Length effect on the stress concentration factor of a perforated orthotropic composite plate under in-plane loading

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Abstract. In this manuscript, a comprehensive numerical analysis is conducted to assess the accuracy of the Tan's model of obtaining the stress concentration factor, for a plate with finite dimensions containing an open hole. The influence of plate length on the stress distribution around the hole is studied. It is demonstrated that the plate length has a significant impact on the degree of accuracy of the method. Therefore, a critical length is proposed for this approach. Critical length is defined as the minimum length the plate requires, to ensure that the SCF which is obtained from the Tan's model will have sufficient accuracy. Finally, the approach of finite-width correction factor is adapted to develop a new model which is applicable for plates under biaxial loading conditions. In this method, biaxial loading is considered as a dominant axial force along the x-direction and lambda times the load ($-1\leq\lambda\leq1$), along the y-direction. A comparison between the SCFs obtained from the proposed analytical method and the SCFs obtained from the extensive FE studies, revealed an excellent agreement when the plate-width to hole-diameter ratio is more than 3 and the lambda is between -0.5 and 1.

Keywords: stress concentration factor (SCF); composite plate; circular hole; correction factors; finite element analysis

1. Introduction

Composites unique properties such as high stiffness and strength to weight ratio, and considerable corrosion and fatigue resistance, have led to their extensive use in a variety of industries including aerospace, transportation and off-shore structures. Therefore, a comprehensive understanding of their behavior is absolutely crucial. One of the important structural components is a plate containing a hole subjected to in-plane loading, because hole in the plate causes stress concentration which reduces structures strength and fatigue life.

Various experimental, analytical, approximate and hybrid methods can be employed to obtain the SCF in a plate. For the first time, Lekhnitskii (1968) proposed a closed-form analytical solution of the stress field in an infinite anisotropic plate containing a hole. Today, this method is known as the "Lekhnitskii Formalism".

Konish and Whitney (1975) provided two approximate solutions for stress field in an infinite orthotropic plate with a circular hole. Tan (1987) expanded this approach and presented two approximate solutions for infinite orthotropic plates with elliptical holes.

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Ukadgaonker and Rao (1999) expanded Savin’s solution (1961) and presented a closed-form solution for the stress around a triangular hole with a general shape in an infinite anisotropic plate. This method can be used for a plate under biaxial and shear loading (without using the superposition principle) as well as the hole under shear load and pressure. In their proceeding work, they developed closed-form solutions for the stress field around a hole with arbitrary shape in an infinite symmetric laminate under in-plane loads (Ukadgaonker 2000a) which indicated a satisfactory agreement with the previous results from literature and FE analysis (Ukadgaonker 2005). Distribution of moments around a hole with arbitrary shape in an infinite symmetric laminate under bending moments is also presented in (Ukadgaonker 2000b).

Hufenbach *et al.* (2008) employed the first-order shear deformation theory and expanded the Mindlin-Reissner plate theory to derive a system of coupled partial differential equations for the plate-bending and the membrane plate problem. The system of equations is solved by the Ritz method. This model provides a layer-by-layer stress analysis of thick-walled multilayered composites. They later proposed a method (Hufenbach 2010) which is based on the complex-valued displacement functions and solved the set of coupled PDEs by the boundary collocation and least square methods to provide a layer-by-layer analysis of stress field in a multilayered anisotropic composite.

Bambill *et al.* (2009) investigated the effect of different in-plane loading conditions, loading directions and fiber orientations in an orthotropic plate with square holes. Rao *et al.* (2010) also investigated the effect of fiber orientation, stacking sequence, biaxiallity ratio and loading direction in an infinite symmetric laminate with square and rectangular holes under in-plane loading.

Yang *et al.* (2010) applied the double U-transformation technique to the finite element governing equations of an infinite plate with a rectangular hole which is subjected to a bending load. They used a 12-DOF plate bending element with four nodes to analytically study the SCF.

Dai *et al.* (2010) proposed a theoretical solution for the three-dimensional stress field in an infinite plate with a through the thickness hole under in-plane loads. They employed the method in order to specifically investigate the effects of the plate thickness, Poisson's ratio and the far-field in-plane loads on the 3-dimensional stress field.

Mahi *et al.* (2014) developed a procedure based on the finite-difference method to evaluate the stress concentrations that occur at the edges of an FRP plate in strengthened beams under thermal loading. They investigated the effect of tapered edges on the SCFs.

Sharma *et al.* (2014a) provided the moment distribution around a polygonal hole, circular, elliptical and triangular holes (Patel 2015) and square holes (Sharma 2015). They (Sharma 2014b) also utilized a genetic algorithm to optimize the fiber orientation and stacking sequence in a laminate containing an elliptical hole which is under in-plane loading.

For the first time, Lin and Ko (1988) studied the stress field around an elliptical hole in a finite plate under in-plane loading. They used the Laurent series as a general form of the complex potential function and employed the *boundary collocation points* method to impose boundary conditions and calculate general form’s unknown coefficients.

Tan (1988) proposed a finite-width correction (FWC) factor to obtain the stress field in a finite plate containing an elliptical hole under uniaxial loading. For this purpose, it is assumed in this approach that the stress profile of the finite plate is identical to that of the same plate with infinite dimensions.

Xu *et al.* (1995a) believed that the method proposed by Lin and Ko (1988) can be time consuming and inaccurate. Therefore, they used the Faber series as a general form of the complex potential function. With this assumption, *boundary collocation points* method can be used to define any kind of boundary condition. In their proceeding works, they expanded this method for plates with multiple not loaded (Xu 1995b)/loaded (Xu 1999) holes.

Xiong (1999) used the Laurent series as a general form for the potential function to determine stress field in a finite plate containing one joint fastener. He believed that in order to use the *boundary collocation points* method, one needs to have an extensive knowledge on the subject since using this method includes choosing the collocation points on the plate edge and hole(s) boundary. If these points are not been chosen wisely, the solution may become incorrect. Therefore, Xiong developed a method based on the minimum potential energy principle to obtain the unknown coefficients of the Laurent series.

ESP (2007) adapted Xu's approach (Xu 1995a) to obtain the stress field for a finite plate with several loaded/not loaded holes. The author used the least square *boundary collocation points* method to apply both internal and external boundaries. By using different orders for the positive and negative terms in complex potential function, the author further improved accuracy of the Xu's method.

Russo and Zuccarello (2007) investigated the results from boundary element analysis and demonstrated that Tan's assumption about stress profiles in finite and infinite plates is not always accurate. They proposed a hybrid analytical-numerical method based on Tan's FWC factor and numerical studies, for determining stress field in a finite width laminates with a circular hole under axial loading. This model uses a correction function which adjusts the stress profile of the infinite plate.

Sevenois (2013) expanded the method proposed by Xiong (1999), to obtain stress field in a finite size rectangular orthotropic plate subjected to in-plane loading and has several elliptical (loaded/not loaded) holes.

Jain and Mittal (2008) conducted a FE analysis in order to investigate the hole-diameter to plate-width ratio on the SCF and deflection of composite plates under transverse loadings. Mao and Xu (2013) used the complex variable method along with boundary collocation method to develop the stress state in a finite composite plate weakened by multiple elliptical holes subjected to bending.

Zappalorto (2015) proposed an engineering formulae for obtaining SCFs in plates with shallow lateral or central notches, and sharp deep lateral notches under tensile loading. The author showed that although from strictly theoretical point of view his formulae are only valid for infinite or semi-infinite plates, they can be used for some special cases of finite plates as well.

Tan's finite width correction factor (Tan 1988) is obtained under the assumption of a remote uniaxial load (sufficiently long plate). In the second section of the manuscript, first, the plate length's effect on the accuracy of SCFs obtained from the Tan's method is studied. Then, a critical length as the minimum required plate length is proposed. It will be demonstrated that the SCF calculated by this method will have sufficient accuracy if the length of the plate is longer than the proposed critical length.

Furthermore, in the third section, Tan's approach is adapted to develop a new analytical correction factor that can account for the finite dimensions of plates under biaxial loading conditions. Then, parameters that may affect the accuracy of the proposed model will be identified. Subsequently, an extensive finite element analysis for different plate configurations and layups will be conducted to investigate effects of those parameters on the accuracy of the model. Finally, the validity of the model for a wide range of orthotropic laminated composite plates containing a circular hole subjected to in-plane loadings will be demonstrated

2. Stress concentration factor in plates under uniaxial loading

2.1. Finite-width correction factors

Tan (1988) proposed the concept of finite width correction (FWC) factors in an orthotropic plate with an elliptical hole under uniaxial loading. If hole's major diameter to minor diameter ratio (which is identified with ξ in the following formulation) is more than 4, the exact FWC factor (Eq. (1)) and approximate FWC factor (Eq. (2)) are recommended:

$$\begin{aligned}\frac{K_T{}^{\infty}}{K_T} = 1 - \frac{2a}{W} + Re\left\{\frac{1}{\mu_1-\mu_2}\left[\frac{\mu_2}{1+i\mu_1\xi}\left(1-\frac{2a}{W}-i\mu_1\xi\left(\frac{2a}{W}\right)-\sqrt{1-(1+\mu_1^2\xi^2)\,(2a/W)^2}\right)\right.\right.\\ \left.\left. - \frac{\mu_1}{1+i\mu_2\xi}\left(1-\frac{2a}{W}-i\mu_2\xi\left(\frac{2a}{W}\right)-\sqrt{1-(1+\mu_2^2\xi^2)(2a/W)^2}\right)\right]\right\}\end{aligned} \tag{1}$$

$$\begin{aligned}\frac{K_T{}^{\infty}}{K_T} = \frac{\xi^2}{(1-\xi)^2} + \frac{1-2\xi}{(1-\xi)^2}\sqrt{1+(\xi^2-1)(2a/W)^2} - \frac{\xi^2}{1-\xi}\frac{(2a/W)^2}{\sqrt{1+(\xi^2-1)(2a/W)^2}}\\ + \frac{\xi^7}{2}\left(\frac{2a}{W}\right)^6\left(K_T{}^{\infty}-1-\frac{2}{\xi}\right)\left\{[1+(\xi^2-1)(2a/W)^2]^{-5/2}\right.\\ \left. - \left(\frac{2a}{W}\right)^2[1+(\xi^2-1)(2a/W)^2]^{-7/2}\right\}\end{aligned} \tag{2}$$

However, if the hole's major diameter to minor diameter ratio is less than 4, the improved exact FWC factor (Eq. (3)) and improved approximate FWC factor (Eq. (4)) are recommended:

$$\begin{aligned}\frac{K_T{}^{\infty}}{K_T} = 1 - \frac{2a}{W}M + Re\left\{\frac{1}{\mu_1-\mu_2}\left[\frac{\mu_2}{1+i\mu_1\xi}\left(1-\frac{2a}{W}M-i\mu_1\xi\left(\frac{2a}{W}M\right)\right.\right.\right.\\ \left. - \sqrt{1-(1+\mu_1^2\xi^2)\left(\frac{2a}{W}M\right)^2}\right)\\ \left.\left. - \frac{\mu_1}{1+i\mu_2\xi}\left(1-\frac{2a}{W}M-i\mu_2\xi\left(\frac{2a}{W}M\right)-\sqrt{1-(1+\mu_2^2\xi^2)\left(\frac{2a}{W}M\right)^2}\right)\right]\right\}\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{K_T{}^{\infty}}{K_T} = \frac{\xi^2}{(1-\xi)^2} + \frac{1-2\xi}{(1-\xi)^2}\sqrt{1+(\xi^2-1)(2aM/W)^2} - \frac{\xi^2}{1-\xi}\frac{(2aM/W)^2}{\sqrt{1+(\xi^2-1)(2aM/W)^2}}\\ + \frac{\xi^7}{2}\left(\frac{2a}{W}M\right)^6\left(K_T{}^{\infty}-1-\frac{2}{\xi}\right)\left\{[1+(\xi^2-1)(2aM/W)^2]^{-5/2}\right.\\ \left. - \left(\frac{2a}{W}M\right)^2[1+(\xi^2-1)(2aM/W)^2]^{-7/2}\right\}\end{aligned} \tag{4}$$

In Eqs. (1)-(4), $K_T{}^{\infty}/K_T$ is the FWC factor, $K_T{}^{\infty} = 1 + n$ is the SCF of the infinite plate (n is given by Eq. (20)), K_T is SCF of the finite-width plate, $2a$ is the major diameter of the elliptical hole, W is the plate's width, μ_1 and μ_2 are principal roots of characteristic equation of the basic

Table 1 Material properties (Tan 1990)

Name	$E_x\ (GPa)$	$E_y\ (GPa)$	$G_{xy}\ (GPa)$	ν_{xy}
$AS_4/3502$	143.3	10.7	6.1	0.29
CFRP	173.90	22.49	7.15	0.26

differential equation of the 2-D problem of elasticity, and M is magnification factor which is used to magnify opening-to-width ratio $2a/W$, and it is defined by Eq. (5) (see 3.2.3.).

$$M^2 = \frac{\sqrt{1 - 8\left[\frac{3\,(1-2a/W)}{2+(1-2a/W)^3} - 1\right]} - 1}{2\,(2a/W)^2} \tag{5}$$

2.2. Plate length's influence on SCF

It is assumed in the Tans's model that the loads are applied far away from the hole boundary, thereby, the influence of *the end effects* and *boundary conditions* on the stress field in the vicinity of the hole are ignored. This issue was first pointed out by Troyani *et al.* (2002) in the context of isotropic materials. It is very well known that typical composite laminates have characteristic decay lengths of several times their width therefore, end effects cannot be ignored in them. In spite of the importance of the influence of length, some researchers have used small length to width ratios in their investigations. As an example, Russo and Zuccarello (2007) have modeled finite square plates (with different width to diameter ratios) by using boundary element method to develop a hybrid numerical-analytical correction factor based on the Tan's model. In this section, extensive finite element analyses are performed to investigate length effects on the accuracy SCFs that are calculated based on Tan's correction factor. It is aimed to assess the accuracy of the method in plates with different length to width ratios.

Finite element analysis using ABAQUS commercial code was employed to investigate effects of different length to width and width to hole diameter ratios on SCFs. For the plate under uniaxial loading, only half-width of the plate has been modeled due to complete material and geometrical symmetry. ABAQUS S8R shell elements have been used to discretize the geometries. For each geometry, several models with increasing number of elements had been studied to ensure that the convergence was achieved. Fig. 1 shows an exemplary meshed geometry and the general boundary conditions of the problem. One side of the plate is fixed, while a distributed shell edge force is applied to the other side, as it is shown in Fig. 1.

Three different laminates, $0_6, (0_2/90)_s, (0_4/\pm45/90_2)_s$ have been studied here. The mechanical properties of each lamina that were used in the analyses are available in table 1. The first two laminates are made from $AS_4/3502$ and the last laminate is made from the CFRP.

Figs. 2 and 3 indicate changes in SCF of 0_6 and $(0_2/90)_s$ laminates with different length to width ratios. These two charts show that the material properties, composite lay-up and width-to-diameter ratio can influence how the plate length affects SCF. However, it can be concluded that the plate length-to-width ratio has a significant effect on the value of SCF and consequently, on the accuracy of the Tan's method (also concluded in (Bakhshandeh 2007, Sanchez 2014)). As an

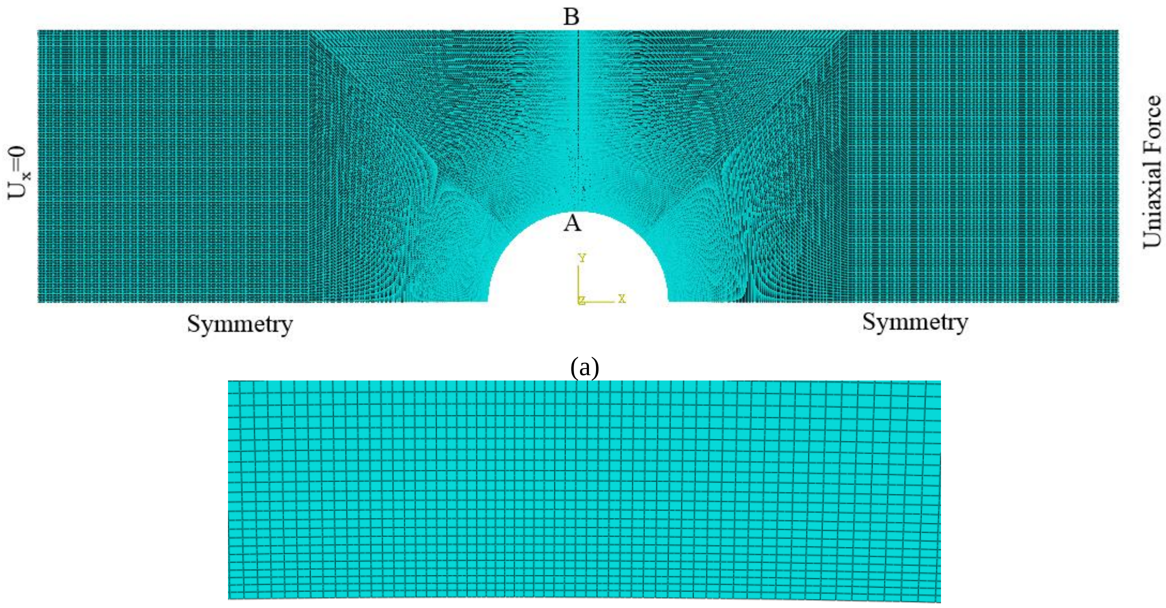

Fig. 1 A: an exemplary meshed geometry and the general B.C. of uniaxial loading, B: Magnifica tion around point "A"

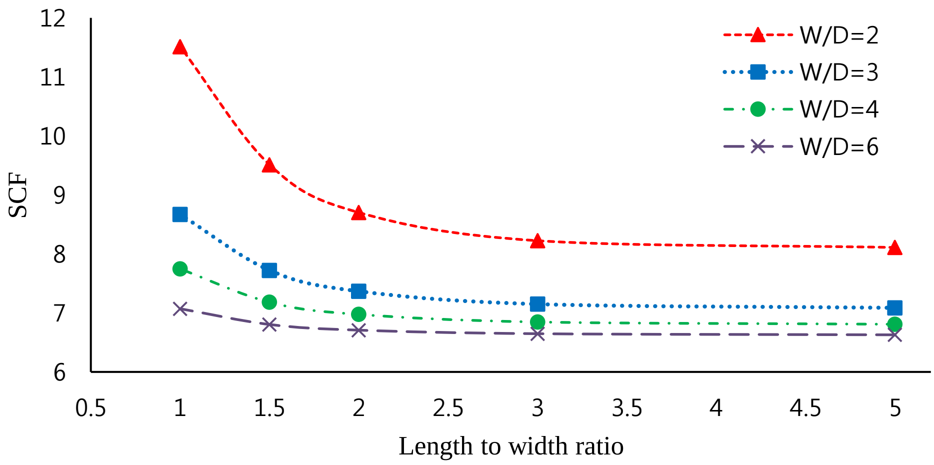

Fig. 2 SCF against length to width ratio for 0_6 Laminate

example, one can see that in 0_6 laminate with $width/diameter = 2$ and $length/width = 1$ (square plate) SCF is 42.2% higher than the same plate with $length/width = 5$.

If a plate containing a hole is long enough, the stress field will be able to become uniform before it reaches the discontinuity. Thereby, when force lines reach the hole, they will gradually change direction and revolve around the hole. However, when the plate length is not long enough (with respect to the plate width), stress field does not reach the discontinuity uniformly. Therefore, the density of force lines is higher in the middle of the plate-width before reaching the

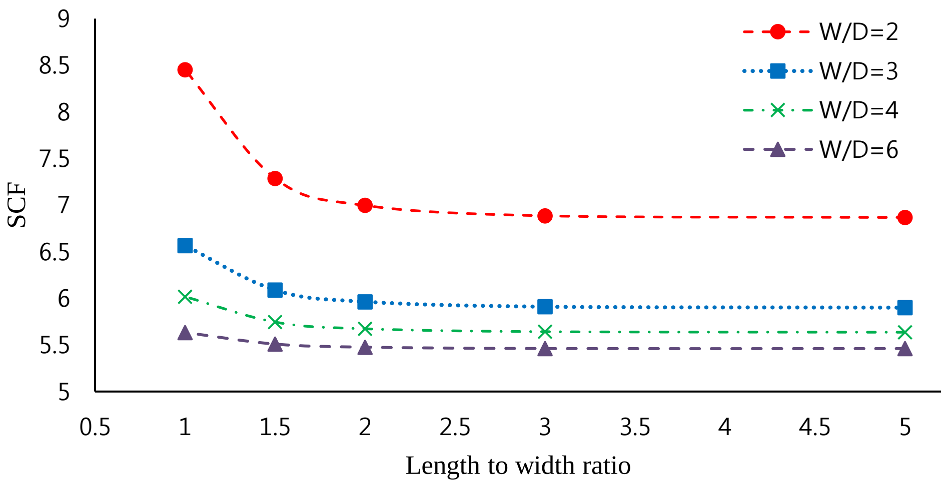

Fig. 3 SCF against length to width ratio for $(0_2/90)_s$ Laminate

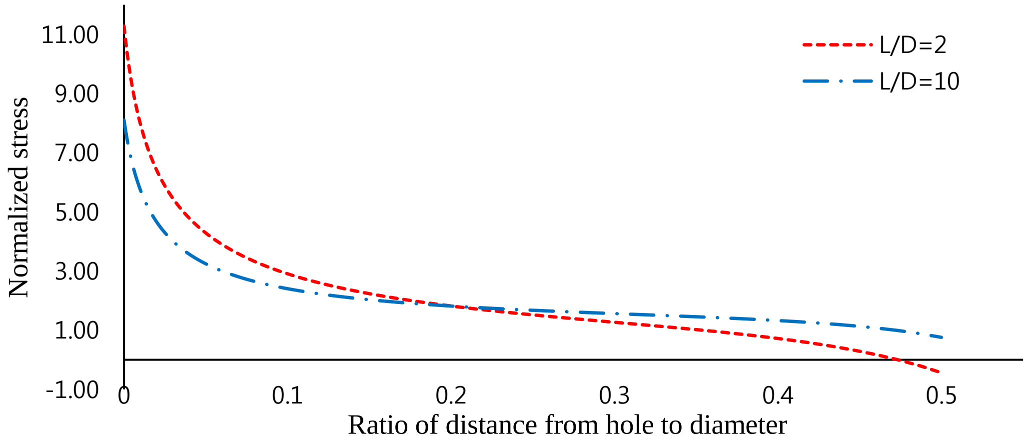

Fig. 4 Normalized stress distribution of a 0_6 laminate with $width/diameter = 2$ along line $\overline{AB}$

discontinuity. As a result, in a section which the hole exists, SCF will be higher in comparison with the former situation. Therefore, stress near the plate edge will be lower in the same cross-section. Fig. 4 shows the normalized stress distribution of a 0_6 laminate with $width/diameter = 2$ along line AB (definition of line AB is available in Fig. 1). The upper line belongs to the plate with $length/diameter = 2$ and the other line belongs to the plate with $length/diameter = 10$. In the case with $length/diameter = 2$, SCF is higher and stress near the plate-edge is compressive.

2.3 Critical length of a plate

As shown in the previous section and pointed out previously in (Bakhshandeh 2007, Sanchez 2014), using the Tan's FWC factor may lead to significant errors in plates with small length-to-width ratio. In order to find a suitable range for the application of Tan's model, Bakhshandeh and

Rajabi (2007) presented a transition length for an orthotropic plate. Following Troyani *et al.* (2002) they defined the transition length as the length of the member for which the SCF calculated by Tan's model remains within 1 percent of the corresponding long member. In 2014, Sánchez and Troyani demonstrated that the results presented in (Bakhshandeh 2007) are only acceptable for the specific material properties given in that work. This is because, unlike the isotropic case, theoretical SCF values and their corresponding transition lengths are a function of four independent constants of a two-dimensional orthotropic material. It is aimed in this section of the present work to use the very well established concepts regarding the Saint-Venant principle in composite materials to propose a critical (i.e. transition) length that can account for a general orthotropic material.

Horgan (1982) used the analogy of Papkovich-Fadle Eigen functions to obtain the characteristic decay length for a semi-infinite rectangular strip. He analytically proved that:

$$\chi \propto b\sqrt{\frac{E_1}{G}} \tag{6}$$

In this equation, χ is the characteristic decay length, E_1 and G are laminate's engineering constants and b is plate width. Characteristic decay length is an axial distance over which the stress decays to fraction $1/e$ of its value at the end of strip.

In a plate containing a hole, $(W-D)/W$ is a key parameter for the stress distribution in the vicinity of the hole. It is also evident that in a given laminate, critical length should converge to a fixed value when plate-width mathematically approaches infinity. Based on the mentioned considerations and the finite element studies, Eq. (6) is adapted and the following critical length is proposed:

$$L_C = 2\sqrt{\frac{E_1}{G}\frac{(W-D)}{W}}D \tag{7}$$

In Eq. (7), L_C is the critical length, W is plate width, D is hole diameter, E_1 is laminate's engineering constant in the loading direction and G is laminate's shear engineering constant in the same coordinate. Error of the analytical method (i.e. Tan's model) is used for the presentation of data in the figures of the rest of the study. This value is used to demonstrate the functionality of the proposed critical length as well. This value is calculated as $Error\,(\%) = 100 \times \left(K_T^{theory} - K_T^{FEM}\right)/K_T^{FEM}$ through out the research. It was aimed in developing the critical length that the Tan's model produces *less than 5% error*. The normalized critical lengths (L_C/W) for the exemplary laminates are tabulated in table 2. For each finite element model, the minimum length of the model is equal to its width (square plate). So for the plates with $width/diameter > 6$ model's minimum length will be definitely longer than corresponding critical length and Tan's method will have sufficient accuracy. Figs. 5 to 7 indicate application of the critical length. In these Figures, errors of the SCF in comparison with the finite element results are plotted against the normalized plate length. Vertical lines are values of the normalized critical lengths. As an example, the vertical line $L_C/W = 2.4$ in Fig. 5 is the normalized critical length for the plates with $width/diameter = 2$. When the normalized length is more than 2.4, error is less than 4%, but the error significantly increases up to 30% when the plate length is less than the critical length. In some cases like $D/W = 1/6$ of Fig. 7, critical length ($L_C/W = 0.7$) is lower than the minimum plate length ($L/W = 1$) and clearly, error is acceptable for all of the plates with $D/W = 1/6$. The

Table 2 Critical lengths normalized to plate width

Lay-up	$D/W = \frac{1}{2}$	$D/W = \frac{1}{3}$	$D/W = \frac{1}{4}$	$D/W = \frac{1}{6}$
0_6	2.4	2.1	1.8	1.3
$(0_2/90)_s$	2.0	1.8	1.5	1.1
$(0_4/\pm45/90_2)_s$	1.3	1.1	1.0	0.7

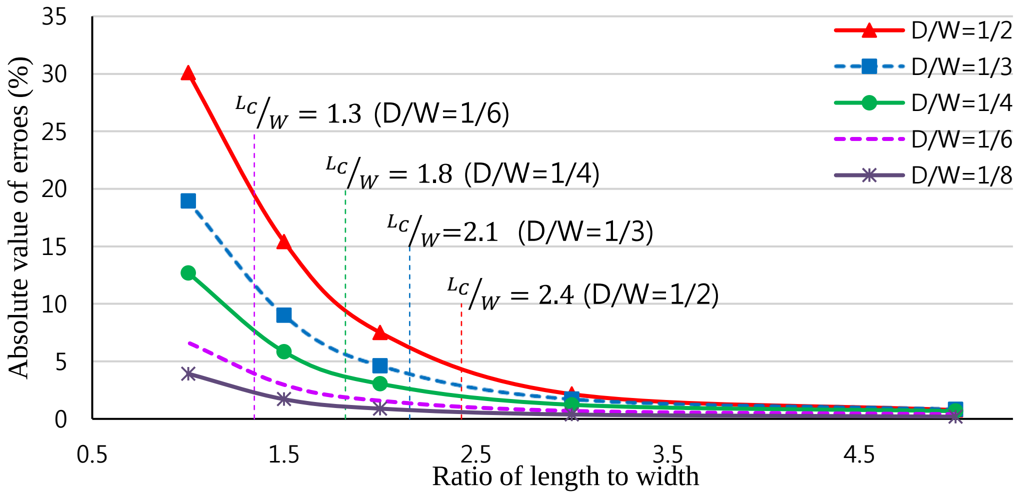

Fig. 5 Error of SCFs obtained from the exact solution (Eq. (1)) against length to width ratio for 0_6 laminate

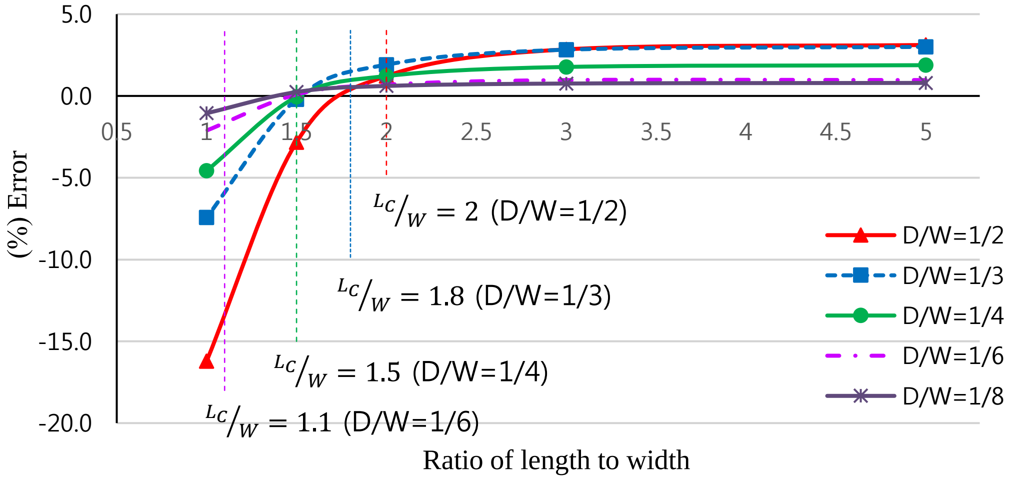

Fig. 6 Error of SCFs obtained from the improved approximate solution (Eq. (4)) against length t o width ratio for $(0_2/90)_s$ laminate

only exception is the case of $[0_4/\pm45/90_2]_s$ laminate (Fig. 7) where the combined effects of the specific material *orthotropy ratio* and the specific *width to diameter ratio* produces relatively

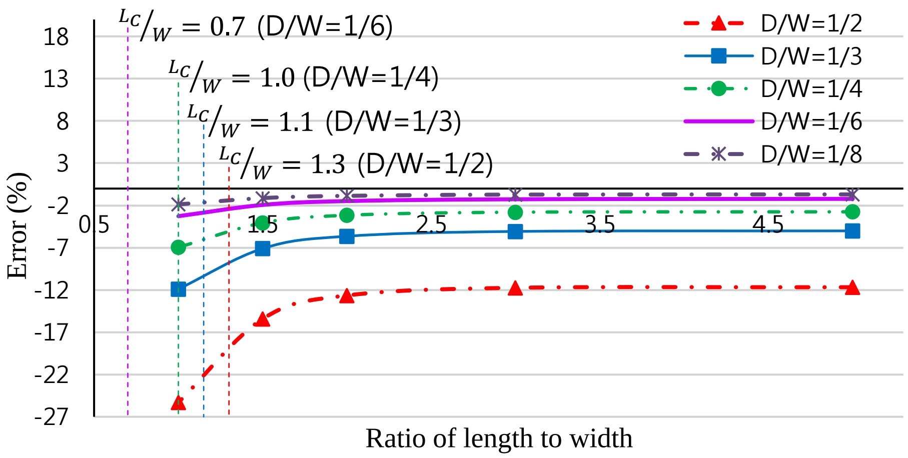

Fig. 7 Error of SCFs obtained from the exact solution (Eq. (1)) against length to width ratio for $(0_4/\pm 45/90_2)_s$ laminate

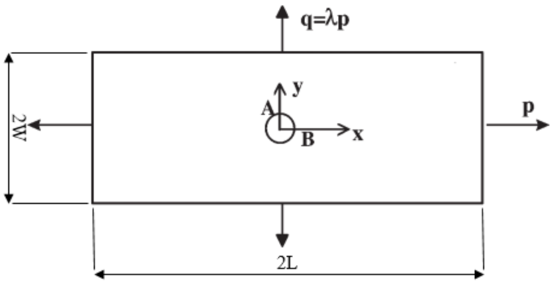

Fig. 8 A finite plate under biaxial loading

higher errors, regardless of the plate length (as also discussed in (Bakhshandeh 2007)). But again, even in this case, Tan's model produces substantially lower errors in plates that are longer than the critical length of the plate. For instance, the error for the plate with $L/W = 1.5$ is 15%, versus the 27% error for the square plate (normalized critical length is $L_C/W = 1.3$ in this case).

It was demonstrated in this section that the application of Tan's FWC factor, can result in erroneous values for the SCF of plates with low length to width ratios (depending upon the degree of orthotropy and the hole diameter to width ratio) which is in agreement with the previous results and discussions presented in (Bakhshandeh 2007, Sanchez 2014). This is due to the fact that all the edge effects are neglected in the development of this model, since it has been developed under the assumption of a remote uniaxial load. A new critical load was proposed in this section that shall be employed to determine the validity of the Tan's model for a specific application with an *arbitrary orthotropic material.*

3. Stress concentration factor for plates under biaxial loading

In this section, Tan's approach is adapted to develop a new analytical model for obtaining the SCF in a plate containing a hole subjected to biaxial loading. Lekhnitskii's (1968) has provided the exact stress distribution in an infinite anisotropic plate containing an elliptical hole, and Soutis (1998) has proposed an approximate stress distribution in an infinite orthotropic plate containing a circular hole. In this section, these two representations of the stress field in infinite plates are employed to obtain the correction factors for the plate under biaxial loading.

3.1 Stress distribution in an infinite plate

A plate under biaxial loading is defined as Fig. 8. It is assumed that a dominant load is applied along the x-direction and lambda (called biaxiallity ratio) times the force ($-1 \leq \lambda \leq 1$) is applied along y-direction.

Soutis (1998) has assumed that stress distribution in an infinite orthotropic plate is approximately equal to the summation of stress distribution of an isotropic plate, with two polynomial terms of orders -6 and -8. He proposed the following approximate relations:

$$\frac{\sigma_{xx}^{ortho}(0,y)}{P} = 1 + \frac{\lambda+1}{2}\left(\frac{R}{y}\right)^2 + \frac{3\,(1-\lambda)}{2}\left(\frac{R}{y}\right)^4 - (3-\lambda)\frac{[H_A-1]}{2}\left[5\left(\frac{R}{y}\right)^6 - 7\left(\frac{R}{y}\right)^8\right] \tag{8}$$

$$\frac{\sigma_{yy}^{ortho}(x,0)}{P} = \lambda + \frac{\lambda+1}{2}\left(\frac{R}{x}\right)^2 + \frac{3\,(1-\lambda)}{2}\left(\frac{R}{x}\right)^4 - (3\lambda-1)\frac{[H_B-1]}{2}\left[5\left(\frac{R}{x}\right)^6 - 7\left(\frac{R}{x}\right)^8\right] \tag{9}$$

In Eqs. (8) and (9):

$$H_B = \frac{K_B^{ortho}}{K_B^{iso}}, \qquad K_B^{iso} = 3\lambda - 1 \tag{10}$$

$$H_A = \frac{K_A^{ortho}}{K_A^{iso}}, \qquad K_A^{iso} = 3 - \lambda \tag{11}$$

Where K_A^{ortho} and K_B^{ortho} are SCFs of the orthotropic plate at points A and B, respectively.

3.2 Finite width correction factors

A finite width correction factor is a scale factor which is applied to multiply the notched infinite plate solution to obtain the notched finite plate result (Tan 1988). It is assumed that the normal stress profiles (in both x and y directions) of a finite plate is identical to that of an infinite plate except for a FWC factor. The following relations mathematically present the definition:

$$\sigma_x(0,y) = \frac{K_T^x}{K_T^{x\infty}}\,\sigma_x{}^{\infty}(0,y) \tag{12}$$

$$\sigma_y(x,0) = \frac{K_T^y}{K_T^{y\infty}}\,\sigma_y{}^{\infty}(x,0) \tag{13}$$

In these equations, $K_T^x/K_T^{x\infty}$ and $K_T^y/K_T^{y\infty}$ are FWC factors in x and y directions, respectively. Infinite superscripts are for the terms regarding the infinite plate. FWC factors presented in Eqs. (12) and (13) can be calculated by solving the equation of static equilibrium in each direction.

$$\sum F_x = 0 \rightarrow 2\int_R^W \sigma_x\,dy = 2W.P \tag{14}$$

$$\sum F_y = 0 \rightarrow 2\int_R^L \sigma_y \, dx = 2L.\lambda P \tag{15}$$

In these equations, R is the circular hole's radius, W is half of the plate width, L is half of the plate length (according to Fig. 8), P is the dominant force, and λ is the biaxiallity ratio. By substituting Eqs. (12) and (13) into Eqs. (14) and (15), one can calculate inverse of the FWC factors.

$$\frac{K_T^{x\infty}}{K_T^x} = \frac{\int \sigma_x{}^\infty(0,y)dy}{PW}, \qquad \frac{K_T^{y\infty}}{K_T^y} = \frac{\int \sigma_y{}^\infty(x,0)dx}{\lambda PL} \tag{16}$$

By substituting the stress distributions in the infinite plate into Eq. (16), basic FWC factors will be calculated. The FWC factors that are derived based on the exact representation of the stress field in the infinite plate (Lekhnitskii solution) will be denoted by *exact FWC factor*. On the other hand, those which are derived based on the approximate representations (Soutis solution) are denoted by *approximate FWC factors* in the rest of the paper.

It is worthwhile to mention that the FWC factors are always applicable in the axial loading conditions since the maximum tangential stress around the hole always occurs at point A. But in the biaxial loading conditions the location of maximum tangential stress around the hole can change due to the laminate's lay-up and biaxiallity ratio. Clearly, if the maximum tangential stress does not occur at points A or B, using the proposed FWC factors will lead to erroneous results. When $\Delta \geq 0$ (Δ is defined in Eq. (17)) maximum stress always occurs at one of the points A or B (Russo 2007).

$$\Delta = (\frac{E_x}{G_{xy}}) - 2(\vartheta_{xy} + \sqrt{\frac{E_x}{E_y}}) \tag{17}$$

In this equation E_x, E_y and G_{xy} are laminate's engineering constants in loading directions and ϑ_{xy} is Poisson's ratio in the same coordinate system.

In laminates which $\Delta < 0$, FWC factors are applicable only if the maximum stress around the hole occurred at A or B. Lekhnitskii (1968) has presented the following relations for obtaining the tangential stress around the hole.

$$\sigma_\vartheta = P\frac{E_\vartheta}{E_1}\{[-kcos^2\vartheta + (1+n)\sin^2\vartheta] + \lambda k[(k+n)cos^2\vartheta - \sin^2\vartheta]\} \tag{18}$$

Where in this equation:

$$\frac{1}{E_\vartheta} = \frac{\sin^4\vartheta}{E_1} + \left(\frac{1}{G} - \frac{2\nu_{21}}{E_1}\right)\sin^2\vartheta\cos^2\vartheta + \frac{\cos^4\vartheta}{E_2} \tag{19}$$

$$n = \sqrt{2\left(\sqrt{E_1/E_2} - \nu_1\right) + E_1/G} \tag{20}$$

$$k = \frac{E_1}{E_2} \tag{21}$$

According to the Eq. (17), $\Delta = 0$ is for the quasi-isotropic laminates. It can be easily shown that for $(0_m/90_n)_s$ laminates with $m, n \geq 0$, and $\left(0_m/\pm 45_p/90_n\right)_s$ laminates with $\frac{m+n}{m+n+p} \geq 0.5$, Δ is positive. It's good to note that, it is shown in (Russo 2007) that if the laminate's lay-up is optimized for the loading along x direction, Δ will not be negative.

3.2.1 Basic exact FWC factors

With using Lekhnitskii's exact stress distribution along with Eq. (16), basic exact FWC factors are defined.

$$\frac{K_T^{x\infty}}{K_T^x} = 1 - \frac{R}{W} + Re\left\{\frac{1}{\mu_1 - \mu_2}\left[\frac{(\lambda\mu_2 - i)\mu_1^2}{1 + i\mu_1}\left(1 - \frac{R}{W} - \frac{1}{\mu_1}\sqrt{\mu_1^2 - \left(\frac{R}{W}\right)^2 (1 + \mu_1^2)} + \frac{i}{\mu_1}\frac{R}{W}\right) + \frac{(i - \lambda\mu_1)\mu_2^2}{1 + i\mu_2}\left(1 - \frac{R}{W} - \frac{1}{\mu_2}\sqrt{\mu_2^2 - \left(\frac{R}{W}\right)^2 (1 + \mu_2^2)} + \frac{i}{\mu_2}\frac{R}{W}\right)\right]\right\} \tag{22}$$

$$\frac{K_T^{y\infty}}{K_T^y} = 1 - \frac{R}{W} + Re\left\{\frac{1}{(\mu_1 - \mu_2)\lambda}\left[\frac{\lambda\mu_2 - i}{1 + i\mu_1}\left(1 - \frac{R}{L} - \sqrt{1 - \left(\frac{R}{L}\right)^2 (1 + \mu_1^2)} + \frac{R}{L}\sqrt{-\mu_1^2}\right) + \frac{i - \lambda\mu_1}{1 + i\mu_2}\left(1 - \frac{R}{L} - \sqrt{1 - \left(\frac{R}{L}\right)^2 (1 + \mu_2^2)} + \frac{R}{L}\sqrt{-\mu_2^2}\right)\right]\right\} \tag{23}$$

3.2.2 Basic approximate FWC factors

By substituting the approximate stress distribution (Eqs. (8) and (9)) into Eq. (16), basic approximate FWC factors are defined.

$$\frac{K_T^{x\infty}}{K_T^x} = 1 - \frac{R}{W} - \frac{\lambda + 1}{2}\frac{R}{W}\left(\frac{R}{W} - 1\right) - \frac{1 - \lambda}{2}\frac{R}{W}\left(\left(\frac{R}{W}\right)^3 - 1\right) - \frac{(3 - \lambda)(H_A - 1)}{2}\left(\frac{R}{W}\right)^6\left(\left(\frac{R}{W}\right)^2 - 1\right) \tag{24}$$

$$\frac{K_T^{y\infty}}{K_T^y} = 1 - \frac{R}{L} - \frac{1}{2}\frac{\lambda + 1}{\lambda}\frac{R}{L}\left(\frac{R}{L} - 1\right) - \frac{1}{2}\frac{1 - \lambda}{\lambda}\frac{R}{L}\left(\left(\frac{R}{L}\right)^3 - 1\right) - \frac{(3\lambda - 1)(H_B - 1)}{2\lambda}\left(\frac{R}{L}\right)^6\left(\left(\frac{R}{L}\right)^2 - 1\right) \tag{25}$$

3.2.3 Improved exact FWC factors

Since the accuracy of the basic FWC factor for plates containing elliptical holes with $\frac{major\ diameter}{minor\ diameter} < 4$ was not satisfactory, Tan (1988) developed a magnification factor (M) to improve the accuracy of the model. Considering the fact that Heywood correction factor has an excellent accuracy for the isotropic materials, Tan magnified the opening-to-width ratio by the factor M, and defined this factor in such a way that the anisotropic solution reduces to the Heywood formula under the isotropic condition.

Biaxial loading condition reduces to axial loading in the case of $\lambda = 0$. Since the present method must agree with the Tan's FWC factor under this condition, the same magnification factor is employed to improve the accuracy of the proposed model. All R/L and R/W ratios in Eqs.

(22) and (23) are multiplied by M (M is defined in Eq. (5)) to obtain the improved exact FWC factors for biaxial loing condition.

$$\frac{K_T^{x\infty}}{K_T^x} = 1 - \frac{R}{W}M + Re\left\{\frac{1}{\mu_1 - \mu_2}\left[\frac{(\lambda\mu_2 - i)\mu_1^2}{1 + i\mu_1}\left(1 - \frac{R}{W}M - \frac{1}{\mu_1}\sqrt{\mu_1^2 - \left(\frac{R}{W}M\right)^2(1 + \mu_1^2)} + \frac{i}{\mu_1}\frac{R}{W}M\right) + \frac{(i - \lambda\mu_1)\mu_2^2}{1 + i\mu_2}\left(1 - \frac{R}{W}M - \frac{1}{\mu_2}\sqrt{\mu_2^2 - \left(\frac{R}{W}M\right)^2(1 + \mu_2^2)} + \frac{i}{\mu_2}\frac{R}{W}M\right)\right]\right\} \tag{26}$$

$$\frac{K_T^{y\infty}}{K_T^y} = 1 - \frac{R}{W} + Re\left\{\frac{1}{(\mu_1 - \mu_2)\lambda}\left[\frac{\lambda\mu_2 - i}{1 + i\mu_1}\left(1 - \frac{R}{L}M - \sqrt{1 - \left(\frac{R}{L}M\right)^2(1 + \mu_1^2)} + \frac{R}{L}M\sqrt{-\mu_1^2}\right) + \frac{i - \lambda\mu_1}{1 + i\mu_2}\left(1 - \frac{R}{L}M - \sqrt{1 - \left(\frac{R}{L}M\right)^2(1 + \mu_2^2)} + \frac{R}{L}M\sqrt{-\mu_2^2}\right)\right]\right\} \tag{27}$$

3.2.4 Improved approximate FWC factors

To obtain the improved approximate FWC factors, all R/L and R/W ratios in Eqs. (24) to (25) are multiplied by M.

$$\frac{K_T^{x\infty}}{K_T^x} = 1 - \frac{R}{W}M - \frac{\lambda + 1}{2}\frac{R}{W}M(\frac{R}{W}M - 1) - \frac{1 - \lambda}{2}\frac{R}{W}M\left((\frac{R}{W}M)^3 - 1\right) - \frac{(3 - \lambda)(H_A - 1)}{2}(\frac{R}{W}M)^6\left((\frac{R}{W}M)^2 - 1\right) \tag{28}$$

$$\frac{K_T^{y\infty}}{K_T^y} = 1 - \frac{R}{L}M - \frac{1}{2}\frac{\lambda + 1}{\lambda}\frac{R}{L}M(\frac{R}{L}M - 1) - \frac{1}{2}\frac{1 - \lambda}{\lambda}\frac{R}{L}M\left((\frac{R}{L}M)^3 - 1\right) - \frac{(3\lambda - 1)(H_B - 1)}{2\lambda}(\frac{R}{L}M)^6\left((\frac{R}{L}M)^2 - 1\right) \tag{29}$$

3.3 Results and discussion

Finite element analysis using ABAQUS was employed to investigate the influence of length to diameter ratio, width to diameter ratio and biaxiallity ratio, on the accuracy of the proposed method. For a plate under biaxial loading, the whole plate has been modeled. ABAQUS S8R shell elements have been used to obtain results. For each geometry, several models with increasing number of elements had been studied to ensure that the convergence was achieved. Fig. 9 shows one of the final meshed models with general boundary condition of the problem. In this section $(0/\pm45/90)_s$, $(0_4/90_3)_s$, $(0_2/90)_s$ and 0_6 laminates are studied. $AS_4/3502$'s mechanical properties (Table 1) have been used for the simulation.

Generally, for the case of equal axial loads ($\lambda = 1$), improved theories are more accurate. Although in small width to diameter ratios (less than 3) finite element results deviate from

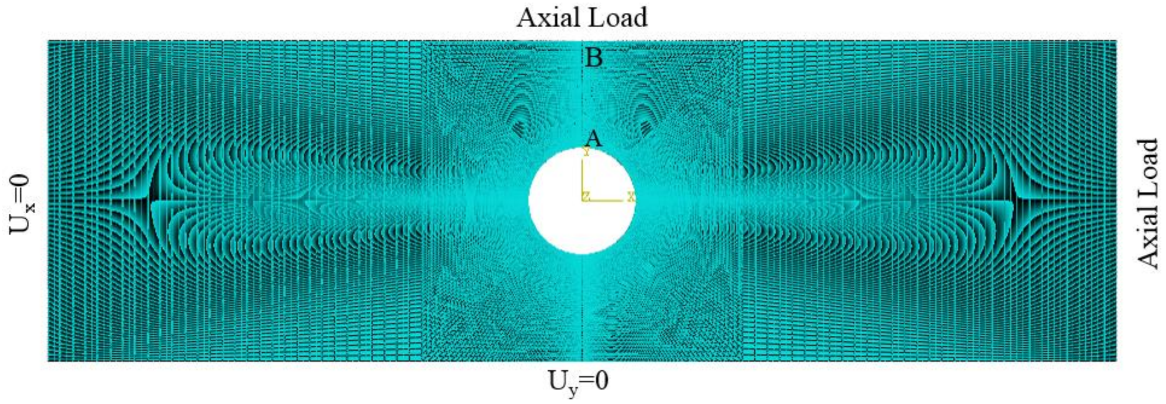

Fig. 9 An exemplary meshed model and general B.C. of biaxial loading

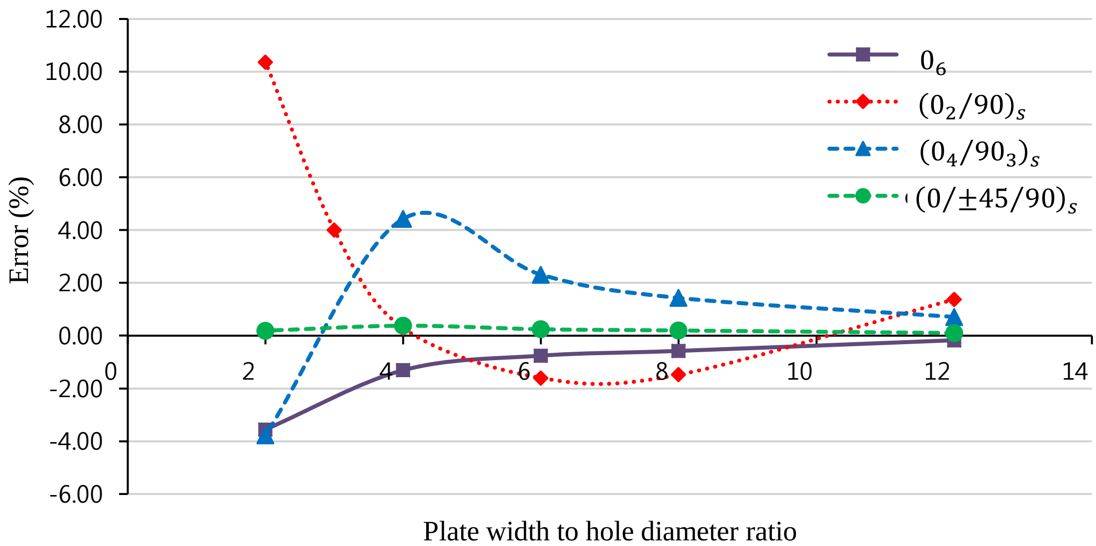

Fig. 10 Error of Eq. (26) versus plate width to hole diameter ratio ($\lambda = 1$)

analytical results due to dominant edge effects (Fig. 10). This phenomenon is clearly depicted in Fig. 11. This figure shows the normal stress profile of the $(0_2/90)_s$ laminate with $w/d = 2$ along the line $\overline{AB}$. The trend of the profile is not descending on the entire path. The maximum normal stress occurs at the hole boundary, it decreases as the distance from the hole boundary increases and the trend reverses after a while. As a result, stress has a considerable value at the edge of the plate (point B) which causes a drop of stress at the hole boundary.

Lambda's value have a significant effect on the results too (Figs. 12 and 13). Generally, when lambda increases from -1 to +1, the absolute value of error decreases, consistently. When $0 \leq \lambda$ analytical values of SCF have a great agreement with finite element results (errors within 4%). For negative lambdas, when $-0.5 \leq \lambda \leq 0$, analytical values of SCF are still in good agreement with the FE solution and errors remain within 8%. However, as the lambda decreases from -0.5 to -1, finite element results further deviate from analytical results.

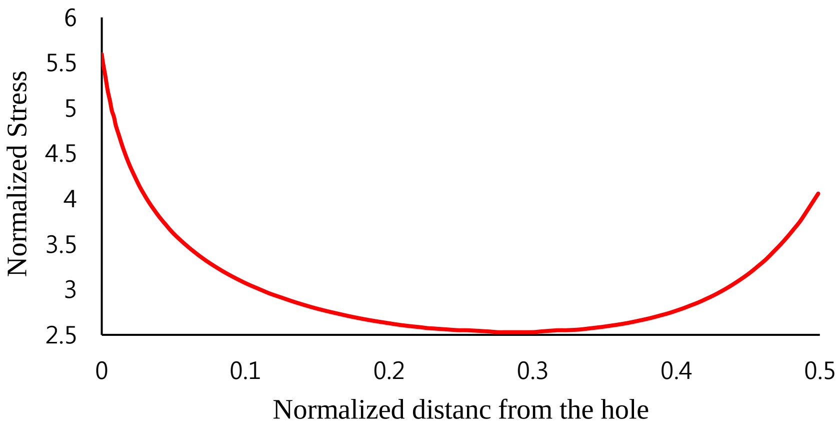

Fig. 11 Stress profile of the $(0_2/90)_s$ laminate with $width/diameter = 2$ and $\lambda = 1$ along theli ne $\overline{AB}$

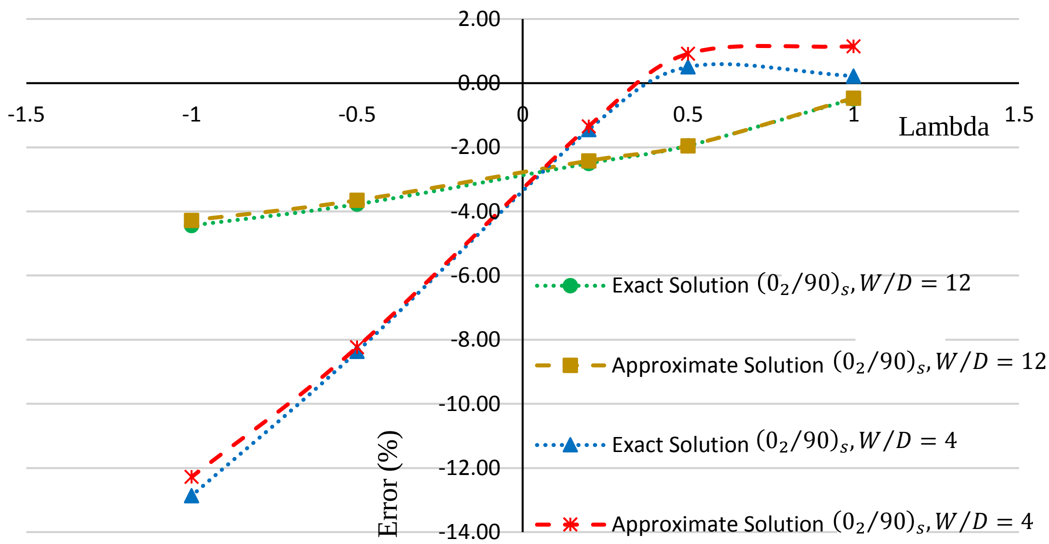

Fig. 12 Error versus lambda for $(0_2/90)_s$ laminates

Studying models with different lengths and widths in Figs. 14 and 15 shows that in a specific width to diameter ratio, with increasing the length to diameter ratio errors will be a more acceptable range and they converge to a constant value, as expected. In the $(0_2/90)_s$ laminate with $\lambda = 1$ (Fig. 14) error of the basic equations remain in a very good range of 4% for plates with $L/D \geq 4$. In addition, even for the smallest plate dimension to diameter ratio (i.e. the max error) the error is 9.2%. The same trend also holds for other cases, e.g. for the case of $\lambda = -0.5$ (Fig. 15), the deviation of the improved equations from the FE solution remains within 8% for $L/D \geq 4$.

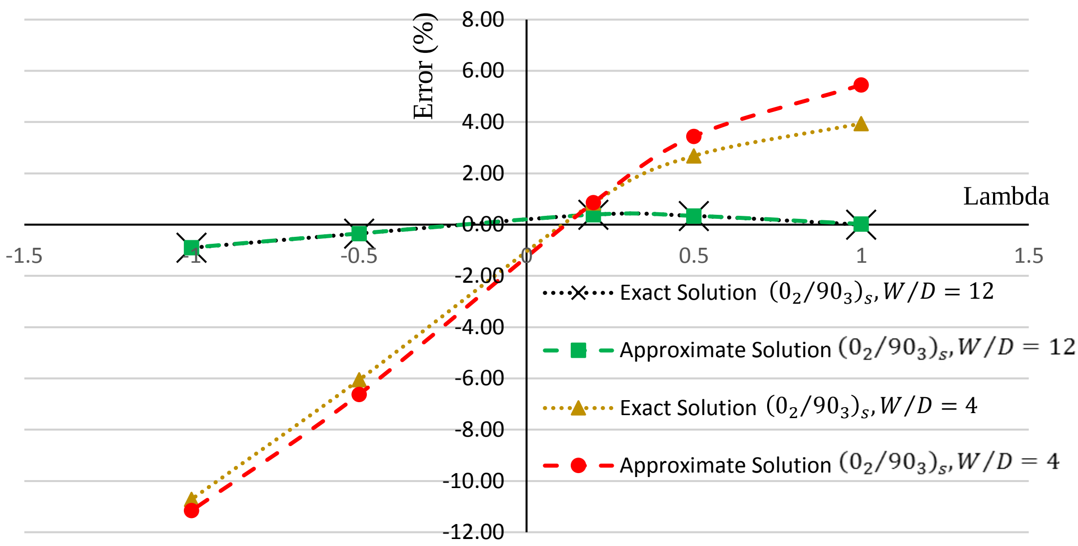

Fig. 13. Error versus lambda for $(0_2/90_3)_s$ laminates

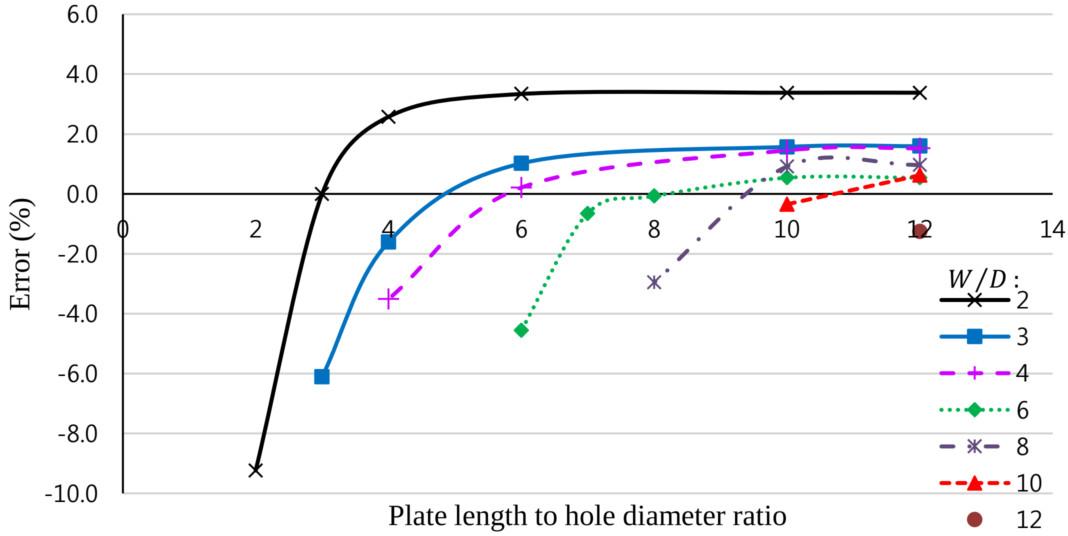

Fig. 14. Error of the basic exact solution (Eq. (22)) versus length to diameter ratio in $(0_2/90)_s$ laminates with $\lambda = 1$

4. Conclusions

In the present manuscript, an extensive numerical analysis is conducted using ABAQUS to investigate the effect of plate length on the accuracy of the stress concentration factor which is calculated using Tan's finite-width correction factor in a plate containing a circular hole. It is

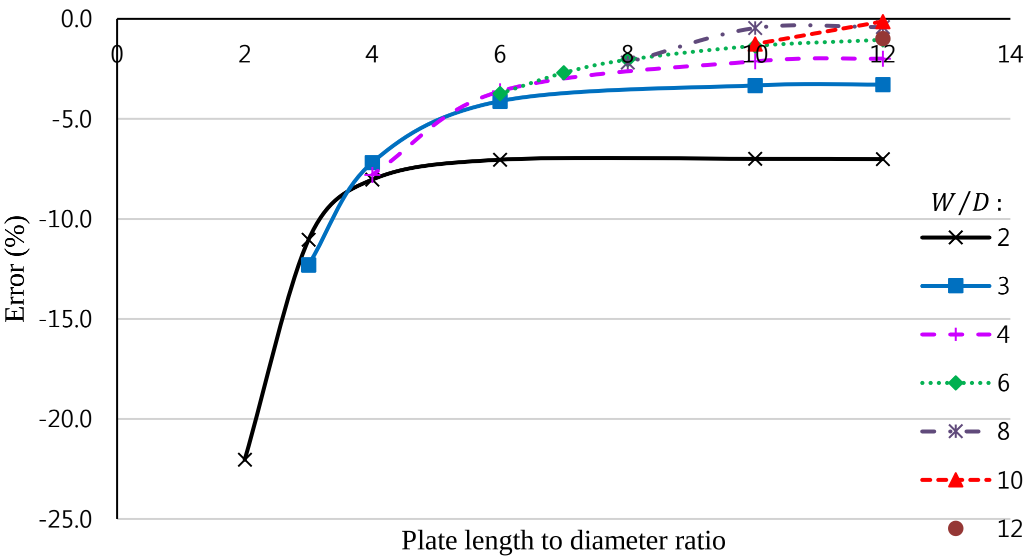

Fig. 15. Error of the improved exact solution (Eq. (26)) versus length to diameter ratio in $(0_2/90)_s$ laminates with $\lambda = -0.5$

demonstrated that the plate length has a significant impact on the degree of the accuracy of this method. Horgan's analytical solution for obtaining the characteristic decay length in a composite plate is adapted to propose a critical length for the plate. It is demonstrated that the stress concentration factor which is calculated by Tan's model will have sufficient accuracy, only if the plate length is longer than the proposed critical length.

Since Tan's analytical method is only valid for the plates subjected to uniaxial loading, this approach was adapted to develop a new model which is applicable for plates under biaxial loading conditions. Comparison between the analytical results from the proposed model and results from the finite element analysis for several different plate configurations and layups, revealed an excellent agreement for the plates with $W/D > 3$ and $-0.5 \leq \lambda \leq 1$.

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