10×10 fluid finite elements and 10 fluid hyper-elements. Fig. 4 compares the results.

As it is usually predicted, the accuracy of the FEM decreased with increasing exciting frequency ω. If these analyses are continued for the higher frequencies, without any change in *N* and FE mesh fineness, the output becomes not so accurate.

The obvious difference between the exact and FEM results in Fig. 5 could be reduced by increasing the number of elements in the FE model of the system. If the beam is discretized by 20 elements and the fluid domain by 20 × 20 elements, the convergence of the numerical results to the exact one is visibly enhanced. In Fig. 6, '*N*' equals 10.

Subsequently, some special cases were investigated. First, the empty reservoir situation was simulated by considering a very low value (approximately zero) for the mass density of the fluid. Fig. 7 shows the results of the both exact and finite element methods:

Due to the elimination of the radiation damping in the case of the empty reservoir, Fig. 7 displays more sharp peaks than Fig. 6. Showing both the full and empty reservoir cases in a single chart is useful, as in Fig 8.

The other special case is the rigid structure condition for the dam, which was modeled by assuming a very high value (approximately infinite) for the bending stiffness (EI) of the beam. Obviously, in this case, the free end displacement of the beam becomes zero for all exciting frequencies. Therefore, the selected output to obtain the FRF should be changed. Herein, the frequency response function was attained for the fluid dynamic pressure at the origin of the Cartesian coordinates in the reservoir, i.e., the point with $x=0$ and $y=0$. See Fig. 9.

In addition to the good agreement of the exact and finite element method, Fig. 9 shows the first natural frequency of the reservoir in the absence of the structure, which is usually called the reservoir "cut off frequency". The exact value of this cut off frequency is $\pi c/2\mathrm{H}=11.31$ rad/sec, which was detected with both methods.

Fig. 9 shows the frequency response function of the fluid dynamic pressure for the special case of the rigid structure. In the next step, the FRF was obtained for the ordinary situation of the system, i.e., a flexible structure with a full reservoir. In Fig. 10, the exact result was calculated with *N* = 10. The finite element model contained 10 beam elements, 100 fluid elements and 10 fluid hyper-elements.

Similar to Fig. 8, comparing the FRF of the fluid dynamic pressure for both the special and ordinary cases may be helpful. The first natural frequency of the fluid-structure system ("Flex. curve") is clearly lower than the fluid "cut off frequency," which could be observed in the "Rigid curve". Herein, the values are 11.31 and 1.03 rad/sec, respectively.

By considering the interaction effect between the fluid and the structure, the combined system

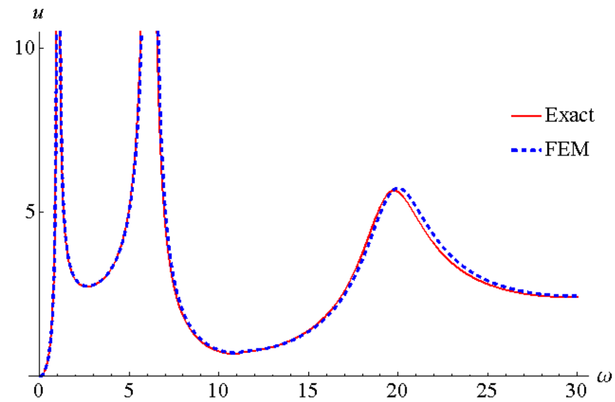

Fig. 4 The FRF of the beam free end displacement

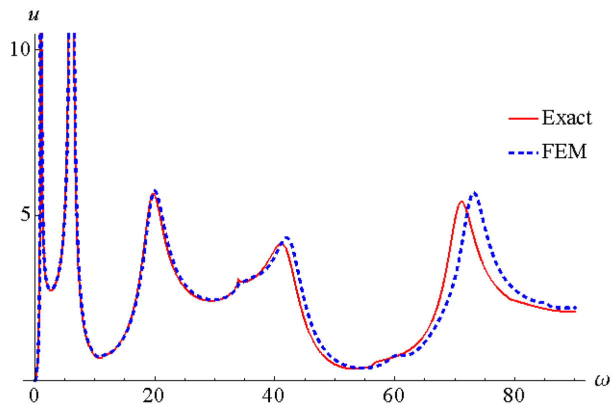

Fig. 5 The FRF of the beam free end displacement

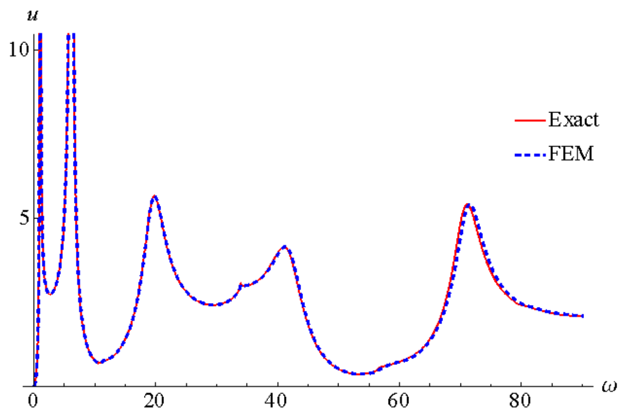

Fig. 6 The FRF of the beam free end displacement

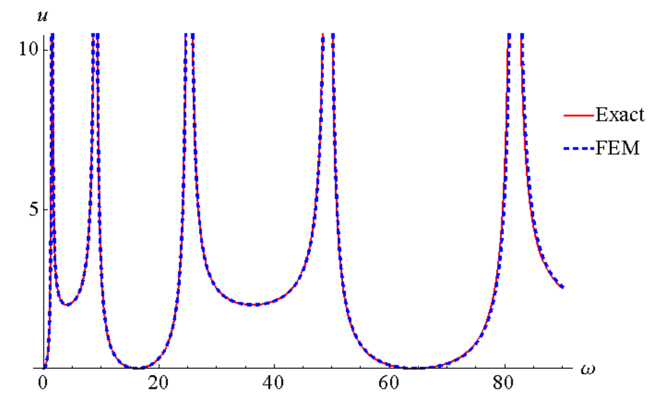

Fig. 7 The FRF of the beam free end displacement for the empty reservoir situation

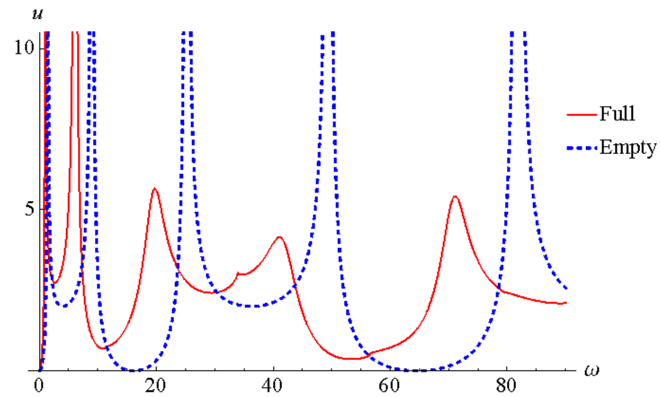

Fig. 8 The FRF of the beam free end displacement for both the full and empty reservoir cases

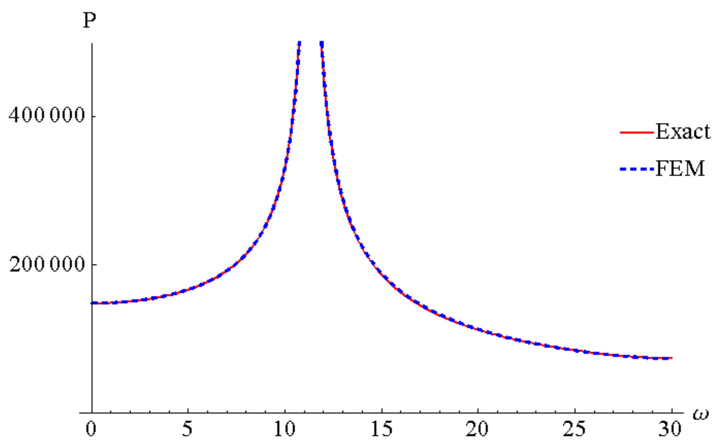

Fig. 9 The FRF of the fluid dynamic pressure at the point with $x = 0$ and $y = 0$ for the special case of the rigid structure

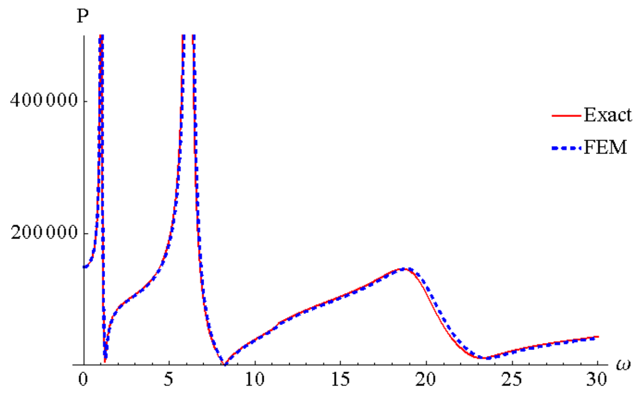

Fig. 10 The FRF of the fluid dynamic pressure at the point with $x = 0$ and $y = 0$ for the ordinary situation of the flexible structure

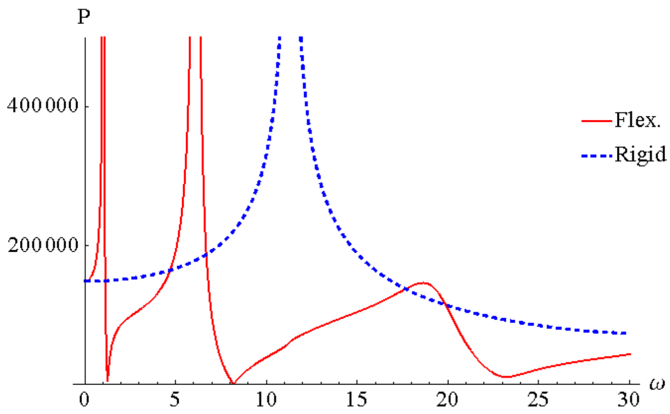

Fig. 11 The FRF of the fluid dynamic pressure at the point with $x = 0$ and $y = 0$ for both the flexible and rigid structure cases

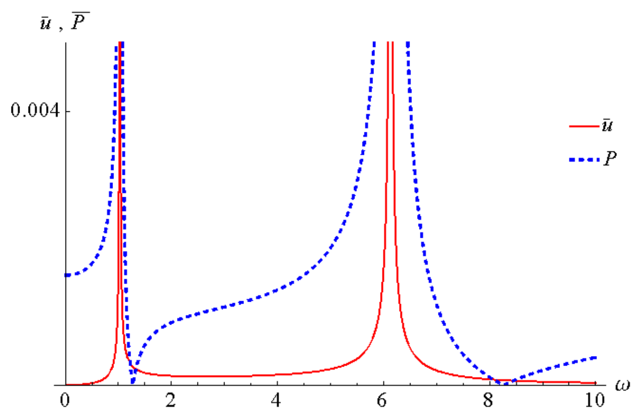

Fig. 12 The normalized FRF for the displacement of the structure and the dynamic pressure of the fluid

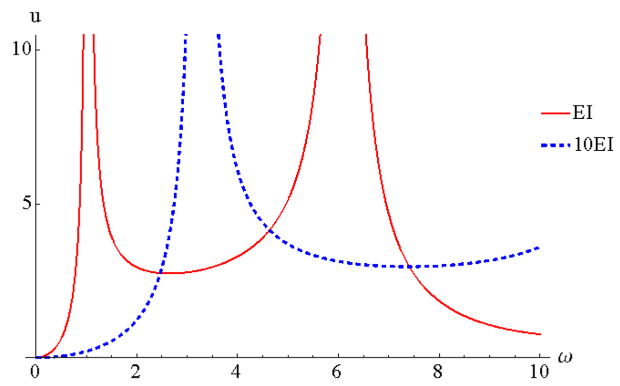

Fig. 13 Effect of increasing EI on the FRF of the beam free end displacement

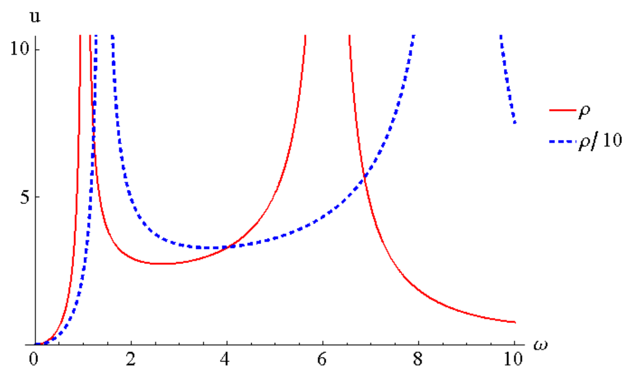

Fig. 14 Effect of the beam mass density on the FRF of the beam free end displacement

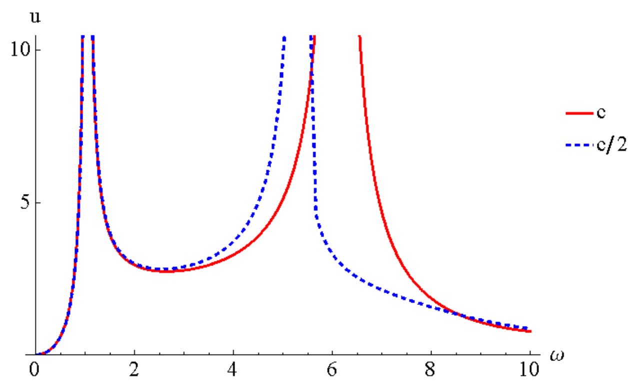

Fig. 15 Effect of the fluid wave velocity on the FRF of the beam free end displacement

contains some common natural frequencies. Therefore, the value of 1.03 rad/sec, which was listed in the previous paragraph, is the natural frequency of the "fluid-structure" system. At this common frequency, the frequency response functions of both the fluid and structure outputs (i.e., the FRF of the displacement in the structure and the dynamic pressure in the fluid) have a clear peak. For illustration, both outputs are shown in a single graph. Although these two outputs have different concepts and dimensions, it is possible to investigate them as normalized values. In Fig. 12, the FRF for the displacement of the structure and the dynamic pressure of the fluid are shown together, after dividing all values of each curve by their maximum amount in the exciting frequency interval selected. It is clear that both the fluid and structure have the same “resonance frequency”.

Obviously, changing the physical properties of the system affects the dynamic responses. The following figures, i.e., Figs. 13-16, which were obtained by the proposed exact method, demonstrate these effects clearly.

Thus, the contour of the fluid dynamic pressure in the reservoir was investigated for some exciting frequencies (Figs. 17 and 18). It should be mentioned that these results were obtained by both the exact and finite element methods for the “fluid-structure” system. In the exact method, *N* equal 10, but, in the FEM, it is assumed that L/H = 2. In addition, the FE mesh was made finer to obtain more accurate contours. Herein, the fluid mesh was 40×20 .

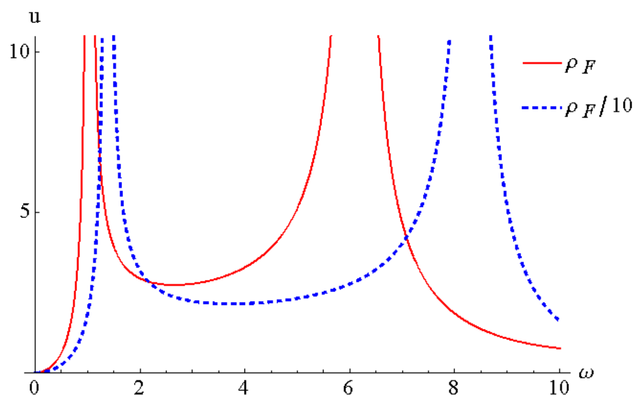

Fig. 16 Effect of the fluid mass density on the FRF of the beam free end displacement

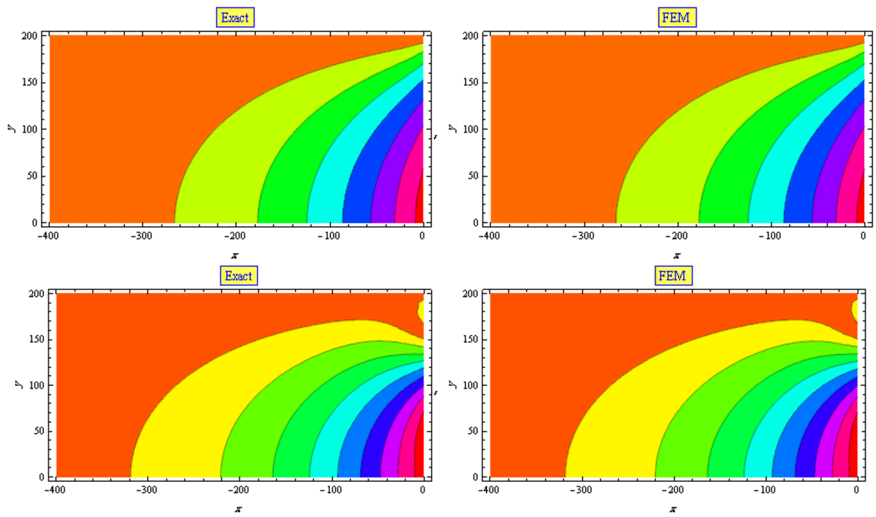

Fig. 17 The contours of the fluid dynamic pressure for $\omega = 0$ *and* $5\,{}^{rad}\!/_{sec}$

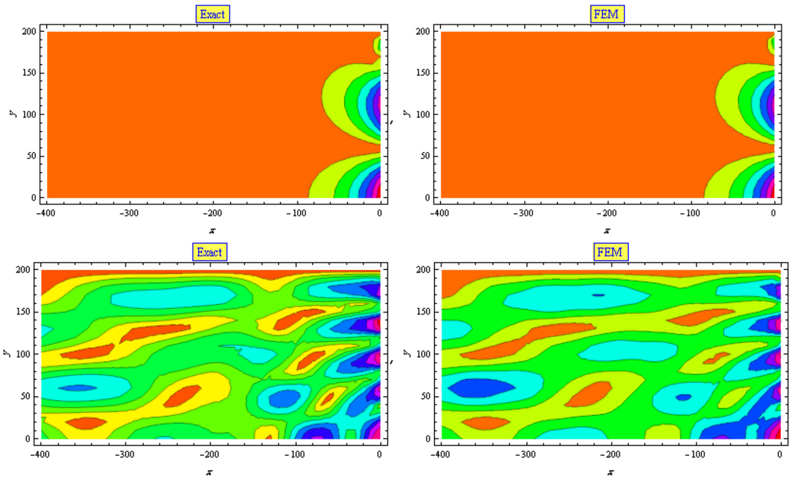

Fig. 18 The contours of the fluid dynamic pressure for $\omega = 10$ *and* $100\, {}^{rad}\!/\!_{\sec}$

6. Conclusions

This paper gives a new analytical solution for a well-known type of fluid-structure interacting system, a slender beam in the vicinity of a semi-infinite field of a compressible fluid. The closed-form solution was obtained directly by solving the related boundary value problem, utilizing some mathematical techniques. Thus, the unknown functions of the system including beam displacement and fluid dynamic pressure were expressed as infinite series. Moreover, these two dynamic quantities were numerically evaluated in an actual example, and the results were shown by using two well-known concepts: the frequency response functions and the field contours. In addition, the finite element method was implemented to solve the problem and to compare the closed-form results. It is concluded that the proposed method is accurate and especially efficient. The dimensions of the matrix that should be inversed in the closed-form method are much less than those of the similar one in the numerical methods, such as FEM. Furthermore, in terms of accuracy, all numerical methods involve some approach to solve the boundary value problem approximately, but the proposed method solves the problem exactly.

References

Aznárez, J.J., Maeso, O. and Domínguez, J. (2006), “BE analysis of bottom sediments in dynamic fluid-structure interaction problems”, *Eng. Anal. Boundary Elements*, **30**(2), 124-136.

Baudille, R. and Biancolini, M.E. (2008), “A general approach for studying the motion of a cantilever beam interacting with a 2D fluid flow”, *Interact. Multiscale Mech.*, **1**(4), 275-293.

Biswal, K.C., Bhattacharyya, S.K. and Sinha, P.K. (2004), “Dynamic response analysis of a liquid-filled cylindrical tank with annular baffle”, *J. Sound and Vib.*, **274**(1-2), 13-37.

Bouaanani, N., Paultre, P. and Proulx, J. (2003), “A closed-form formulation for earthquake-induced hydrodynamic pressure on gravity dams”, *J. Sound Vib.*, **261**(3), 573-582.

Failla, G. and Santini, A. (2008), “A solution method for Euler–Bernoulli vibrating discontinuous beams”, *Mech. Res. Commun.*, **35**(8), 517-529.

Ghaemian, M. and Ghobarah, A. (1999). "Nonlinear seismic response of concrete gravity dams with dam–reservoir interaction", *Eng. Struct.*, **21**(4), 306-315.

Hariri-Ardebili, M.A., Mirzabozorg, H. and Ghasemi, A. (2013), "Strain-based seismic failure evaluation of coupled dam-reservoir-foundation system", *Coupled Syst. Mech.*, **2**(1), 85-110.

Jeong, K.H. (2003), "Free vibration of two identical circular plates coupled with bounded fluid", *J. Sound Vib.*, **260**(4), 653-670.

Lotfi, V. (2004), "Direct frequency domain analysis of concrete arch dams based on FE-(FE-HE)-BE technique", *Comput. Concrete*, **1**(3), 285-302.

Lotfi, V. and Samii, A. (2012), "Dynamic analysis of concrete gravity dam-reservoir systems by wavenumber approach in the frequency domain", *Earthq. Struct.*, **3**(3), 533-548.

Maity, D. and Bhattacharyya, S.K. (2003), "A parametric study on fluid–structure interaction problems", *J. Sound Vib.*, **263**(4), 917-935.

Neild, S.A., McFadden, P.D. and Williams, M.S. (2003), "A review of time-frequency methods for structural vibration analysis", *Eng. Struct.*, **25**(6), 713-728.

Panza, M.J. (2004), "Closed form solution for acoustic wave equation between two parallel plates using Euler–Maclaurin sum formula", *J. Sound Vib.*, **277**(1-2), 123-132.

Pekau, O.A. and Yuzhu, C. (2004), "Failure analysis of fractured dams during earthquakes by DEM", *Eng. Struct*, **26**(10), 1483-1502.

Tong, Z., Zhang, Y., Zhang, Z. and Hua, H. (2007), "Dynamic behavior and sound transmission analysis of a fluid–structure coupled system using the direct-BEM/FEM", *J. Sound Vib.*, **299**(3), 645-655.

Xing, J.T. (2007), "Natural vibration of two-dimensional slender structure–water interaction systems subject to Sommerfeld radiation condition", *J. Sound and Vib.*, **308**(1-2), 67-79.

Zilian, A., Dinkler, D. and Vehre, A. (2009), "Projection-based reduction of fluid-structure interaction systems using monolithic space-time modes", *Comput. Meth. Appl. Mech. Eng.*, **198**(47-48), 3795-3805.