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Stoneley wave propagation in an orthotropic thermoelastic media with fractional order theory

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Abstract. The present paper deals with the study of Stoneley wave propagation at the interface of two dissimilar homogeneous orthotropic thermoelastic solids with three phase lags in the context of fractional order theory of thermoelasticity. By using appropriate boundary conditions the secular equations of Stoneley waves are derived in the form of the determinant. The wave characteristics like phase velocity, attenuation coefficient are computed numerically. The numerical simulated results have shown with the help of graphs to show the effect of fractional parameter on the phase velocity, attenuation coefficient, displacement components, stress components and temperature change.

Keywords: orthotropic medium; fractional order; Stoneley wave propagation; phase velocity; attenuation coefficient; phase lags

1. Introduction

During the last few decades, the study of surface waves propagating in different media along the interface of two dissimilar half spaces in perfect contact is one of the interested and important areas of research in the scientific community. These elastic waves propagate through earth's surface have variable properties when they travel through different interfaces or mediums. These waves not only give us the information about the internal structure of the earth but also helpful in the study of materials like minerals, crystals and metals, etc. The interface waves require at least one of the two mediums is solid while the other may be a vacuum, air, a liquid or solid. However, the boundary or interface wave which occurs at the interface of two solid mediums is known as Stoneley wave. The penetration depth of these waves is similar as that of Rayleigh wave and are highly dispersive in nature. Stoneley waves are well known in the study of geophysics, ocean acoustics and non-destructive evaluation, etc.

The wave that can propagate along a fluid-solid interface is referred to as Scholte wave. Stoneley waves have high intensity at the boundary and decreases exponentially far away from it. Sonic tool generates a wave in a borehole is an example of Stoneley wave. The dispersion equation for the propagation of Stoneley waves was derived by Stoneley (1994). Tajuddin (1995) investigated the existence of Stoneley waves at an interface between two micropolar elastic spaces. Moreover, the

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fractional order theory of generalized thermoelasticity is an important branch of solid mechanics. Fractional calculus is used by several researchers and scientists to find the solution of many practical problems, which contains differential equations of non-integer order. Fractional calculus has been applied in many fields like in quantum mechanics, nuclear physics, chemistry, astrophysics, control theory, etc. The growing popularity of fractional calculus is due to its global dependency, which is more appropriate to solve some particular problems of physical processes. Caputo (1967) was the one who gave the definition of fractional derivative of order 'α' where $0 < \alpha \leq 1$. Ezzat (2010) proposed a model by using Taylor's series expansion in fractional heat conduction equation with fractional order.

Abbas (2018) studied the effect of fractional parameter in a two dimensional problem in the context of thermal shock with three types of conductivity weak, normal and strong conductivity. Abbas and Youssef (2015) studied the two dimensional problem with the help of fractional order theory with one relaxation time for porous materials. Ezzat and Ezzat (2016) formulated fractional thermoelasticity applications for porous asphaltic materials. Kumar and Gupta (2013) studied the plane wave propagation in anisotropic thermoelastic medium with fractional order derivative and voids with two-phase-lag and three-phase-lag model of heat transfer. Abd-Alla and Ahmed (2003) studied the Stoneley and Rayleigh waves in a non-homogeneous orthotropic elastic medium under the effect of gravity. Kumar *et al*. (2013) investigated the propagation of Stoneley waves at the boundary of two couple stress thermoelastic medium by using LS and GL theories. Tomar and Singh (2006) investigated the propagation of Stoneley waves at an interface between two microstretch elastic half-spaces. Markov (2009) discussed the propagation of Stoneley elastic wave at the boundary of two fluid-saturated porous media. Mahmoud (2014) studied the Rayleigh wave propagation of an initially stressed non-homogeneous orthotropic solid under the effect of magnetic field, gravity field, and rotation. Kumar *et al*. (2013) investigated the propagation of Stoneley waves at the boundary of two couple stress thermoelastic medium by using LS and GL theories. Markov (2009) discussed the propagation of Stoneley elastic wave at the boundary of two fluid-saturated porous media. Mahmoud (2014) studied the Rayleigh wave propagation of an initially stressed non-homogeneous orthotropic solid under the effect of magnetic field, gravity field and rotation.

Saeed *et al*. (2020) investigated the effect of thermal relaxation times in a poroelastic material by using the finite element method. Othman *et al*. (2019) studied the effect of gravity field on the fibre-reinforced thermoelastic medium with two temperature and three phase-lag model of heat transfer by using GN-II and GN-III theory. Kumar *et al*. (2017) solved the Stoneley wave propagation problem at the interface of two dissimilar transversely isotropic thermoelastic solids without energy dissipation and with two temperatures. Ahmed and Abo-Dahab (2012) studied the propagation of Rayleigh and Stoneley waves in a thermoelastic orthotropic granular half-space under the effect of initial stress and gravity. Abbas and Marin (2018) found the analytical solutions of two dimensional problem of half space due to laser pulse. Mohamed *et al*. (2009) analysed the flow, chemical reaction and mass transfer of a steady laminar boundary layer of an electrically conducting and heat generating fluid driven by a continuously moving porous surface embedded in a non-Darcian porous medium in the presence of a transfer magnetic field. Abbas *et al*. (2009) studied the effect of thermal dispersion on free convection in a fluid saturated porous medium. Abbas (2014) also studied the thermoelastic interaction in a semi-infinite medium due to ramp-type heat. Abbas (2014) studied the thermoelastic interactions in a three dimensional isotropic solid with temperature dependent material properties. Chadwick and Borejko (1994) studied the existence-uniqueness theory for Stoneley waves propagating along a plane interface between different isotropic elastic media. Kumar (2018) also studied thepropagation of Stoneley waves at the boundary surface of thermoelastic diffusion solid and microstretch thermoelastic diffusion solid. Including this several researchers worked in different areas by using thermoelasticity theories as Marin (1996, 1997a, b), Othman and Marin (2017), Lata and Zakhmi (2019, 2020), Lata *et al*. (2017), Kaur and Lata (2020), Biswas and Abo-Dahab (2018), Abd-Alla (1999), Ezzat (2020).

Inspite of all the above investigations, we see that propagation of Stoneley waves using Fractional order theory of thermoelasticity in orthotropic medium with three phase lags has not been studied yet. The aim of this paper is to examine the propagation of Stoneley waves in an orthotropic media with Fractional order theory of generalized thermoelasticity with three phase lags. In three-phase lag model the heat conduction equation consists of three phase lags namely τ_t, τ_v and τ_q, i.e. (phase lag of temperature gradient, phase lag of thermal displacement and phase lag of heat flux vector. The effect of fractional parameter on the various components has been computed numerically. The variations in the normal stress, normal displacement and temperature change have been depicted through graphs.

2. Basic equations

Following Kumar and Chawla (2014), the constitutive relations and basic equations for anisotropic media in the absence of body forces and heat sources with three-phase-lag thermoelastic model are the following.

$$\sigma_{ij} = c_{ijkm}e_{km} - \beta_{ij}\,T \tag{1}$$

$$\sigma_{ij,j} = \rho\ddot{u}_i \tag{2}$$

$$K_{ij}\left(1+\frac{\tau_t^{\alpha}}{\alpha!}\frac{\partial}{\partial t^{\alpha}}\right)\dot{T}_{,ji} + K_{ij}^{*}\left(1+\frac{\tau_v^{\alpha}}{\alpha!}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)T_{,ji} = \left[\left(1+\frac{\tau_q^{\alpha}}{\alpha!}\right)+\frac{\tau_q^{2\alpha!}}{2\alpha!}\right][\rho C_E\ddot{T}+\beta_{ij}T_0\ddot{e}_{ij}] \tag{3}$$

In Eqs. (1)-(3), c_{ijkm} (= c_{kmij} = c_{jikm} = c_{ijmk}) is the tensor of elastic constant, ρ is the density, T_0 is the reference temperature such that $|\frac{T}{T_0}| \ll 1$, u_i are the components of displacement vector u, C_E is the specific heat at constant strain, σ_{ij} = (σ_{ji}) and $e_{ij} = \frac{1}{2}\,(u_{i,j}\;+u_{j,i})$ are the components of stress and strain tensors respectively. $T(x, y, z, t)$ is the temperature distribution from the reference temperature T_0 and and τ_q, τ_t, τ_v is respectively the phase lag of the heat flux, the phase lag of the temperature gradient and the phase lag of the thermal displacement, β_{ij} are tensor of thermal moduli, K_{ij} and K_{ij}^{*} are the components of thermal conductivity and material characteristic constant respectively. Also in all above equations dot (.) represents the partial derivative w.r.t time and (,) denote the partial derivative w.r.t spatial coordinate.

Here, the symmetries of elastic parameters C_{ijkm} is due to

i. The stress tensor is symmetric, which is only possible if ($C_{ijkm} = C_{jikm}$).

ii. If a strain energy density exists for the material, the elastic stiffness tensor must satisfy $C_{ijkm} = C_{kmij}$.

iii. From stress tensor and elastic stiffness tensor symmetries infer ($C_{ijkm} = C_{ijmk}$) and $C_{ijkm} = C_{jikm} = C_{ijmk}$.

Following Kumar and Chawla (2014), the Eq. (1) for an orthotropic media in Cartesian coordinate system (x, y, z) in component form can be written as

$$\sigma_{xx} = C_{11}e_{xx} + C_{12}e_{yy} + C_{13}e_{zz} - \beta_1 T$$
$$\sigma_{yy} = C_{12}e_{xx} + C_{22}e_{yy} + C_{23}e_{zz} - \beta_2 T$$
$$\sigma_{zz} = C_{13}e_{xx} + C_{23}e_{yy} + C_{33}e_{zz} - \beta_3 T$$
$$\sigma_{yz} = 2\,C_{44}e_{yz}, \qquad \sigma_{xz} = 2\,C_{55}e_{xz}, \qquad \sigma_{xy} = 2\,C_{66}e_{xy}$$

where

$$e_{xx} = \frac{\partial u}{\partial x}, \qquad e_{zz} = \frac{\partial w}{\partial z}, \qquad e_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{4}$$

Now Eqs. (2) and (3) with the help of and above equations can be written as

$$C_{11}\frac{\partial^2 u}{\partial x^2} + C_{66}\frac{\partial^2 u}{\partial y^2} + C_{55}\frac{\partial^2 u}{\partial z^2} + (C_{12} + C_{66})\frac{\partial^2 v}{\partial x \partial y} + (C_{13} + C_{55})\frac{\partial^2 w}{\partial x \partial z} - \beta_1 \frac{\partial T}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \tag{5}$$

$$(C_{12} + C_{66})\frac{\partial^2 u}{\partial x \partial y} + C_{66}\frac{\partial^2 v}{\partial x^2} + C_{22}\frac{\partial^2 v}{\partial y^2} + C_{44}\frac{\partial^2 v}{\partial z^2} + (C_{23} + C_{44})\frac{\partial^2 w}{\partial y \partial z} - \beta_2 \frac{\partial T}{\partial y} = \rho \frac{\partial^2 v}{\partial t^2} \tag{6}$$

$$(C_{13} + C_{55})\frac{\partial^2 u}{\partial x \partial z} + (C_{23} + C_{44})\frac{\partial^2 v}{\partial y \partial z} + C_{55}\frac{\partial^2 w}{\partial x^2} + C_{44}\frac{\partial^2 w}{\partial y^2} + C_{33}\frac{\partial^2 w}{\partial z^2} - \beta_3 \frac{\partial T}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2} \tag{7}$$

$$K_1\left(1 + \frac{\tau_t^\alpha}{\alpha!}\frac{\partial}{\partial t^\alpha}\right)\dot{T}_{,11} + K_2\left(1 + \frac{\tau_t^\alpha}{\alpha!}\frac{\partial}{\partial t^\alpha}\right)\dot{T}_{,22} + K_3\left(1 + \frac{\tau_t^\alpha}{\alpha!}\frac{\partial}{\partial t^\alpha}\right)\dot{T}_{,33} + K_1^*\left(1 + \frac{\tau_v^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)T_{,11} + K_2^*\left(1 + \frac{\tau_v^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)T_{,22} + K_3^*\left(1 + \frac{\tau_v^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)T_{,33} = \left[1 + \frac{\tau_q^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha!}}{2\alpha!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right]\left[\rho C_E \ddot{T} + T_0\left(\beta_1 \ddot{u}_{1,1} + \beta_2 \ddot{u}_{2,2} + \beta_3 \ddot{u}_{3,3}\right)\right] \tag{8}$$

3. Formulation of the problem

We consider a perfectly conducting homogeneous orthotropic thermoelastic half-pace M_1 overlying another homogeneous orthotropic thermoelastic half-space M_2. The origin of the coordinate system (x, y, z) is taken on $(z = 0)$. We choose x-axis in the direction of wave propagation in such a way that all the particles on a line parallel to the y-axis are equally displaced, so that $v = 0$ and u, w, T are independent of y. Medium M_2 occupies the region $-\infty < x \leq 0$ and the medium M_1 occupies the region $0 \leq x < \infty$. The plane $x_3 = 0$ represents the interface between the two media M_1 and M_2. We define all the quantities without bar for the medium M_1 and with bar for medium M_2. For the two dimensional problem in xz-plane, we take

$$\vec{u} = u\,(x, z, t), \qquad \vec{v} = 0, \qquad \vec{w} = w(x, z, t) \quad \text{and} \quad T = T(x, z, t) \tag{9}$$

With the aid of Eq. (9), Eqs. (5)-(8) reduce to the form

$$C_{11}\frac{\partial^2 u}{\partial x^2} + C_{55}\frac{\partial^2 u}{\partial z^2} + (C_{13} + C_{55})\frac{\partial^2 w}{\partial x \partial z} - \beta_1 \frac{\partial T}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \tag{10}$$

$$(C_{13} + C_{55})\frac{\partial^2 u}{\partial x \partial z} + C_{55}\frac{\partial^2 w}{\partial x^2} + C_{33}\frac{\partial^2 w}{\partial z^2} - \beta_3 \frac{\partial T}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2} \tag{11}$$

$$K_1\left(1+\frac{\tau_t^\alpha}{\alpha!}\frac{\partial}{\partial t^\alpha}\right)\dot{T}_{,11}+K_3\left(1+\frac{\tau_t^\alpha}{\alpha!}\frac{\partial}{\partial t^\alpha}\right)\dot{T}_{,33}+K_1^*\left(1+\frac{\tau_v^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)T_{,11}+K_3^*(1+\frac{\tau_v^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha})T_{,33}$$
$$=\left[1+\frac{\tau_q^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}+\frac{\tau_q^{2\alpha!}}{2\alpha!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right][\rho C_E\ddot{T}+T_0\left(\beta_1\ddot{u}_{1,1}+\beta_3\ddot{u}_{3,3}\right)] \tag{12}$$

Also,

$$\sigma_{xx}= C_{11}\,e_{xx}+C_{13}\,e_{zz}-\beta_1\mathrm{T} \tag{13}$$

$$\sigma_{zz}= C_{13}\,e_{xx}+C_{33}\,e_{zz}-\beta_3\mathrm{T} \tag{14}$$

$$\sigma_{xz}=2\,C_{55}\,e_{xz} \tag{15}$$

And $\beta_{ij}=\beta_i\delta_{ij}, K_{ij}=K_i\,\delta_{ij}, K_{ij}^*=K_i^*\delta_{ij}$, i is not summed; i =1, 2, 3 and δ_{ij} is Kronecker delta.

The following dimensionless quantities are used to find the solution

$$x'=\frac{x}{L},\quad z'=\frac{z}{L},\quad u'=\frac{\rho c_1^2}{LT_0\beta_1}u,\quad w'=\frac{\rho c_1^2}{LT_0\beta_1}w$$
$$t'=\frac{C_1}{L}t,\quad \sigma_{33}'=\frac{\sigma_{33}}{T_0\beta_1},\quad \sigma_{31}'=\frac{\sigma_{31}}{T_0\beta_1},\quad T'=\frac{T}{T_0} \tag{16}$$

where $c_1^2=\frac{c_{11}}{\rho}$ and L is a constant of dimension of length.

With the help of dimensionless quantities given by Eq. (16) in Eqs. (10)-(12) and suppressing the primes we get

$$\frac{\partial^2 u}{\partial x^2}+\delta_1\frac{\partial^2 u}{\partial z^2}+\delta_2\frac{\partial^2 w}{\partial x\partial z}-\frac{\partial T}{\partial x}=\frac{\partial^2 u}{\partial t^2} \tag{17}$$

$$\delta_3\frac{\partial^2 w}{\partial z^2}+\delta_1\frac{\partial^2 w}{\partial x^2}+\delta_2\frac{\partial^2 u}{\partial x\partial z}-\varepsilon\frac{\partial T}{\partial z}=\frac{\partial^2 w}{\partial t^2} \tag{18}$$

$$\epsilon_1\tau_t\frac{\partial}{\partial t}\left(\frac{\partial^2 T}{\partial x^2}\right)+\epsilon_2\tau_t\frac{\partial}{\partial t}\left(\frac{\partial^2 T}{\partial z^2}\right)+\epsilon_3\tau_v\left(\frac{\partial^2 T}{\partial x^2}\right)+\epsilon_4\tau_v\left(\frac{\partial^2 T}{\partial z^2}\right)=\tau_q\left[\frac{\partial^2 T}{\partial z^2}+\epsilon_5\frac{\partial^2}{\partial t^2}\left(\frac{\partial u}{\partial x}+\varepsilon\frac{\partial w}{\partial z}\right)\right] \tag{19}$$

where

$$\delta_1=\frac{c_{55}}{c_{11}},\quad \delta_2=\frac{c_{13}+c_{15}}{c_{11}},\quad \delta_3=\frac{c_{33}}{c_{11}},\quad \tau_t=\left(1+\frac{\tau_t^\alpha}{\alpha!}\frac{\partial}{\partial t^\alpha}\right)$$
$$\tau_v=\left(1+\frac{\tau_v^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right),\quad \tau_q=\left(1+\frac{\tau_q^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}+\frac{\tau_q^{2\alpha!}}{2\alpha!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right),\quad \epsilon_1=\frac{K_1}{\rho\,L\,C_1C_E}$$
$$\epsilon_2=\frac{K_3}{\rho\,LC_1C_E},\quad \epsilon_3=\frac{K_1^*}{\rho c_1^2C_E},\quad \epsilon_4=\frac{K_3^*}{\rho c_1^2C_E},\quad \epsilon_5=\frac{\beta_1^2T_0}{\rho^2c_1^2C_E},\quad \varepsilon=\frac{\beta_3}{\beta_1}$$

4. Solution of the problem

We assume the Stoneley wave solution of the form

$$(u, w, T) = (u^*, w^*, T^*)(z) e^{i\xi(x-ct)} \tag{20}$$

where c = ω/ξ the phase velocity, ξ is a wave number and ω is the angular frequency of the wave. Using Eq. (20) in Eqs. (17)-(19) and satisfying the radiation conditions u, w, $T \to 0$ as z $\to \infty$, the values of u, w and T are obtained for the medium M_1.

On substituting the values in Eqs. (17)-(19), we get

$$u^*[p_1 + \delta_1 D^2] + w^*[p_3 D] + [i\xi]T^* = 0 \tag{21}$$

$$u^*[p_3 D] + w^*[p_2 + \delta_3 D^2] - [\varepsilon D]T^* = 0 \tag{22}$$

$$u^*[p_7 \varepsilon_5 \tau'_q] + w^*[p_6 \varepsilon \varepsilon_5 \tau'_q D] + [\ p_4 \varepsilon_1 \tau'_t - p_5 \varepsilon_2 \tau'_t D^2 - \xi^2 \varepsilon_3 \tau'_v + \varepsilon_4 \tau'_v D^2 + p_6 \tau'_q]T^* = 0 \tag{23}$$

where

$$D = \frac{d}{dz}, \qquad p_1 = \xi^2(c^2 - 1), \qquad p_2 = \xi^2(c^2 - \delta_1)$$

$$p_3 = i\,\xi\,\delta_2, \qquad p_4 = i\xi^3 c, \qquad p_5 = i\xi c$$

$$p_6 = \xi^2 c^2, \qquad p_7 = i\xi^3, \qquad \tau'_t = 1 + \frac{\tau_t^\alpha}{\alpha!}(-i\xi c)^\alpha$$

$$\tau'_v = 1 + \frac{\tau_v^\alpha}{\alpha!}(-i\xi c)^\alpha, \qquad \tau'_q = 1 + \frac{\tau_q^\alpha}{\alpha!}(-i\xi c)^\alpha + \frac{\tau_q^{2\alpha}}{2\alpha!}(-i\xi c)^{2\alpha}$$

These above resulting Eqs. (21)-(23) have non-trivial solution if the determinant of the coefficients (u^*, w^*, T^*) vanishes, which give the following characteristic equation.

$$(D^6 + QD^4 + RD^2 + S)\,(u^*, w^*, T^*) = 0 \tag{24}$$

where

$$D = \frac{d}{dz}, \qquad P = [\,\tau'_v \delta_3 \delta_1 \varepsilon_4 - \varepsilon_2 \delta_1 \delta_3 p_5 \tau'_t\,]$$

$$Q = \tau'_t\,[-p_1 p_5 \varepsilon_2 \delta_3 - p_2 p_5 \delta_1 \varepsilon_2 + p_3^2 p_5 \varepsilon_2 + \varepsilon_1 \delta_1 \delta_3 p_4] + \tau'_v[\,p_1 \delta_3 \varepsilon_4 + p_2 \varepsilon_4 \delta_1 - \xi^2 \delta_1 \delta_3 \varepsilon_3 - \varepsilon_4 p_3^2] + \tau'_q\,[\,p_6\,\delta_1\,\delta_3 + p_6 \varepsilon_5 \delta_1 \varepsilon^2\,]$$

$$R = \tau'_t[-p_1 p_2\, p_5 \varepsilon_2 + \varepsilon_1 \delta_3 p_1\, p_4 - \varepsilon_1\, p_3^2 p_4 + \varepsilon_1 \delta_1 p_2\, p_4] + \tau'_v[\,p_1 p_2 \varepsilon_4 - \xi^2 p_1\,\delta_3 \varepsilon_3 + p_3^2 \xi^2 \varepsilon_3 - \delta_1\,\xi^2 p_2 \varepsilon_3] + \tau'_q\,[\,p_1\,p_6\,\delta_3 + p_1 p_6\,\varepsilon_5 \varepsilon^2 - p_3^2 p_6 - p_3\,p_7\,\varepsilon_5\,\varepsilon - i\xi\varepsilon\varepsilon_5\,p_3 p_6 + i\xi\delta_3\varepsilon_5\,p_7 + \delta_1\,p_2 p_6\,]$$

$$S = \tau'_t\,[\,p_1 p_2\,p_4 \varepsilon_1\,] + \tau'_v\,[\,-p_1\,p_2\,\xi^2 \varepsilon_3] + \tau'_q\,[\,p_1 p_2\,p_6 + i\xi\varepsilon_5\,p_2\,p_7]$$

For the medium M_1

$$(u, w, T) = \sum_{j=1}^{3} A_j\,(1, d_j, l_j) e^{-m_j z} e^{i\xi(x-ct)} \tag{25}$$

Thus, from Eqs. (20) and (25)

$$u^* = \sum_{j=1}^{3} A_j\,e^{-m_j z}, \qquad w^* = \sum_{j=1}^{3} d_j A_j e^{-m_j z}, \qquad T^* = \sum_{j=1}^{3} l_j A_j e^{-m_j z}$$

where

$$d_j = \frac{m_j^4 A^* + m_j^2 B^* + C^*}{m_j^4 A' + m_j^2 B' + C'}, \qquad j = 1, 2, 3$$

$$l_j = \frac{m_j^4 P^* + m_j^2 Q^* + R^*}{m_j^4 A' + m_j^2 B' + C'}, \qquad j = 1, 2, 3$$

where

$$A^* = \tau_t'[-\delta_1 \; p_5 \varepsilon_2] + \tau_v'[\delta_1 \; \varepsilon_4 \,],$$

$$B^* = \tau_t'[-p_1 p_5 \varepsilon_2 + \delta_1 \; \varepsilon_1 p_4] + \tau_v'[p_1 \varepsilon_4 - \delta_1 \xi^2 \varepsilon_3 \,] + \tau_q'[\, \delta_1 p_6]$$

$$C^* = \tau_t'[\, p_1 p_4 \, \varepsilon_1] + \tau_v'[-p_1 \xi^2 \varepsilon_3 \,] + \tau_q' \, [\, p_1 p_6 \; - \varepsilon_5 \, p_6 \; \xi^2]$$

$$A' = \tau_t'[-\, p_5 \; \varepsilon_2 \delta_3 \,] + \tau_v'[\, \varepsilon_4 \delta_3 \,]$$

$$B' = \tau_t'[-\, p_2 p_5 \; \varepsilon_2 + \; p_4 \; \varepsilon_1 \delta_3 \,] + \tau_v'[\, p_2 \; \varepsilon_4 - \delta_3 \varepsilon_3 \xi^2] + \tau_q'[\, \delta_3 p_6 \; + \varepsilon^2 \varepsilon_5 p_6 \,]$$

$$C' = \tau_t'[\, p_2 p_4 \, \varepsilon_1] + \tau_v'[\, -\epsilon_3 \xi^2 p_2 \,] + \tau_q'[\, p_2 p_6], \qquad P^* = [\delta_1 \delta_3]$$

$$Q^* = [p_1 \delta_3 + p_2 \delta_1 - \; {p_3}^2], \qquad R^* = \; [p_1 p_2]$$

For medium M_2 ($z > 0$)
We attach bars for the medium M_2 i.e.,

$$(\bar{u}, \bar{w}, \bar{T}) = \left(1, \overline{d_j}, \overline{l_j}\right) e^{\overline{m_j} z} \overline{A_j} e^{i\xi(x-ct)} \tag{26}$$

where quantities $\bar{u},\ \bar{w},\ \bar{T},\ \overline{d_j},\ \overline{l_j},\ \overline{A_j},\ \overline{m_j}$ are obtained by attaching bars in the above expressions.

5. Boundary conditions

Following Kaur and Lata (2020), we assume that the half spaces are in perfect contact. Thus, there is continuity of components of displacement vector, normal stress vector, tangential stress vector, temperatures and temperature change at the interface.

$$\begin{gathered} \sigma_{zz} = \bar{\sigma}_{zz}, \qquad \sigma_{zx} = \bar{\sigma}_{zx}, \qquad T = \bar{T}, \qquad u = \bar{u} \\ w = \bar{w}, \qquad K_3^* \frac{\partial T}{\partial z} = \overline{K_3^*} \frac{\partial \bar{T}}{\partial z} \qquad \text{at } z = 0 \end{gathered} \tag{27}$$

6. Derivations of the secular equations

By using the values of $u,\ w,\ T,\ \bar{u},\ \bar{w}, \bar{T}$ in Eq. (27), we get six linear equations as

$$\sum_{j=1}^{3} \eta_{qj} A_j + \sum_{j=1}^{3} \eta_{q(j+3)} \bar{A}_j = 0\,, \qquad q = 1, 2, 3, 4, 5, 6$$

where

$$
\begin{aligned}
&\eta_{1j} = i\xi \tfrac{c_{13}}{\rho c_1^2} - \tfrac{c_{33}}{\rho c_1^2} d_j m_j - \varepsilon\, l_j, && j = 1,2,3\\
&\eta_{1(j+3)} = -i\xi\, \frac{\overline{c_{13}}}{\rho c_1^2} + \frac{\overline{c_{33}}}{\rho c_1^2}\overline{d_j}\,\overline{m_j} + \bar{\varepsilon}\overline{l_j}, && j = 1,2,3\\
&\eta_{2j} = -\tfrac{c_{55}}{\rho c_1^2} m_j + \tfrac{c_{55}}{\rho c_1^2} d_j i\xi\,, && j = 1,2,3\\
&\eta_{2(j+3)} = \frac{\overline{c_{55}}}{\rho c_1^2} m_j - \frac{\overline{c_{55}}}{\rho c_1^2} i\xi \overline{d_j}, && j = 1,2,3\\
&\eta_{3j} = l_j, && j = 1,2,3\\
&\eta_{3(j+3)} = -\overline{l_j}, && j = 1,2,3 \qquad (28)\\
&\eta_{4j} = 1, && j = 1,2,3\\
&\eta_{4(j+3)} = -1, && j = 1,2,3\\
&\eta_{5j} = d_j, && j = 1,2,3\\
&\eta_{5(j+3)} = -\overline{d_j}, && j = 1,2,3\\
&\eta_{6j} = -K_3^* m_j l_j\\
&\eta_{6(j+3)} = \overline{K_3^*}\,\overline{m_j}\,\overline{l_j}
\end{aligned}
$$

The system of Eq. (28) has a non-trivial solution if the determinant of unknowns $A_j, \bar{A}_j, j = 1,2,3$ vanishes i.e.,

$$\left|\eta_{ij}\right|_{6\times 6} = 0$$

The whole information regarding the wavenumber, phase velocity and attenuation coefficient of Stoneley waves are described by secular equations.

7. Particular cases

1) If we put $K_1^* = K_3^* = 0$ in Eq. (12), we get the expressions for Stoneley wave propagation in orthotropic magneto-thermoelastic medium without energy dissipation with fractional order theory of thermoelasticity.

2) If $C_{11} = C_{33}$, $2C_{44} = C_{11} - C_{33}$, we get the expressions for Stoneley wave propagation in transversely isotropic magneto-thermoelastic medium with and without energy dissipation with three phase lags and fractional order theory of thermoelasticity.

3) If $C_{11} = C_{33} = \lambda + 2\mu$, $C_{13} = \lambda$, $C_{55} = \mu$, $\beta_1 = \beta_3 = \beta$, $K_1 = K_3 = K$, $K_1^* = K_3^* = K^*$, we obtain the expressions for Stoneley wave propagation for isotropic materials with and without energy dissipation for three phase-lag model of heat transfer with fractional order theory of thermoelasticity.

4) If we put $\tau_t = \tau_v = \tau_q = 0$ and $K_1^* = K_3^* = 0$ in Eq. (12), then the resulting equation represents heat equation for coupled theory of thermoelasticity.

5) If we put $K_1^* = K_3^* = 0$ in Eq. (12), we obtain the heat equation for dual-phase-lag model with fractional order theory of thermoelasticity.

Table 1 Following Biswas *et al*. (2017), cobalt material has been taken for the purpose of numerical computation for the medium M_1 with non-dimensional parameter L = 1

Quantity	Value	Unit
c_{11}	3.071×10^{11}	$\mathrm{kgm^{-1}s^{-2}}$
c_{13}	1.650×10^{11}	$\mathrm{kgm^{-1}s^{-2}}$
c_{33}	3.581×10^{11}	$\mathrm{kgm^{-1}s^{-2}}$
c_{55}	1.510×10^{11}	$\mathrm{kgm^{-1}s^{-2}}$
c_E	4.27×10^{2}	$\mathrm{JKg^{-1}K^{-1}}$
β_1	7.04×10^{6}	$\mathrm{Nm^{2}K^{-1}}$
β_3	6.90×10^{6}	$\mathrm{Nm^{2}K^{-1}}$
T_0	298	K
K_1	6.90×10^{2}	$\mathrm{Wm^{-1}K^{-1}}$
K_3	7.01×10^{2}	$\mathrm{Wm^{-1}K^{-1}}$
K_1^*	1.313×10^{2}	$\mathrm{Ws^{-1}}$
K_3^*	1.54×10^{2}	$\mathrm{Ws^{-1}}$
ρ	8.836×10^{3}	$\mathrm{kgm^{-3}}$
τ_t	1.5×10^{-7}	s
τ_v	1.0×10^{-7}	s
τ_q	2.0×10^{-7}	s

Table 2 Following Kumar and Chawla (2014), we take the following values of the relevant parameter for an orthotropic thermoelastic material for numerical computations for the medium M_2 with non-dimensional parameter L = 1

Quantity	Value	Unit
$\bar{c}_{11}$	18.78×10^{10}	$\mathrm{kgm^{-1}s^{-2}}$
$\bar{c}_{13}$	8.0×10^{10}	$\mathrm{kgm^{-1}s^{-2}}$
$\bar{c}_{33}$	10.2×10^{10}	$\mathrm{kgm^{-1}s^{-2}}$
$\bar{c}_{55}$	10.06×10^{10}	$\mathrm{kgm^{-1}s^{-2}}$
$\bar{\mathrm{T}}_0$	293	K
$\bar{\beta}_1$	1.96×10^{-5}	$\mathrm{Nm^{-2}K^{-1}}$
$\bar{\beta}_3$	1.4×10^{-5}	$\mathrm{Nm^{-2}K^{-1}}$
$\bar{\rho}$	8.836×10^{3}	$\mathrm{kgm^{-3}}$
$\bar{C}_E$	4.27×10^{2}	$\mathrm{JKg^{-1}K^{-1}}$
$\bar{K}_1{}^*$	1.313×10^{2}	$\mathrm{Ws^{-1}}$
$\bar{K}_3{}^*$	1.54×10^{2}	$\mathrm{Ws^{-1}}$
$\bar{K}_1$	0.12×10^{3}	$\mathrm{wm^{-1}k^{-1}}$
$\bar{K}_3$	0.33×10^{3}	$\mathrm{wm^{-1}k^{-1}}$
$\bar{\tau_t}$	1.5×10^{-7}	s
$\bar{\tau_v}$	1.0×10^{-8}	s
$\bar{\tau_q}$	2.0×10^{-7}	s

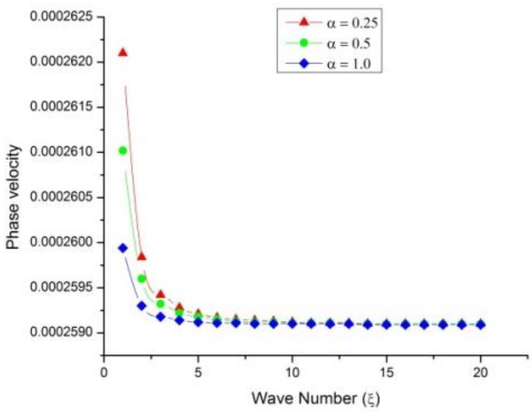

Fig. 1 Variation of phase velocity with wave number ξ

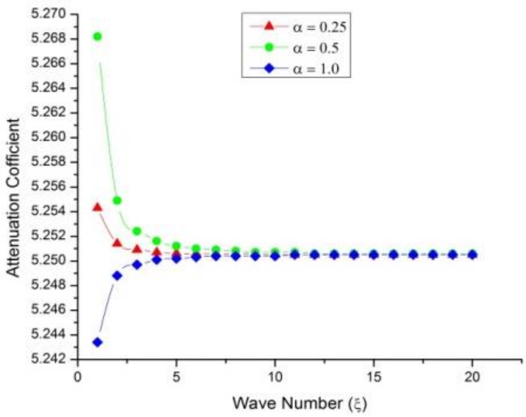

Fig. 2 Variation of Attenuation coefficient with wave number ξ

8. Numerical results and discussion

Using above values of parameters, the graphical representation of phase velocity, attenuation coefficient depth of Stoneley wave, stress component and temperature change with wave number 'ξ' has been made for an orthotropic body by using different values of fractional parameter $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 1.0$ respectively.

• **Effect of fractional parameter**:

(1) The red dashed line with centre symbol triangle (Δ) for an orthotropic material corresponds to $\alpha = 0.25$.

(2) The green dashed line with centre symbol plus (o) for an orthotropic material corresponds to $\alpha = 0.5$.

(3) The blue dashed line with centre symbol circle (◊) for an orthotropic material corresponds to $\alpha = 1.0$.

Fig. 1 gives the variation of phase velocity w.r.t. 'ξ' for $\alpha = 0.25$, $\alpha = 0.5$ and $\alpha = 0.1$, respectively.

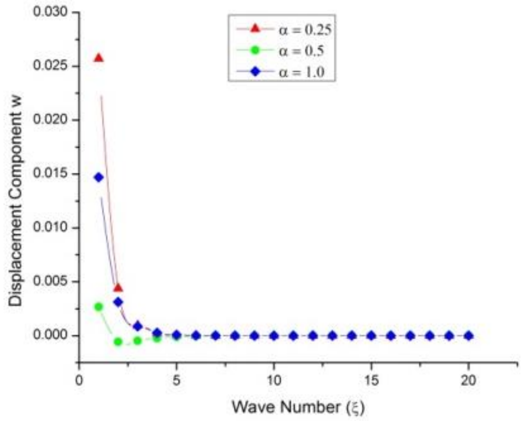

Fig. 3 Variation of normal displacement w with wave number ξ

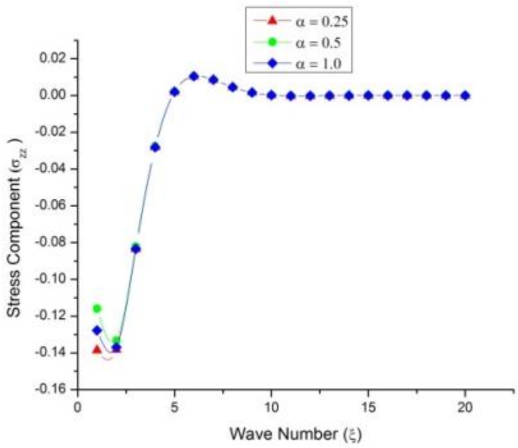

Fig. 4 Variation of stress component σ_{zz} with wave number ξ

It is clear from the graphs that the value of phase velocity decreases sharply in the range $0 \leq \xi \leq 2.5$ then exhibits a steady state behaviour in the rest of the range for all values of $\alpha = 0.25$, $\alpha = 0.5$ and $\alpha = 0.1$, respectively. Fig. 2 demonstrates the Stoneley wave's attenuation coefficient w.r.t. 'ξ' for different values of fractional parameter α. It can be noticed that for $\alpha = 0.25$ and $\alpha = 0.5$, the value of attenuation coefficient declines in the range $0 \leq \xi \leq 5$ then remains constant in the rest of the range. Moreover, for $\alpha = 0.1$ near the interface of the two material the magnitude values of attenuation coefficient of Stoneley wave shows the opposite behaviour in the range $0 \leq \xi \leq 5$. Fig. 3 displays the behaviour of displacement component w w.r.t. 'ξ'. We see that for $\alpha = 0.25$, $\alpha = 0.5$ and $\alpha = 0.1$ in the range $0 \leq \xi \leq 5$ it decreases then shows a steady state behaviour in the remaining range. The variation of stress component σ_{zz} with wave number 'ξ' has shown in Fig. 4. It can be noticed that in the starting range $0 \leq \xi \leq 2.5$ for $\alpha = 0.25$, $\alpha = 0.5$ and $\alpha = 0.1$ the value of stress component σ_{zz} declines after that it increases in the range $2.5 \leq \xi \leq 5$. The peak value attains near $\xi = 5.3$ then it comes to steady state in the rest of the range for all values of α. Fig. 5 gives the variation of temperature change T w.r.t. ξ for $\alpha = 0.25$, $\alpha = 0.5$ and $\alpha = 0.1$, respectively. In the starting range $0 \leq \xi \leq 4$ its value decreases sharply then remains same in the rest of the range.

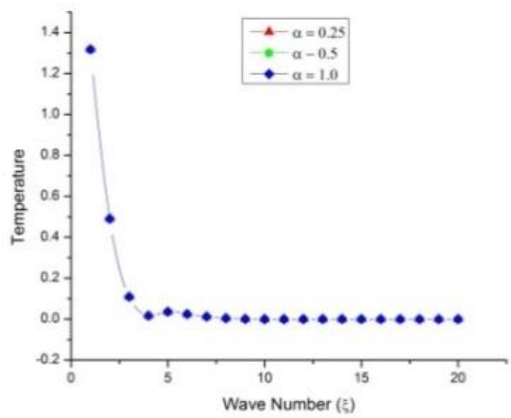

Fig. 5 Variation of temperature change T with wave number ξ

9. Conclusions

In the present work, the propagation of Stoneley waves in a homogeneous orthotropic solid by using fractional order heat conduction equation with three phase lags has been studied. The effect of fractional parameter on the Stoneley wave phase velocity, attenuation coefficient, displacement component and stress component as well as on temperature change has been investigated. It can be observed that for small value of non-dimensional wave number, the effect of fractional parameter has a significant impact on dispersion curve for lower value and negligible effect is observed for higher value. It is also observed that the velocity of surface (Stoneley) waves not only influenced by the direction of wave propagation but as well as on the elastic properties and density of materials. Stoneley wave's analysis provides information about the positions of fractures and permeability of the formation. These waves not only deliver better information about the internal structure of the earth but are also helpful in the assessment of valuable materials under the earth's surface. The results of this research may provide useful information for experimental scientists, researchers and seismologists which are working in this field.

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