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# Analytical approximate solution for Initial post-buckling behavior of pipes in oil and gas wells

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**Abstract.** This paper presents analytical approximate solutions for the initial post-buckling deformation of the pipes in oil and gas wells. The governing differential equation with sinusoidal nonlinearity can be reduced to form a third-order-polynomial nonlinear equation, by coupling of the well-known Maclaurin series expansion and orthogonal Chebyshev polynomials. Analytical approximations to the resulting boundary condition problem are established by combining the Newton's method with the method of harmonic balance. The linearization is performed prior to proceeding with harmonic balancing thus resulting in a set of linear algebraic equations instead of one of non-linear algebraic equations, unlike the classical method of harmonic balance. We are hence able to establish analytical approximate solutions. The approximate formulae for load along axis, and periodic solution are established for derivative of the helix angle at the end of the pipe. Illustrative examples are selected and compared to "reference" solution obtained by the shooting method to substantiate the accuracy and correctness of the approximate analytical approach.

**Keywords:** analytical approximation; buckling; shooting method

## 1. Introduction

Many aspects of petroleum engineering (such as the functions of pipes, safety and surveying accuracy of down-hole instruments, and so on) are related with buckling of drill-strings. Because of the high frequency of drill-string failure, drill-string lock up, and casing wear, the stability of drill-strings has been a serious problem in oil/gas field operations for many years (Gulyayev *et al*. 2009 and Tan *et al*. 2009). On the other hand, with the development of drilling technology, oil/gas wells become very long currently, even more than ten kilometers. Furthermore, some wells have a very complex geometrical configuration, such that parts of wells may be inclined, vertical, horizontal, just plane curved, and even 3-D curved. Therefore, it is important and meaningful to investigate the buckling behavior of pipes for the science and technologies in petroleum engineering and related fields.

In despite of the complexity of the problem, many results are still reported. Paslay and Bogy first studied the problem of sinusoidal buckling of the tube (Paslay and Bogy 1964). Based on the principle of minimum potential energy, the problem of helical buckling of a vertical tube was first

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analyzed by Lubinski *et al*. (1962) Since then, Cheatham and Pattillo (1984), He and Kyllingstad (1993), Miska and Cunha (1995) and so on, have studied helical buckling of tubes in vertical, horizontal or inclined wellbores, based on the energy method. Experimental study of helical buckling of a horizontal rod in a tube was performed by McCann and Suryanarayana (1994). Wicks *et al*. (2008) reviewed available analytical and experimental results on the structural behavior of constrained horizontal cylinders subjected to axial compression, torsion, and gravity.

The view about the effect of torsional loads on the buckling loads of drill-strings has no conclusion. When neglecting friction, the influence of torsion on axial compression force was relatively small, it is a good approximation by neglecting the torsion effect (Wu 1997, and Qiu *et al*. 1998). However, some researches present that torsional loads have a significant effect on buckling loads (Timoshenko and Gere 1961, Benecke and van Vuuren 2005, Tan *et al*. 2006). Thus the effect of torsional loads on buckling loads of drill-strings deserves further study.

In this paper, having not considered the effect of torsion, we present an alternative approach to solve initial post-buckling deformation of the pipe. The solution is based on the governing equations presented in Ref. Gao *et al*. (2002). The proposed approach is an extension of recent work of finding analytical approximate solutions to non-linear oscillations (Wu *et al*. 2006) and constructing analytical approximations to large post-buckling deformation of elastic rings (Wu *et al*. 2007) and large hygrothermal buckling deformation of a beam (Yu *et al*. 2008).By coupling of the well-known Maclaurin series expansion and orthogonal Chebyshev polynomials, and then combining the Newton linearization of the governing equation with the method of harmonic balance, we establish analytical approximate solutions to initial post-buckling deformation of the pipe in terms of derivative of the helix angle at the end of the pipe. Illustrative examples are selected and compared to those analytical formulae and "reference" solutions obtained by the shooting method to substantiate the accuracy and correctness of the approximate analytical approach.

## 2. Formulation

For the title problem, the weightless pipe within a curved wellbore is assumed to be slender, and the friction in system and the effect of torsion are negligible. The buckled pipe which is keeping

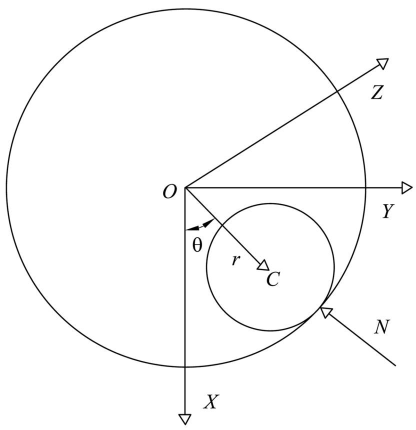


Fig. 1 Schematic view of buckled pipe

contact with the wellbore is displayed in Fig. 1, where $\theta$ is the helix angle of the pipe, and $r$ is the radial clearance between the pipe and the wellbore.

The buckling governing equation of a pipe with a curved wellbore in dimensionless mode can be expressed as in the form (Gao *et al*. 2002)

$$\theta^{(4)} - 6\theta'^2\theta'' + 2\theta'' + Q_0 \sin\theta = 0 \tag{1}$$

and

$$P = 4\theta'''\theta' + 3\theta''^2 - \theta'^4 + 2\theta'^2 + Q\cos\theta \tag{2}$$

where

$\theta' = d\theta/d\tau$, $Q = F/(EIRr\omega^4)$ and $P = N/EIr\omega^4$

$\omega = \sqrt{F/2EI}$, $F$ is the axial load, $EI$ is the bending stiffness, and $N$ is the contact force, $R$ is the radius of curvature of the wellbore, and $r$ is the distance between the center of the wellbore and the that of the pipe. $\tau = \omega s$ is dimensionless length and $\tau \in [0, 2\pi]$. At the simple support ends of the beam, we have the boundary conditions

$$\theta(0) = \theta(2\pi) = \theta''(0) = \theta''(2\pi) = 0 \tag{3}$$

Once $\theta(\tau)$ and $Q$ are achieved from differential and integration formulations in Eqs. (1) and (3), the normalized contact force $P$ can then be calculated from the following relations Eq.(2).

For details of the content in this section, we refer readers to Gao *et al*. (2002).

## 3. Solution methodology

In this section we will establish the analytical approximate solution to Eqs. (1) and (3) in terms of $\theta'(0) = a$. Along with the Maclaurin series expansion and the Chebyshev polynomials (Denman 1969, Jonckheere 1971, Li *et al*. 2008, Beléndez *et al*. 2009), we arrive at a new nonlinear equation with no circular functions. Introducing a variable $u = \theta/a$ (Denman 1969, Jonckheere 1971, Li *et al*. 2008, Beléndez *et al*. 2009) to Eqs. (1) and (3), and applying the Maclaurin series representation for the functions $\sin(au)/a$ by taking the first five terms yield a series of equations. Expressing the powers of $u$ in the resulting equations in the form of Chebyshev polynomials as $T_k(k = 1,2,\cdots)$, and then neglecting all terms associated with those Chebyshev polynomials for $T_i(i > 3)$ yield

$$u^{(4)} - 6a^2u'^2u'' + 2u'' + Q[B_1u + B_2u^3] = 0 \tag{4}$$

$$u(0) = u(2\pi) = u''(0) = u''(2\pi) = 0, \; u'(0) = 1 \tag{5}$$

where

$$B_1 = 1 - \frac{a^4}{24} + \frac{a^6}{180} - \frac{a^8}{2880}, \; B_2 = -\frac{2a^2}{3} + \frac{a^4}{6} - \frac{a^6}{60} + \frac{a^8}{1080}$$

A reasonable and simple initial approximation satisfying conditions in Eq. (5) can be taken as

$$u_0(\tau) = sin\tau,\ \ \tau \in [0, 2\pi] \tag{6}$$

Here, $u_0(\tau)$ is a periodic function of $\tau$, of period $2\pi$.

Substituting Eq. (6) into Eq. (4) and setting the resulting coefficient of term $\sin\tau$ equal zero, give

$$(4B_1 + 3B_2)Q - 4 + 6a^2 = 0 \tag{7}$$

From Eq. (7), the first analytical approximation for $Q$ is solved and expressed as a function of $a$, as

$$Q_0(a) = \frac{4 - 6a^2}{4B_1 + 3B_2} \tag{8}$$

and the corresponding analytical approximate solution is given by Eq. (6). Applying Eq. (2), we can obtain the first analytical approximations for $P$ as

$$P_0 = -2a^2cos^2\tau - a^4cos^4\tau + Q_0cos(asin\tau) + 3a^2sin^2\tau \tag{9}$$

And the first analytical approximate periodic solution can be expressed as

$$\theta_0(\tau) = asin\tau,\ \ \tau \in [0, 2\pi] \tag{10}$$

Next, we express the solution $(u(\tau), Q)$ of Eqs. (4) and (5) as

$$u(\tau) = u_0(\tau) + \Delta u_0(\tau),\ Q = Q_0 + \Delta Q_0 \tag{11}$$

Here, $(u_0(\tau), Q_0)$ is the principal part and $(\Delta u_0(\tau), \Delta Q_0)$ is the correction part. Substituting Eq. (11) into Eqs. (4) and (5) and linearizing with respect to $(\Delta u_0(\tau), \Delta Q_0)$ lead to

$$\begin{aligned} &u_0^{(4)} + \Delta u_0^{(4)} + 2u_0'' + 2\Delta u_0'' - 6a^2[2u_0''u_0'\Delta u_0' + u_0'^2(u_0'' + \Delta u_0'')] \\ &+ Q_0[B_1(u_0 + \Delta u_0) + B_2(u_0^3 + 3u_0^2\Delta u_0)] + \Delta Q_0(B_1u_0 + B_2u_0^3) = 0 \end{aligned} \tag{12}$$

$$\Delta u_0(0) = \Delta u_0(2\pi) = \Delta u_0''(0) = \Delta u_0''(2\pi) = 0,\ \Delta u_0'(0) = 0 \tag{13}$$

Where $\Delta u_0(\tau)$, a periodic function of period $2\pi$, $\Delta Q_0$ is a unknown quantity. The second approximate solution can be obtained by solving via the method of harmonic balance the resulting linear Eqs. (12) and (13) in $\Delta u_0(\tau)$, and $\Delta Q_0$.

$\Delta u_0(\tau)$ satisfying Eq. (13) is taken of the form

$$\Delta u_0(\tau) = z_0[\sin\tau - (\sin3\tau)/3] \tag{14}$$

Substituting Eqs. (6) and (14) into Eq. (12), expanding the expression into a trigonometric series and setting the resulting coefficients of the items $\sin\tau$ and $\sin3\tau$ to zeros, respectively, yield

$$\xi_1 \times \Delta Q_0 + \xi_2 \times z_0 + \xi_3 = 0 \tag{15a}$$
$$B_2 \times \Delta Q_0 + \eta_1 \times z_0 + \eta_2 = 0 \tag{15b}$$

Solving Eqs. (15 (a) and (b)) gives $z_0$, and $\Delta Q_0$:

$$\Delta Q_0 = (\xi_3 \eta_1 - \xi_2 \eta_2)/(\xi_2 B_2 - \xi_1 \eta_1) \tag{16a}$$

$$z_0 = (\xi_1 \eta_2 - \xi_3 B_2)/(\xi_2 B_2 - \xi_1 \eta_1) \tag{16b}$$

Where

$$\xi_1 = 4B_1 + 3B_2, \;\; \xi_2 = -4 + 12a^2 + (4B_1 + 10B_2)Q_0, \;\; \xi_3 = -4 + 6a^2 + (4B_1 + 3B_2)Q_0$$
$$\eta_1 = B_2 Q_0 - 6a^2, \;\; \eta_2 = [252 + 54a^2 + (4B_1 + 15B_2)Q_0]/3$$

Then we get the second analytical approximation to the post-buckling deformation as

$$Q_1(a) = Q_0(a) + \Delta Q_0(a) \tag{17a}$$

$$u_1(\tau) = \sin\tau + z_0[\sin\tau - (\sin 3\tau)/3], \;\; \tau \in [0, 2\pi] \tag{17b}$$

and get the second analytical approximate periodic solution $\theta(\tau)$ as

$$\theta_1(\tau) = a\{\sin\tau + z_0[\sin\tau - (\sin 3\tau)/3]\}, \;\; \tau \in [0, 2\pi] \tag{18a}$$

$$P_1 = 4\theta_1''\theta_1' + 3\theta_1''^2 - \theta_1'^4 + 2\theta_1'^2 + Q\cos\theta_1 \tag{18b}$$

It should be clear how the procedure works for constructing further analytical approximate solutions. It will be shown in the next section that Eqs. (18 (a) and (b)) provide excellent analytical approximations with respect to the "reference" solution obtained by the shooting method for *a*.

## 4. Results and discussion

In this section, the accuracy of the proposed analytical approximations will be illustrated by comparing with the "reference" solutions obtained by the shooting method (Seydel 1994).

The "reference" solutions $\theta_r$ ($\tau$, *a*), $P_r$ ($\tau$, *a*) and $Q_r$ (*a*) can be obtained by solving Eqs. (1)-(3) in terms of $\theta'(0) = a$. For comparison, the variations of the "reference" and approximate values of *Q* with *a* is shown in Fig. 2.

Fig. 2 indicates that Eq. (17(a)) is able to provide excellent approximate values of *Q* to numerical solution, while Eq. (8) may give acceptable results for small *a*.

The variations of the "reference" and approximate values of the normalized contact force *P* with *a* is shown in Fig. 3. Fig. 3 indicates that Eq. (18(b)) is able to provide excellent approximate values of the normalized contact force *P* to numerical solution for small as well as large value of *a*.

The "reference" solution $\theta_r$, approximate solutions $\theta_0$, and $\theta_1$ given in Eqs. (10) and (18(a)), respectively, are plotted in Figs. 4 and 5. These figures correspond to $a = 0.1$ and $a = 0.56$, respectively. Figs.

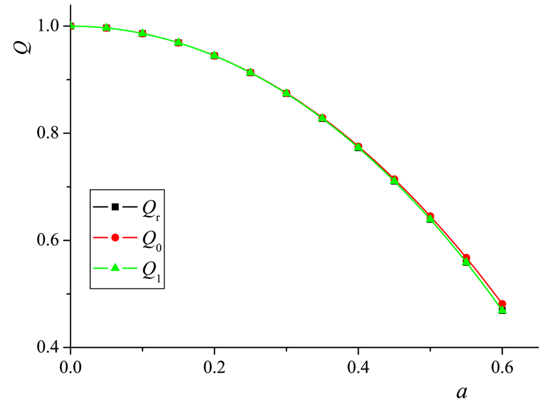


Fig. 2 Variations of the “reference” and approximate values of the normalized axial load $Q$ with $a$

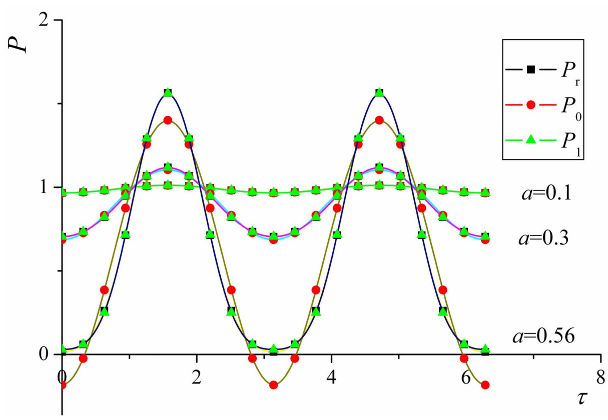


Fig. 3 Variations of the “reference” and approximaorce $P$ with $a$

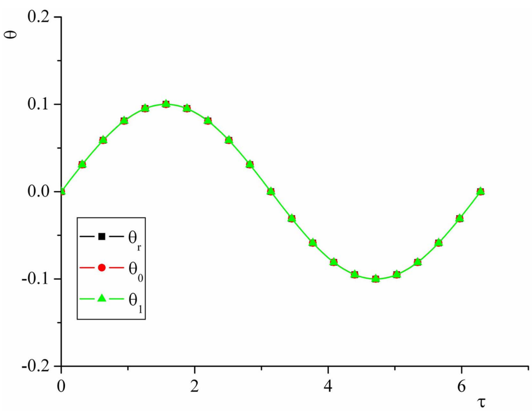


Fig. 4 Comparison of “reference” and approximate solution for $a = 0.1$

4 and 5 demonstrates that Eq. (18(b)) provides the best approximations with respect to the “reference” solution for both small and large $a$ despite significant increases in nonlinear effects.

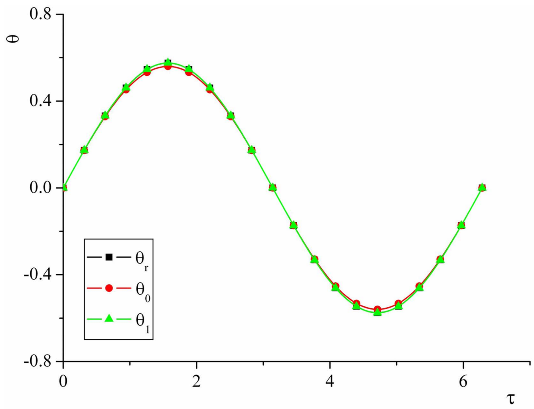


Fig. 5 Comparison of “reference” and approximate solution for $a = 0.56$

## 5. Conclusions

In this paper, having not considered the effect of torsion, we present an alternative, accurate approach to solve initial post-buckling behavior of a pipe in wellbore. Using Maclaurin series expansion and Chebyshev polynomials, the post-buckling deformation equation of the pipe can be transformed to a simple nonlinear system. The new approach combines linearization of governing equation and the method of harmonic balance to establish excellent analytical approximate solutions to initial post-buckling deformation of the pipe in terms of the positive derivative of helix angle at the end of the pipe. The advantage of the approximating procedure is its capability of predicting accurate periodic solutions spontaneously by solving the linear non-homogeneous differential equation in each approximating step. We are hence able to establish analytical approximate solutions. At the same time, it is an alternative approach for solving the initial post-buckling response problem of a pipe without using the Bessel functions. Furthermore, the present analysis demonstrates excellent results as compared to the “reference” solutions.

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## Notations

The following symbols are used in the paper:

| | | |
|---|---|---|
| $F$ | : | the axial load |
| $I$ | : | Moment of inertia |
| $\theta$ | : | The helix angle of the pipe |
| $u$ | : | $\theta/a$ |
| $N$ | : | the contact force |

| | | |
|---|---|---|
| $E$ | : | Young's modulus |
| $\tau$ | : | dimensionless length of the pipe |
| $R$ | : | the radius of curvature of the wellbore |
| $r$ | : | the distance between the center of the wellbore and the that of the pipe |
| $Q$ | : | $F/(EIRr\omega^4)$ |
| $P$ | : | $N/EIr\omega^4$ |
| $a$ | : | derivative of $\theta$ at the end $\tau = 0$ of the pipe |
| $u_i$, $Q_i$ | : | The $(i+1)$th analytical approximation to $u$, $Q$, respectively |
| $\Delta u_i$, $\Delta Q_i$ | : | correction to $u_i$, $Q_i$, respectively |
| $z_i$ | : | coefficient to be determined in the method of harmonic balance |
| $u_r$, $Q_r$ | : | "reference" value of $u$, $Q$, respectively |